ON THE ASYMPTOTIC BIEBERBACH CONJECTURE

MAURISO ALVES and ARMANDO J.P. CAVALCANTE

Department of Mathematics
Universidade Federal de Pernambuco
Recife, Pe, 50.000 BRASIL

(Received December 11, 1981)

ABSTRACT. The set $S$ consists of complex functions $f$, univalent in the open unit disk, with $f(0) = f'(0) - 1 = 0$. We use the asymptotic behavior of the positive semidefinite FitzGerald matrix to show that there is an absolute constant $N_0$ such that, for any $f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S$ with $|a_3| \leq 2.58$, we have $|a_n| < n$ for all $n > N_0$.

KEY WORDS AND PHRASES. Univalent functions.

1980 AMS SUBJECT CLASSIFICATION CODES. 30A32, 30A34.

1. INTRODUCTION.

Let $S$ denote the class of all normalized univalent functions $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ in the open unit disc $D$. The Bieberbach conjecture states that, for functions in $S$, one has $|a_n| \leq n$ for all $n \in \mathbb{N}$. It is known to be true for $n \leq 6$. The best known estimate for all coefficients is $|a_n| \leq (1.066)n$ (Horowitz [1]).

On the other hand, Hayman's Regularity Theorem (Hayman [2]) states that $\lim_{n \to \infty} \frac{n}{a_n} = 1$ for each $f \in S$, and that equality holds only for the Koebe function $K(z) = \frac{z}{(1-\theta z)^2}$, $|n| = 1$, for which $|a_n| = n$. This implies that $|a_n| \leq n$ for $n \geq n_0(f)$.

Hayman [3] also proved that $A_n/n$ tends to a limit, where $A_n$ is the maximum of $|a_n|$ for all $f \in S$. It is still an open question as to whether this limit is equal to 1. The asymptotic Bieberbach conjecture asserts that $\lim_{n \to \infty} A_n/n = 1$, where $A_n = \max_{f \in S} |a_n|$.
Ehrig [4] has proved via the FitzGerald Inequality [5] that if \( f \in S \) and \( |a_3| \leq C < 2.43 \), then \( |a_n| < n \) for all \( n \geq N_0 \), where \( N_0 \) depends only on \( C \) and not (as in Hayman's Regularity Theorem) on \( f \). This result is a proof of the Asymptotic Bieberbach Conjecture for a subclass of \( S \).

In this paper, we apply the Asymptotic FitzGerald Inequalities to get, by elementary means, an improvement of Ehrig's result (Theorem 1) and the result in [6], (see Remark 2).

2. PRELIMINARY RESULTS.

**THEOREM A.** (FitzGerald Inequality, [5]). Let

\[
f(z) = z + \sum_{k=2}^{\infty} a_k(f) z^k
\]

be in \( S \) and define

\[
q_{mn}(f) = q_{nm}(f) = \left( \sum_{j=1}^{n+m-1} \beta_j(m,n) b_j^2(f) \right) - b_m^2(f) b_n^2(f)
\]

where \( b_j(f) = |a_j(f)|, \beta_j(m,n) = \beta_j(n,m), j \in \mathbb{N}, \) and for \( m < n \):

\[
\beta_j(m,n) = \begin{cases} m-|j-n| & \text{for } |j-n| < m \\ 0 & \text{if otherwise.} \end{cases}
\]

Then the FitzGerald matrix

\[
Q(f) = (q_{mn}(f))_{m,n} \in \mathbb{N}
\]

is positive semi-definite.

**THEOREM B.** (Asymptotic FitzGerald Inequalities [7]). Let \( \{f_n\}, n \in \mathbb{N}, \) be a sequence of functions in \( S \), such that

a) \( f_n \) converges locally uniformly to \( f \in S \)

b) \( \lim \inf b_n(f_n)/n \leq \beta \leq \lim \sup b_n(f_n)/n \)

c) \( \alpha(f) = \lim b_n(f)/n \)

d) \( d = \lim \alpha(f_n) \).

Then \( A = Q(j_1,j_2,\ldots,j_{r-1}, \alpha(f),\ldots,\alpha(j),\beta,d,\ldots,d) \) \( (f) \), defined below, is a positive semi-definite matrix.

Denote by \( E_{mn} \) the \( m \times n \) matrix whose elements are all equal to one. Moreover,
let $H_{mn}(f)$ be the $m \times n$ matrix defined by its elements $h_{st}(f) = j_t^2 = b_t^2(f)$. We use the notation

$$Q(j_1, \ldots, j_{r-1})(f) = (q_{j_t} j_{s-t}(f))_{1 \leq s, t \leq p}, M_p(x) = (m_{st}(x))_{1 \leq s, t \leq p}$$

where

$$m_{st}(x) = \begin{cases} \frac{7x^2}{6} - x^4 & \text{for } s = t \\ x^2(1 - x^2) & \text{for } s \neq t \end{cases}$$

and $\delta = \limsup_{n \to \infty} \delta_n$ where $\delta_n = \sup_{k} b_n(j_k)/n$. Then matrix $A$ has the following form:

