A MINIMIZATION THEOREM IN QUASI-METRIC SPACES
AND ITS APPLICATIONS

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We prove a new minimization theorem in quasi-metric spaces, which improves the results of Takahashi (1993). Further, this theorem is used to generalize Caristi’s fixed point theorem and Ekeland’s $\varepsilon$-variational principle.

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1. Introduction. Caristi [1] proved a fixed point theorem on complete metric spaces which generalizes the Banach contraction principle. Ekeland [3] also obtained a non-convex minimization theorem, often called the $\varepsilon$-variational principle, for a proper lower semicontinuous function, bounded from below, on complete metric spaces. Later Takahashi [4] proved the following minimization theorem: let $X$ be a complete metric space and let $f : X \to (-\infty, \infty]$ be a proper lower semicontinuous function, bounded from below. Suppose that, for each $u \in X$ with $f(u) > \inf_{x \in X} f(x)$, there exists $v \in X$ such that $v \neq u$ and $f(v) + d(u, v) \leq f(u)$. Then there exists $x_0 \in X$ such that $f(x_0) = \inf_{x \in X} f(x)$. These theorems are very useful tools in nonlinear analysis, control theory, economic theory, and global analysis.

2. Main results. Throughout this note, we denote by $\mathbb{N}$ the set of all positive integers and by $\mathbb{R}$ the set of all real numbers.

DEFINITION 2.1. A real-valued function $\Phi$ defined on a topological space $X$ is said to be lower semicontinuous at $x$ in $X$ if and only if \{ $x_\lambda$ \} is a net in $X$ and $\lim_{\lambda} x_\lambda = x$ implies $\Phi x \leq \liminf \Phi x_\lambda$.

DEFINITION 2.2 [2]. A real-valued function $\Phi$ defined on a topological space $X$ is said to be weak lower semicontinuous at $x$ in $X$ if and only if \{ $x_\lambda$ \} is a net in $X$ and $\lim_{\lambda} x_\lambda = x$ implies $\Phi x \leq \limsup \Phi x_\lambda$. A mapping $\Phi$ is said to be a weak lower semicontinuous on $X$ if and only if it is weak lower semicontinuous for every $x \in X$.

DEFINITION 2.3. A pair $(X,d)$ of a set $X$ and a mapping $d$ from $X \times X$ into real numbers is said to be a quasi-metric space if and only if

\[
d(x, y) \geq 0, \quad d(x, y) = 0 \quad \text{iff} \quad x = y,
\]

\[
d(x, z) \leq d(x, y) + d(y, z) \quad \forall x, y, z \in X.
\]
**Definition 2.4.** A sequence \{x_n\} in X is said to be a left \(k\)-Cauchy sequence if for each \(k \in \mathbb{N}\) there is an \(N_k\) such that
\[
d(x_n, x_m) < \frac{1}{k} \quad \forall \, m \geq n \geq N_k.
\] (2.2)

A quasi-metric space is left \(k\)-sequentially complete if each left \(k\)-Cauchy sequence is convergent.

**Theorem 2.5.** Let \((X, d)\) be left \(k\)-sequentially complete quasi metric space such that for each \(x \in X\) the mapping \(u \to d(x, u)\) is a lower semicontinuous on \(X\). Let \(f : X \to (-\infty, \infty]\) be a proper weak lower semicontinuous function bounded from below such that for any \(u \in X\) with \(\inf_{x \in X} f(x) < f(u)\), there exists \(v \in X\) with \(v \neq u\) and \(f(v) + d(u, v) \leq f(u)\). Then there exists \(x_0 \in X\) such that \(\inf_{x \in X} f(x) = f(x_0)\).

**Proof.** Suppose that \(\inf_{x \in X} f(x) < f(y)\) for every \(y \in X\). For each \(y \in X\), we define \(S(y)\) by
\[
S(y) = \{z \in X : f(z) + d(y, z) \leq f(y)\}. \quad (2.3)
\]

From (2.3) and hypotheses of the theorem we have the following:

(*) For each \(y \in X\), there exists \(v \in X\) with \(v \neq y\) such that \(v \in S(y)\), and for each \(z \in S(y)\), \(S(z) \subseteq S(y)\).

