ON BELLMAN-BIHARI INTEGRAL INEQUALITIES

EUTIQUIO C. YOUNG
Florida State University
Tallahassee, Florida 32306

(Received October 15, 1980)

ABSTRACT. Integral inequalities of the Bellman-Bihari type are established for
integrals involving an arbitrary number of independent variables.

KEY WORDS AND PHRASES. Integral inequalities, differential inequalities.

1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34A40, 35B45.

1. INTRODUCTION.

In a number of recent papers, Dhongade and Deo [1] and Pachpatte [2,3,4] have
generalized the well known Bellman inequality [5] and Bihari's generalization of
it [6] in several different directions. Although the results concern only functions
of a single variable, it was shown in [7] that corresponding inequalities also hold
for functions of several independent variables. The purpose of this note is to
show that the technique employed in [7] can be profitably utilized to establish
more general integral inequalities of the Bellman-Bihari type in any number of
independent variables. We present here some of the results along this line.

As in [7] we assume that all the functions under discussion are defined in a
bounded domain $\mathbb{R}$ of $\mathbb{R}^n$ which, for convenience, is assumed to contain the origin.
The symbol $x < y$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are any two
points of $\mathbb{R}$, means $x_i < y_i$ for $i = 1, \ldots, n$. We also adopt the notation

$$\int_0^\tau f(s)ds = \int_0^{x_n} \ldots \int_0^{x_1} f(s_1, \ldots, s_n)ds_1 \ldots ds_n$$
2. MAIN RESULTS.

Our first result is a variation of Theorem 3 of [7].

THEOREM 1. Let \( u, f, \) and \( g \) be continuous and nonnegative in \( \mathbb{R} \) and let \( a \) be continuous, positive and nondecreasing in \( \mathbb{R} \). Let \( W: [0, \infty) \to [0, \infty) \) be continuously differentiable and nondecreasing such that

\[
W^{-1}(v) - W^{-1}(u) \leq W^{-1}(v - u), \quad u \geq 0, \quad v > 0
\]  

(2.1)

Then the inequality

\[
u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)W(u)dt]ds
\]

(2.2)

implies

\[
u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds]
\]

if \( g(x) \leq f(x) \) or

\[
u(x) \leq a(x)[1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s g(t)dt)ds]
\]

if \( f(x) \leq g(x) \), where \( G^{-1} \) is the inverse of the function

\[
G(w) = \int_{w_0}^w \frac{dr}{r+\overline{W}(r)}, \quad w > w_0 > 0
\]

(2.5)

provided \( G(1) + \int_0^x f(t)dt \) lies in the domain of \( G^{-1} \).

PROOF. Since \( a > 0, \overline{W} > 0 \) and both are nondecreasing, and by (2.1), we may rewrite (2.2) in the form

\[
u(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)W(m)dt]ds
\]

(2.6)

where \( m(x) \leq u(x)/a(x) \). If we set \( v(x) \) equal to the right hand side of (2.6) and differentiate, we find

\[
D_1...D_n v(x) = f(x)(m(x) + \int_0^x g(t)W(m)dt)
\]

\[
\leq f(x)(v(x) + \int_0^x g(t)\overline{W}(v)dt)
\]

(2.7)

where \( D_i \) indicates differentiation with respect to \( x_i, \ i = 1,...,n. \)
Let us define
\[ w(x) = v(x) + \int_0^x g(t)W(v)\,dt \quad (2.8) \]
and assume \( g(x) \leq f(x) \). Then, by differentiating (2.8) and using (2.7), we obtain
\[ D_1\ldots D_n w(x) = D_1\ldots D_n v(x) + g(x)W(v) \leq f(x)w(x) + g(x)W(w) \leq f(x)(w(x) + W(w)) \quad (2.9) \]

Set \( S(x) = w(x) + W(w) \). Following the technique in [7], we observe from (2.9) that
\[ \frac{S(x)D_1\ldots D_n w(x)}{S(x)^2} \leq \frac{D_1S(x)D_2\ldots D_n w(x)}{S(x)^2} \]
or
\[ D_1\left(\frac{D_2\ldots D_n w(x)}{S(x)}\right) \leq f(x) \]