$$A = \begin{pmatrix} Q(j_1, \ldots, j_{r-1})(f), \alpha(f)H_{r-1, q-r}(f), \beta^2 H_{r-1, 1}(f), d^2 H_{r-1, p-q}(f) \\ \alpha(f)H^T_{r-1, q-r}(f), M_{q-r}(\alpha(f)), \beta^2(1-\alpha^2(f))E_{q-r, 1}, d^2(1-\alpha^2(f))E_{q-r, p-q} \\ \beta^2 H_{r-1, 1}(f), \beta^2(1-\alpha^2(f))E_{1, q-r}, (7\delta^2/6 - \beta^4)E_{1, 1}, d^2(1-\beta^2)E_{1, p-q} \\ d^2 H_{r-1, p-q}(f), d^2(1-\alpha^2(j))E_{p-q, q-r}, d^2(1-\beta^2)E_{p-q, 1}, M_{p-q}(d) \end{pmatrix}$$

where $H^T$ is the transposed matrix of $H$.

THEOREM C. [6]. Let $f \in S$; if $|a_3| \leq 2.042$, then $|a_n| < n$ for all $n \geq 2$.

3. MAIN RESULTS.

For the proof of the Theorem 1, we need the following lemmas:

LEMMA 1. Suppose that $n > 1$ and that

$$a_n(H) = \sup_{f \in H, n} |a_n|,$$

where $H$ is a compact subclass of $S$. Let $f(z) = z + a_2z^2 + \ldots$ be in $H$ with $|a_n| = a_n(H)$. Then $f(z)$ satisfies the differential equation

$$z^2(f'(z))^2 a_n(v) f(z)^{-v-1} = n-1 + \sum_{v=1}^{n-1} \left( \frac{va_v}{a_n} z^{-n+v} + \frac{va_v}{a_n} z^{-n-v} \right).$$

Here, $a_n(v)$ are the coefficients of $f(z)^v$, where

$$f(z)^v = \sum_{n=v}^{\infty} a_n(v) z^n.$$

The proof of this lemma is completely similar to that of Theorem 1 in Schaeffer and Spencer ([8], p. 612).
As an application of the Lemma 1, we have the following:

**Lemma 2.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in H \) is the extremal function maximizing \(|a_3|\) such that \( a_3 > 0 \), then \( 2a_3 = a_2^2 + 2 \).

The proof of this lemma is completely similar to that in Garabedian and Schiffer ([9], p. 118).

Hayman [2] showed that for each \( f \in S \), the limits

\[
\alpha(f) = \lim_{r \to 1} (1-r)^2 M_\infty(r,f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n}
\]

exist, where \( M_\infty(r,f) \) is the maximum of \(|f(z)|\) on \(|z| = r\). The number \( \alpha(0 \leq \alpha \leq 1) \) is called the Hayman index of \( f \).

**Lemma 3.** [10]. If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in S \), and \( |a_2| \) is given, then

\[
\alpha(f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n} \leq 4b^{-2} \exp(2 - 4b^{-1})
\]

where \( b = 2 - (2 - |a_2|)^{1/2} \), and this inequality is sharp for \( 0 \leq |a_2| \leq 2 \).

**Lemma 4.** If \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \) satisfies the conditions of the Lemma 2 with \( |a_3| > 1 \), then

\[
\alpha(f) = \lim_{n \to \infty} \frac{|a_n(f)|}{n} \leq 4C^{-2} \exp(2 - 4C^{-1})
\]

where \( C = 2 - \left[ 2 - \sqrt{2}(|a_3| - 1)^{1/2} \right]^{1/2} \).

**Proof.** By Lemma 3, we have

\[
\alpha(f) \leq 4b^{-2} \exp(2 - 4b^{-1})
\]

where \( b = 2 - (2 - |a_2|)^{1/2} \). Since we may assume \( a_3 \) real positive (otherwise, we consider \( e^{-i\theta} f(e^{i\theta} z) \in H \), where \( 0 \leq \theta = -\frac{\arg a_3}{2} \leq 2\pi \), we obtain that

\[
b = 2 - (2 - |a_2|)^{1/2} = 2 - \left[ 2 - \sqrt{2}(|a_3| - 1)^{1/2} \right]^{1/2} = 2 - \left[ 2 - \sqrt{2}(|a_3| - 1)^{1/2} \right]^{1/2} = C.
\]

Hence,

\[
\alpha(f) \leq 4C^{-2} \exp(2 - 4C^{-1}).
\]

**Lemma 5.** [7]. Let \( \{f_n\}, n \in \mathbb{N} \), be a sequence of univalent functions in \( S \),
that converges locally uniformly to a function \( f \) in \( S \) and suppose that \( \alpha(f) > 0 \).

Then \( 7 \delta^2 \alpha^2(f) \geq 6 \beta^4 \), where \( \beta \) and \( \delta \) are chosen as in theorem B.