For each \(y \in X\), we define \(A(y)\) by
\[
A(y) = \inf \{ f(z) : z \in S(y) \}. \quad (2.4)
\]

Choose \(u \in X\) with \(f(u) < \infty\). Then we choose a sequence \(\{u_n\}\) in \(S(u)\) as follows: when \(u = u_1, u_2, \ldots, u_n\) have been chosen, choose \(u_{n+1} \in S(u_n)\) such that
\[
f(u_{n+1}) < A(u_n) + \frac{1}{n}. \quad (2.5)
\]

Thus, we obtain a sequence \(\{u_n\}\) such that
\[
d(u_n, u_{n+1}) \leq f(u_n) - f(u_{n+1}), \quad (2.6)
\]
\[
f(u_{n+1}) - \frac{1}{n} < A(u_n) \leq f(u_{n+1}). \quad (2.7)
\]

By (2.6), \(\{f(u_n)\}\) is a nonincreasing sequence of reals and so it converges. Therefore, by (2.7) there is some \(\alpha\) in \(\mathbb{R}\) such that
\[
\alpha = \lim_{n \to \infty} A(u_n) = \lim_{n \to \infty} f(u_n) = \inf_{n \in \mathbb{N}} f(u_n). \quad (2.8)
\]

Let \(k \in \mathbb{N}\) be arbitrary. By (2.8) there exists some \(N_k\) such that \(f(u_n) < \alpha + 1/k\) for all \(n \geq N_k\). Thus, by monotony of \(\{f(u_n)\}\), for \(m \geq n \geq N_k\), we have
\[
\alpha \leq f(u_m) \leq f(u_n) < \alpha + \frac{1}{k}, \quad (2.9)
\]
and hence
\[
f(u_n) - f(u_m) < \frac{1}{k} \quad \forall \, m > n \geq N_k. \quad (2.10)
\]
From the triangle inequality, (2.6) and (2.10), we get
\[d(u_n, u_m) \leq \sum_{i=n}^{m-1} d(u_i, u_{i+1}) \leq \sum_{i=n}^{m-1} [f(u_i) - f(u_{i+1})]\]
\[\leq f(u_n) - f(u_m) < \frac{1}{k}\] \hspace{1cm} (2.11)
for all \(m > n \geq N_k\).

Therefore, \(\{u_n\}\) is a left \(k\)-Cauchy sequence in \(X\). By completeness, there exists \(z \in X\) such that \(u_n \to z\). Since \(f\) is a weak lower semicontinuous; by (2.8), we have
\[f(z) \leq \limsup_{n \to \infty} f(u_n) = \alpha.\] \hspace{1cm} (2.12)
From (2.11), we obtain
\[f(u_m) \leq f(u_n) - d(u_n, u_m).\] \hspace{1cm} (2.13)
Since \(f\) is a weak lower semicontinuous on \(X\) and \(u \to d(x, u)\) on \(X\) is a lower semicontinuous, we have
\[f(z) \leq \limsup_{m \to \infty} f(u_m) \leq f(u_n) + \limsup_{m \to \infty} [-d(u_n, u_m)]\]
\[= f(u_n) - \liminf_{m \to \infty} d(u_n, u_m) = f(u_n) - d(u_n, z).\] \hspace{1cm} (2.14)
Hence
\[d(u_n, z) \leq f(u_n) - f(z).\] \hspace{1cm} (2.15)
From (2.3) and (2.15), we obtain that \(z \in S(u_n)\) for every \(n \in \mathbb{N}\) and hence
\[A(u_n) \leq f(z) \quad \forall n \in \mathbb{N}.\] \hspace{1cm} (2.16)
Taking the limit when \(n\) tends to infinity, we have
\[\lim_{n \to \infty} A(u_n) \leq f(z).\] \hspace{1cm} (2.17)
From (2.8), (2.12), and (2.17), we have
\[f(z) = \alpha.\] \hspace{1cm} (2.18)
Since \(z \in S(u_n)\) and \(u_n \in S(u)\), by (*), we obtain \(z \in S(u)\). Suppose that \(v_1 \in S(z)\) and \(v_1 \neq z\). Then \(f(v_1) < f(z)\) or by (2.18), \(f(v_1) < \alpha\). Since \(v_1 \in S(z)\), \(z \in S(u_n)\) and \(u_n \in S(u)\), by (*), we have \(S(z) \subseteq S(u_n) \subseteq S(u)\). Hence \(v_1 \in S(u_n)\) and \(v_1 \in S(u)\). Thus
\[A(u_n) \leq f(v_1) \quad \forall n \in \mathbb{N}.\] \hspace{1cm} (2.19)
Taking the limit when \(n\) tends to infinity, we get
\[\alpha \leq f(v_1).\] \hspace{1cm} (2.20)
This is in contradiction with \(f(v_1) < \alpha\). Hence \(S(z) = \{z\}\). But, by (2.3) and hypothesis of a function \(f\) in theorem there exists \(y \in X\) such that \(y \neq z\) and \(\{y, z\} \subseteq S(z)\). This is a contradiction. This completes the proof. \(\square\)
Remark 2.6. Theorem 2.5 is a generalization of Takahashi’s minimization theorem [4].

Theorem 2.7. Let \((X,d)\) be left \(k\)-sequentially complete quasi-metric space such that for each \(x \in X\), the mapping \(u \rightarrow d(x,u)\) is a lower semicontinuous on \(X\). Let \(f : X \rightarrow (-\infty, \infty]\) be a proper weak lower semicontinuous function bounded from below. Assume that there exists a selfmapping \(T : X \rightarrow X\) such that

\[ f(Tx) + d(x,Tx) \leq f(x) \quad \forall x \in X. \tag{2.21} \]

Then \(T\) has a fixed point in \(X\).

Proof. Since \(f\) is proper, there exists \(v \in X\) such that \(f(v) < \infty\). Put

\[ Z = \{ x \in X \mid f(x) \leq f(v) \}. \tag{2.22} \]

Then, since \(f\) is weak lower semicontinuous, \(Z\) is closed. So \(Z\) is left \(k\)-sequentially complete. Let \(x \in Z\). Then, since

\[ f(Tx) + d(x,Tx) \leq f(x) \leq f(v). \tag{2.23} \]

So \(Z\) is invariant under \(T\). Assume that \(Tx \neq x\) for every \(x \in Z\). Then by Theorem 2.5, there exists \(u \in Z\) such that \(f(u) = \inf_{x \in X} f(x)\). Since \(f(Tu) + d(u,Tu) \leq f(u)\) and \(f(u) = \inf_{x \in Z} f(x)\), we have \(f(Tu) = f(u) = \inf_{x \in Z} f(x)\) and \(d(u,Tu) = 0\). Hence \(Tu = u\). This is a contradiction. Therefore \(T\) has a fixed point \(u\) in \(Z\). This completes the proof.

Remark 2.8. Theorem 2.7 is a generalization of Caristi’s fixed point theorem [1].

The following theorem is a generalization of Ekeland’s \(\varepsilon\)-variational principle [3].

Theorem 2.9. Let \((X,d)\) be left \(k\)-sequentially complete quasi-metric space such that for each \(x \in X\) the mapping \(u \rightarrow d(x,u)\) is a lower semicontinuous on \(X\). Let \(f : X \rightarrow (-\infty, \infty]\) be a proper weak lower semicontinuous function bounded from below. Then,

1. for any \(u \in X\) with \(f(u) < \infty\), there exists \(v \in X\) such that \(f(v) \leq f(u)\) and \(f(w) > f(v) - d(v,w)\) for every \(w \in X\) with \(w \neq v\);
2. for any \(\varepsilon > 0\) and \(u \in X\) with \(f(u) < \inf_{x \in X} f(x) + \varepsilon\), there exists \(v \in X\) such that \(f(v) \leq f(u)\), \(d(u,v) \leq 1\) and \(f(w) > f(v) - \varepsilon d(v,w)\) for every \(w \in X\) with \(w \neq v\).

Proof. (1) Let \(u \in X\) be such that \(f(u) < \infty\) and let

\[ Y = \{ x \in X \mid f(x) \leq f(u) \}. \tag{2.24} \]

Then \(Y\) is nonempty and complete. We prove that there exists \(v \in Y\) such that \(f(w) > f(v) - d(v,w)\) for every \(w \in X\) with \(w \neq v\). If not, for every \(x \in Y\), there exists \(w \in X\) such that \(w \neq x\) and \(f(w) + d(x,w) \leq f(x)\). Since \(f(w) \leq f(x) \leq f(u)\), \(w \in X\) is an element of \(Y\). By Theorem 2.5, there exists \(x_0 \in Y\) such that \(f(x_0) = \inf_{x \in Y} f(x)\). For this \(x_0 \in Y\), there exists \(x_1 \in Y\) such that \(x_0 \neq x_1\) and \(f(x_1) + d(x_0,x_1) \leq f(x_0)\).
Thus we have \( f(x_0) = f(x_1) \) and \( d(x_0, x_1) = 0 \). Hence \( x_0 = x_1 \). This is a contradiction. Therefore (1) holds.

(2) Put

\[
Z = \{ x \in X \mid f(x) \leq f(u) - \varepsilon d(u, x) \}. \quad (2.25)
\]

Then \( Z \) is nonempty and complete. Since \( \varepsilon d(u, x) \) is a quasi metric, as in the proof of (1), we have that there exists \( v \in Z \) such that \( f(w) > f(v) - \varepsilon d(v, w) \) for every \( w \in X \) with \( w \neq v \). Since \( v \in Z \), we have \( f(v) \leq f(u) - \varepsilon d(u, v) \leq f(u) \) and

\[
d(u, v) \leq \frac{1}{\varepsilon} \left[ f(u) - f(v) \right] \leq \frac{1}{\varepsilon} \left[ f(u) - \inf_{x \in X} f(x) \right] \leq \frac{1}{\varepsilon} \cdot \varepsilon = 1. \quad (2.26)
\]

This completes the proof of (2).

**Remark 2.10.** Theorem 2.9 is a generalization of Ekeland’s \( \varepsilon \)-variational principle in [3].

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**References**


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