Note that, from the hypotheses, it follows that \( D_1(w(x) + W(w)) \geq 0 \), for \( i = 1, 2, \ldots, n \). Hence, integrating with respect to \( x_1 \) from 0 to \( x_1 \), we find
\[ \frac{D_2\ldots D_n w(x)}{S(x)} \leq \int_0^{x_1} f(s_1, x_2, \ldots, x_n)\,ds_1 \quad (2.10) \]

Similarly, since
\[ \frac{D_2S(x)(D_3\ldots D_n w(x))}{S(x)^2} \geq 0 \]
the left hand side of (2.10) can be replaced by
\[ D_2\left(\frac{D_3\ldots D_n w(x)}{S(x)}\right) \leq \int_0^{x_1} f(s_1, x_2, \ldots, x_n)\,ds_1 \]

By integrating this from 0 to \( x_2 \), we obtain
\[ \frac{D_3\ldots D_n w(x)}{S(x)} \leq \int_0^{x_2} \int_0^{x_1} f(s_1, s_2, x_3, \ldots, x_n)\,ds_1\,ds_2 \]

Continuing in this manner, we have after \( (n-1) \) steps
\[ \frac{D_n w(x)}{S(x)} \leq \int_0^{x_{n-1}} \cdots \int_0^{x_1} f(s_1, \ldots, s_{n-1}, x_n)\,ds_1\cdots ds_{n-1} \quad (2.11) \]
With the function $G(w)$ defined in (2.5), we note that
\[
D_n G(w) = G'(w) D_n w(x) = D_n w(x)/(w(x) + W(w)).
\]
Hence, integration of (2.11) from 0 to $x_n$ yields
\[
G(w(x_1, \ldots, x_n)) - G(w(x_1, \ldots, x_{n-1}, 0)) \leq \int_0^x f(s) \, ds
\]
or
\[
w(x) \leq G^{-1}(G(1) + \int_0^x f(s) \, ds)
\]  \hspace{1cm} (2.12)
since $w(x) = v(x) = 1$ when $x_i = 0$ for any $i, 1 \leq i \leq n$.

From (2.7) and (2.8) we have
\[
D_1 \ldots D_n v(x) \leq f(x) w(x)
\]  \hspace{1cm} (2.13)
Substituting for $w(x)$ from (2.12) and integrating (2.13), we finally obtain
\[
v(x) \leq 1 + \int_0^x f(s) G^{-1}(G(1) + \int_0^s f(t) \, dt) \, ds
\]  \hspace{1cm} (2.14)
The inequality (2.3) follows from (2.6), (2.14), and the fact that $m(x) = u(x)/a(x)$.

If $f(x) \leq g(x)$, then we need only replace $f$ by $g$ in the last line of (2.9) to obtain again (2.12) with $f$ replaced by $g$. The result (2.4) then follows in the same fashion.

Our next theorem combines the feature of Theorems 1 and 2 of [7].

**THEOREM 2.** Let $u, f, g,$ and $h$ be continuous and nonnegative functions in $\mathbb{R}$, and let $a$ be continuous, positive, and nondecreasing in $\mathbb{R}$. Let $Z: [0, \infty) \to [0, \infty)$ satisfy the same conditions as $W$ in Theorem 1 such that $Z$ is submultiplicative. If $u$ satisfies
\[
u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)u(t) \, dt] \, ds + \int_0^x h(s)Z(u) \, ds
\]  \hspace{1cm} (2.15)
then
\[
u(x) \leq a(x)p(x)H^{-1}(H(1) + \int_0^x h(s)Z(p) \, ds)
\]  \hspace{1cm} (2.16)
where
\[
p(x) = 1 + \int_0^x f(s) \exp \left( \int_0^s (f(t) + g(t)) \, dt \right) \, ds
\]  \hspace{1cm} (2.17)
and $H^{-1}$ is the inverse of the function
The proof of this theorem makes use of the following result which we state as a lemma. This was established in [7] as Theorem 1.

**Lemma.** Under the hypotheses of Theorem 2, the inequality

\[ u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)u(t)dt]ds \]

implies

\[ u(x) \leq a(x)[1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dtds]. \]

**Proof of Theorem 2.** As in Theorem 1 we rewrite (2.15) in the form

\[ m(x) \leq 1 + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds \]

\[ + \int_0^x h(s)Z(m)ds \]

If we set

\[ v(x) = 1 + \int_0^x h(s)Z(m)ds \]

then (2.19) becomes

\[ m(x) \leq v(x) + \int_0^x f(s)[m(s) + \int_0^s g(t)m(t)dt]ds. \]

Hence, by the lemma, we have

\[ m(x) \leq v(x)(1 + \int_0^x f(s)\exp \int_0^s (f(t) + g(t))dtds) \]

\[ \leq v(x)p(x) \]

Since \( Z \) is submultiplicative, we note that \( Z(m) \leq Z(v)Z(p) \). Therefore, differentiating (2.20) with respect to \( x_1, \ldots, x_n \), we find

\[ D_1 \ldots D_n v(x) = h(x)Z(m) \]

\[ \leq h(x)Z(v)Z(p) \]
By the same argument as in the proof of Theorem 1, we can integrate (2.22) to obtain

\[ H(v(x_1, \ldots, x_n)) - H(v(x_1, \ldots, x_{n-1}, 0)) \leq \int_0^x h(s)Z(p)ds \]

where \( H(v) \) is defined by (2.18). This gives

\[ v(x) \leq H^{-1}(H(1) + \int_0^x h(s)Z(p)ds) \]  

(2.23)

The substitution of (2.23) in (2.21) yields the inequality (2.16) since \( m(x) = u(x)/a(x) \).

When \( g(x) = 0 \), Theorem 2 reduces to Theorem 3 of [7].

By combining Theorems 1 and 2, we finally have

**THEOREM 3.** Let \( u, a, f, g, h, \) and \( Z \) be as in Theorem 2 and let \( W \) be as in Theorem 1. If \( u \) satisfies

\[ u(x) \leq a(x) + \int_0^x f(s)[u(s) + \int_0^s g(t)W(u)dt]ds \]  

(2.24)

then

\[ u(x) \leq a(x)q(x)H^{-1}(H(1) + \int_0^x h(s)Z(q)ds) \]  

(2.25)

where

\[ q(x) = 1 + \int_0^x f(s)G^{-1}(G(1) + \int_0^s f(t)dt)ds \]  

(2.26)

\( G^{-1} \) is the inverse of the function defined in (2.5) and \( H^{-1} \) is the inverse of the function defined in (2.18).

**PROOF.** We rewrite (2.24) in the form

\[ m(x) \leq v(x) + \int_0^x f(s)[m(s) + \int_0^s g(t)W(m)dt]ds \]  

(2.27)

where

\[ v(x) = 1 + \int_0^x h(s)Z(m)ds \]  

(2.28)
with \( m(x) = u(x)/a(x) \). Then according to Theorem 1, we have

\[
m(x) \leq v(x)[1 + \int_0^x f(s)t^{-1}(t+1) + \int_0^s f(t)dt] ds
\]

\[
< v(x)q(x)
\]

Since \( Z(m) \leq Z(v)Z(q) \), we obtain from (2.28)

\[
D_{1...n}v(x) = h(x)Z(m) \leq h(x)Z(v)Z(q)
\]

With \( H(v) \) defined by (2.18), we obtain as in the proof of Theorem 2

\[
v(x) < H^{-1}(H(1) + h(s)Z(q)ds)
\]

The substitution of this for \( v(x) \) in (2.29) leads to the desired inequality (2.25).

Observe that, when \( h(x) = 0 \), (2.25) reduces to (2.3); when \( v = u \), it agrees with (2.16) with \( g \) replaced by \( f \) in view of the condition \( g \leq f \).

We remark that our Theorems 1, 2, and 3 correspond respectively to Theorems 4, 2, and 5 of [4]. From the argument presented above, we readily see that other more general integral inequalities can also be established for \( n \) independent variables along the lines considered in [1] and [4].

REFERENCES


Special Issue on
Time-Dependent Billiards

Call for Papers

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

**Edson Denis Leonel**, Departamento de Estatística, Matemática Aplicada e Computação, Instituto de Geociências e Ciências Exatas, Universidade Estadual Paulista, Avenida 24A, 1515 Bela Vista, 13506-700 Rio Claro, SP, Brazil; edleonel@rc.unesp.br

**Alexander Loskutov**, Physics Faculty, Moscow State University, Vorob'evy Gory, Moscow 119992, Russia; loskutov@chaos.phys.msu.ru