**PROOF.** Consider the \( (q - r + 1) \times (q - r + 1) \) principal minor

\[
Q(\alpha(f), \ldots, \alpha(f), \beta) = \begin{bmatrix}
M_{q-r}(\alpha(f)) & \beta^2(1-\alpha^2(f))E_{q-r,1} \\
\beta^2(1-\alpha^2(f))E_{1, q-r} & (7\delta^2/6-\beta^4)E_{1,1}
\end{bmatrix}
\]

of the matrix \( A \) in theorem B. A well-known result about positive semidefinite quadratic form is that all principal minor determinants of the matrix of the coefficients of the quadratic form are non-negative. Let \( \alpha = \alpha(f) \) and \( n = q-r \). If we use induction, we obtain:

\[
\det Q(\alpha, \ldots, \alpha, \beta) = (\alpha^2/6)^n (1-\alpha^2) \left[ n(7\delta^2-6\beta^4/\alpha^2) \right] + \alpha^{2n} 6^{-(n+1)} (7\delta^2-6\beta^4/\alpha^2) \geq 0;
\]

hence, for \( 0 \leq \alpha < 1 \)

\[
(7\delta^2 \alpha^2 - 6\beta^4) + \frac{6(1-\alpha^2)\beta^4}{6n(1-\alpha^2) + 1} \geq 0.
\]

Since \( n \) is arbitrary, the result follows. The case \( \alpha = 1 \) is immediate.

**THEOREM 1.** Let \( f(z) = z + \sum_{k=2}^{\infty} a_k z^k \) be in \( S \). If \( 1 \leq |a_3| \leq 2.58 \), then there is an absolute constant \( N_0 \) (independent of \( f \)), such that \( |a_n| < n \) for all \( n > N_0 \).

**PROOF.** Suppose the contrary and take a sequence \( \{g_k\}, k \in N \), of univalent functions in \( S \) such that

i) \( \{g_k\}, k \in N \), converges locally uniformly to a function \( g_0 \in S \),

ii) \( 1 \leq b_3(g_k) = |a_3(g_k)| \leq 2.58 \),

iii) \( 2a_3(g_k) = a_2(g_k)^2 + 2 \),

iv) \( b_n(g_k) \geq n_k \) for sequence \( n_k \) going to infinity.

**REMARK.** The functions \( g_k \) are the extremal functions maximizing \( |a_3(g_k)| \) in the compact subclass \( H_k = \{g \in S; 1 \leq b_3(g) \leq 2.58 \text{ and } b_{n_k}(g) \geq n_k\} \) of \( S \). Applying Lemma 2 to the subclass \( H_k \), we obtain condition (iii).

We pick for each \( n_k \) one of the functions of

\[ \{g_j\}, j = 0, 1, \ldots, \]

which maximizes \( b_n \) and denote it by \( f_{n_k} \); precisely, let \( \{f_{n_k}\}, k \in N \), be a sequence
of the functions in
\[ \{g_j\}, \quad j = 0, 1, \ldots, \]
such that
\[ \sup_{n_k} b_n g_j = b_n f_n. \]

We may assume that \( \{f_n\}, \quad k \in \mathbb{N} \), converges locally uniformly to a function \( f \in S \).

Otherwise, we pick a subsequence of \( \{f_n\}, \quad k \in \mathbb{N} \). Evidently, \( 1 \leq b_3(f) \leq 2.58 \) and \( 2a_3(f) = a_2(f)^2 + 2 \). For this sequence \( \{f_n\}, \quad k \in \mathbb{N} \), we have
\[ b_{n_k} = \sup_{n_k} b_{n_j} f_j/n_k = b_{n_k} f_n/n_k. \]

Thus
\[ \delta = \limsup_{k \to \infty} b_{n_k} f_n/n_k > 1. \]

We take \( \beta = \delta \) in Theorem B. First we show that \( \alpha(f) > 0 \). In fact, the determinant of the 2\times2 submatrix \( Q(\alpha(f), \delta) \) of \( A = Q(j_1, \ldots, j_{r-1}, \alpha(f), \ldots, \alpha(f), \beta, d, \ldots, d)(f) \) is
\[ (7 \alpha^2(f) - 6 \delta^4(f)) (7 \delta^2 - 6 \delta^4)/36 - \delta^4(1 - \alpha^2(f))^2 \geq 0. \]

This excludes \( \alpha(f) = 0 \) because \( \delta \geq 1 \). By Lemma 5, we have
\[ 7 \alpha^2(f) \delta^2 - 6 \delta^4 \geq 0 \quad \text{or} \quad \alpha^2(f) \geq 6 \delta^2/7 \geq 6/7. \]

This implies, by Lemma 4, that \( b_3(f) > 2.58 \) which contradicts the assumptions.

**COROLLARY.** Let \( f(z) = z + a_k z^k \) be in \( S \). If \( |a_3| \leq 2.58 \), then there is an absolute constant \( N_0 \) (independent of \( f \)), such that \( |a_n| < n \) for all \( n > N_0 \).

**PROOF.** The proof of corollary follows immediately from Theorem 1 and Theorem C.

**ACKNOWLEDGEMENT.** This research was supported in part by FINEP and CNPQ.

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

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Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk