ON BRANCHWISE IMPLICATIVE BCI-ALGEBRAS

MUHAMMAD ANWAR CHAUDHRY

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We introduce a new class of BCI-algebras, namely the class of branchwise implicative BCI-algebras. This class contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras (Chaudhry, 1990), and the class of medial BCI-algebras. We investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative BCI-algebras.

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1. Introduction. Iséki and Tanaka [10] defined implicative BCK-algebras and studied their properties. Further, Iséki [7, 8] gave the notion of a BCI-algebra which is a generalization of the concept of a BCK-algebra. Iséki [8] and Iséki and Thaheem [11] have shown that no proper class of implicative BCI-algebras exists, that is, such BCI-algebras are implicative BCK-algebras.

Thus, a natural question arises whether it is possible to generalize the notion of implicativeness in such a way that this generalization not only gives us a proper class of BCI-algebras but also contains the class of implicative BCK-algebras. In this paper, we answer this question in yes by introducing the concept of a branchwise implicative BCI-algebra. This proper class of BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras [4, 6].

2. Preliminaries. A BCI-algebra is an algebra \((X, *, 0)\) of type \((2,0)\) satisfying the following conditions:

\[
(x * y) * (x * z) \leq z * y, \quad \text{where } x \leq y \text{ if and only if } x * y = 0, \quad (2.1)
\]

\[
x * (x * y) \leq y, \quad (2.2)
\]

\[
x \leq x, \quad (2.3)
\]

\[
x \leq y \text{ and } y \leq x \text{ imply } x = y, \quad (2.4)
\]

\[
x \leq 0 \text{ implies } x = 0. \quad (2.5)
\]

If (2.5) is replaced by \(0 \leq x\), then the algebra is called a BCK-algebra. It is well known that every BCK-algebra is a BCI-algebra.

In a BCI-algebra \(X\), the following hold:

\[
(x * y) * z = (x * z) * y, \quad (2.6)
\]

\[
x * 0 = x, \quad (2.7)
\]

\[
x \leq y \text{ implies } x * z \leq y * z \text{ and } z * y \leq z * x, \quad (2.8)
\]
\( (x \ast z) \ast (y \ast z) \leq x \ast y, \quad (2.9) \)
\[ x \ast (x \ast (x \ast y)) = x \ast y \quad \text{(see [8]).} \quad (2.10) \]

**Definition 2.1** (see [9]). A subset \( I \) of a BCI-algebra \( X \) is called an ideal of \( X \) if it satisfies
\[ 0 \in I, \quad x \ast y \in I, \quad y \in I \implies x \in I. \quad (2.11) \]

**Definition 2.2** (see [10]). If in a BCK-algebra \( X \)
\[ (x \ast y) \ast z = (x \ast z) \ast (y \ast z) \quad (2.12) \]
holds for all \( x, y, z \in X \), then it is called positive implicative.

**Definition 2.3** (see [10]). If in a BCK-algebra \( X \)
\[ x \ast (x \ast y) = y \ast (y \ast x) \quad (2.13) \]
holds for all \( x, y \in X \), then it is called commutative.

**Theorem 2.4** (see [10]). A BCK-algebra \( X \) is positive implicative if and only if it satisfies
\[ (x \ast y) = (x \ast y) \ast y \quad \forall x, y \in X. \quad (2.14) \]

It has been shown in [8, 11] that no proper classes of positive implicative BCI-algebras and commutative BCI-algebras exist and such BCI-algebras are BCK-algebras of the corresponding type. That is why we generalized these notions and defined weakly positive implicative BCI-algebras [1] and branchwise commutative BCI-algebras [3] and studied some of their properties. Each class of these proper BCI-algebras contains the class of BCK-algebras of the corresponding type.

**Definition 2.5** (see [1]). A BCI-algebra \( X \) satisfying
\[ (x \ast y) \ast z = ((x \ast z) \ast z) \ast (y \ast z) \quad \forall x, y, z \in X \quad (2.15) \]
is called a weakly positive implicative BCI-algebra.

**Theorem 2.6** (see [1]). A BCI-algebra \( X \) is weakly positive implicative if and only if
\[ x \ast y = ((x \ast y) \ast y) \ast (0 \ast y) \quad \forall x, y \in X. \quad (2.16) \]

A BCI-algebra satisfying \( (x \ast y) \ast (z \ast u) = (x \ast z) \ast (y \ast u) \) is called a medial BCI-algebra.

Let \( X \) be a BCI-algebra and \( M = \{x : x \in X \text{ and } 0 \ast x = 0\} \). Then \( M \) is called its BCK-part. If \( M = \{0\} \), then \( X \) is called \( p \)-semisimple.

It has been shown in [4, 5, 6, 13] that in a BCI-algebra \( X \) the following are equivalent:
\[ \begin{align*}
X \text{ is medial,} & \quad x \ast (x \ast y) = y \quad \forall x, y \in X, \\
0 \ast (0 \ast x) = x & \quad \forall x \in X, \quad X \text{ is } p \text{-semisimple.} \quad (2.17)
\end{align*} \]

We now describe the notions of branches of a BCI-algebra and branchwise commutative BCI-algebras defined and investigated in [2, 3].
Definition 2.7 (see [3]). Let $X$ be a BCI-algebra, then the set $\text{Med}(X) = \{x : x \in X$ and $0 \ast (0 \ast x) = x\}$ is called medial part of $X$.

Obviously, $0 \in \text{Med}(X)$ and thus $\text{Med}(X)$ is nonempty. In what follows the elements of $\text{Med}(X)$ will be denoted by $x_0, y_0, \ldots$. It is known that $\text{Med}(X)$ is a medial subalgebra of $X$ and for each $x \in X$, there is a unique $x_0 = 0 \ast (0 \ast x) \in \text{Med}(X)$ such that $x_0 \leq x$ (see [3]). Further, $\text{Med}(X)$, in general, is not an ideal of $X$. Obviously, for a BCK-algebra $X$, $\text{Med}(X) = \{0\}$ and hence is an ideal of $X$.

Definition 2.8 (see [3]). Let $X$ be a BCI-algebra and $x_0 \in \text{Med}(X)$, then the set $B(x_0) = \{x : x \in X$ and $x \ast x_0 = 0\}$ is called a branch of $X$ determined by the element $x_0$.

The following theorem (proved in [2, 3]) shows that the branches of a BCI-algebra $X$ are pairwise disjoint and form its partition. So the study of branches of a BCI-algebra $X$ plays an important role in investigation of the properties of $X$. Obviously, a BCK-algebra $X$ is a one-branch BCI-algebra and in this case $X = B(0)$.

Theorem 2.9 (see [2, 3]). Let $X$ be a BCI-algebra with medial part $\text{Med}(X)$, then

(i) $X = \cup \{B(x_0) : x_0 \in \text{Med}(X)\}$.

(ii) $B(x_0) \cap B(y_0) = \emptyset$, $x_0, y_0 \in \text{Med}(X)$, and $x_0 \neq y_0$.

(iii) If $x, y \in B(x_0)$, then $0 \ast x = 0 \ast y = 0 \ast x_0 = 0 \ast y_0$ and $x \ast y, y \ast x \in M$.

Definition 2.10 (see [3]). A BCI-algebra $X$ is said to be branchwise commutative if and only if for $x_0 \in \text{Med}(X)$, $x, y \in B(x_0)$, the following equality holds:

$$x \ast (x \ast y) = y \ast (y \ast x).$$

(2.18)

Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is commutative if and only if it is branchwise commutative.

Theorem 2.11 (see [3]). A BCI-algebra $X$ is branchwise commutative if and only if

$$x \ast (x \ast y) = y \ast (y \ast (x \ast y)) \quad \forall x, y \in X.$$ 

(2.19)

3. Branchwise implicative BCI-algebras. In this section, we define branchwise implicative BCI-algebras. We show that this proper class of BCI-algebras contains the class of implicative BCK-algebras [10], the class of weakly implicative BCI-algebras [1] and the class of medial BCI-algebras. We also find necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.

Definition 3.1 (see [10]). A BCK-algebra $X$ is said to be implicative if and only if

$$x \ast (y \ast x) = x \quad \forall x, y \in X.$$ 

(3.1)

It has been shown in [8, 11] that no proper class of implicative BCI-algebras exists. Due to this reason we generalized the notion of implicativeness to weak implicativeness [1] mentioned below.

Definition 3.2 (see [1]). A BCI-algebra $X$ is said to be weakly implicative if and only if

$$x = (x \ast (y \ast x)) \ast (0 \ast (y \ast x)) \quad \forall x, y \in X.$$ 

(3.2)
We further generalize this concept and find a generalization of the following well-known result of Iséki [10].

**Theorem 3.3.** An implicative BCK-algebra is a positive implicative and commutative BCK-algebra.

**Definition 3.4.** A BCI-algebra $X$ is said to be a branchwise implicative BCI-algebra if and only if

$$x \ast (y \ast x) = x \quad \forall x, y \in B(x_0) \text{ and } x_0 \in \text{Med}(X). \quad (3.3)$$

**Example 3.5.** Let $X = \{0, 1, 2, \}$ in which $\ast$ is defined by

\[
\begin{array}{ccc}
0 & 1 & 2 \\
0 & 0 & 0 & 2 \\
1 & 1 & 0 & 2 \\
2 & 2 & 2 & 0 \\
\end{array}
\]

Then $X$ is a branchwise implicative BCI-algebra. This shows that proper branchwise implicative BCI-algebras exist.

**Remark 3.6.** (i) Since a BCK-algebra is a one-branch BCI-algebra, therefore, it is implicative if and only if it is branchwise implicative.

(ii) Let $X$ be weakly implicative and let $x, y \in B(x_0), x_0 \in \text{Med}(X)$, then using **Theorem 2.9(iii)**, we get $y \ast x \in M$. Thus $0 \ast (y \ast x) = 0$. So $x = (x \ast (y \ast x)) \ast (0 \ast (y \ast x))$ reduces to $x = x \ast (y \ast x)$. Hence every weakly implicative BCI-algebra is branchwise implicative BCI-algebra. But the branchwise implicative BCI-algebra $X$ of **Example 3.5** is not weakly implicative because $(1 \ast (2 \ast 1)) \ast (0 \ast (2 \ast 1)) = (1 \ast 2) \ast (0 \ast 2) = 2 \ast 2 = 0 \neq 1$.

(iii) It is known that each branch of a medial BCI-algebra $X$ is a singleton. Let $X$ be a medial BCI-algebra and $x_0 \in \text{Med}(X)$. Then $B(x_0) = \{x_0\}$. Hence $x_0 \ast (x_0 \ast x_0) = x_0 \ast 0 = x_0$, which implies that $X$ is branchwise implicative.

Thus the class of branchwise implicative BCI-algebras contains the class of implicative BCK-algebras, the class of weakly implicative BCI-algebras, and the class of medial BCI-algebras. We now prove the following results.

**Lemma 3.7.** Let $X$ be a BCI-algebra. If $x, y \in X$ and $x \leq y$, then $x, y \in B(x_0)$ for $x_0 \in \text{Med}(X)$.

**Proof.** Let $x \in X$, then there is a unique $x_0 = 0 \ast (0 \ast x) \in \text{Med}(X)$ such that $x \in B(x_0)$. Now $x_0 \ast y = (0 \ast (0 \ast x)) \ast y = (0 \ast y) \ast (0 \ast x) \leq x \ast y = 0$. Hence $x_0 \ast y = 0$, which implies $y \in B(x_0)$.

**Theorem 3.8.** If $X$ is a branchwise implicative BCI-algebra, then it is branchwise commutative.

**Proof.** Let $x, y \in X$, then $x \ast (x \ast y) \leq y$ and **Lemma 3.7** imply that $x \ast (x \ast y)$ and $y \in B(y_0)$ for some $y_0 \in \text{Med}(X)$. Since $X$ is branchwise implicative, therefore
using (3.3), we get
\[(x \ast (x \ast y)) \ast (y \ast (x \ast (x \ast y))) = x \ast (x \ast y).\] (3.4)

Using (2.2) and (2.8), we get
\[x \ast (x \ast y) = (x \ast (x \ast y)) \ast (y \ast (x \ast (x \ast y))) \leq y \ast (y \ast (x \ast (x \ast y))) \leq x \ast (x \ast y).\] (3.5)

Thus
\[x \ast (x \ast y) = y \ast (y \ast (x \ast (x \ast y))),\] (3.6)

which along with Theorem 2.11 implies that X is branchwise commutative.

**Theorem 3.9.** If X is a branchwise implicative BCI-algebra, then it satisfies
\[(x \ast y) \ast (0 \ast y) = ((x \ast y) \ast y) \ast (0 \ast y) \ast (0 \ast y).\] (3.7)

**Proof.** Since X is branchwise implicative, therefore Theorem 3.8 implies that X is branchwise commutative. Let \(x, y \in X\). Since \((x \ast y) \ast (0 \ast y) \leq x\), therefore Lemma 3.7 implies that \((x \ast y) \ast (0 \ast y), \ x \in B(x_0)\). Now branchwise implicativeness of X implies
\[((x \ast y) \ast (0 \ast y)) \ast (x \ast ((x \ast y) \ast (0 \ast y))) = (x \ast y) \ast (0 \ast y),\] (3.8)

which, using (2.6) twice, gives
\[((x \ast ((x \ast y) \ast (0 \ast y))) \ast y) \ast (0 \ast y) = (x \ast y) \ast (0 \ast y).\] (3.9)

Using branchwise commutativeness of X, from (3.9) we get
\[((x \ast y) \ast (0 \ast y)) \ast ((x \ast y) \ast (0 \ast y)) \ast x) \ast y) \ast (0 \ast y) = (x \ast y) \ast (0 \ast y),\] (3.10)

which implies
\[((x \ast y) \ast (0 \ast y) \ast y) \ast (0 \ast y) = (x \ast y) \ast (0 \ast y),\] (3.11)

so
\[((x \ast y) \ast y) \ast (0 \ast y) \ast (0 \ast y) = (x \ast y) \ast (0 \ast y).\] (3.12)

**Remark 3.10.** Since a BCK-algebra is a one-branch BCI-algebra, therefore an implicative BCK-algebra is commutative. Further, for a BCK-algebra \(0 \ast y = 0\) and thus (3.7) reduces to \(x \ast y = (x \ast y) \ast y\), which implies X is positive implicative. So we get Theorem 3.3, a well-known result of Iséki [10], as a corollary from Theorems 3.8 and 3.9.

We now investigate necessary and sufficient conditions for two types of BCI-algebras to be branchwise implicative.
Theorem 3.11. A BCI-algebra $X$, with $\text{Med}(X)$ as an ideal of $X$, is a branchwise implicative BCI-algebra if and only if it is branchwise commutative and satisfies

$$(x \ast y) \ast (0 \ast y) = (((x \ast y) \ast y) (0 \ast y)) \ast (0 \ast y) \quad \forall x, y \in X. \quad (3.13)$$

Proof. ($\Rightarrow$) Sufficiency follows from Theorems 3.8 and 3.9.

($\Leftarrow$) For necessity we consider $x, y \in X$ such that $x, y \in B(x_0)$ for some $x_0 \in \text{Med}(X)$. Now from Theorem 2.9(iii), we get $x \ast y$ and $y \ast x \in M$. So $0 \ast (x \ast y) = 0 \ast (y \ast x) = 0$. Further, $(x \ast (y \ast x)) \ast x = (x \ast x) \ast (y \ast x) = 0 \ast (y \ast x) = 0$, so

$$x \ast (y \ast x) \leq x. \quad (3.14)$$

Now (3.14) along with Lemma 3.7 implies $x \ast (y \ast x)$ and $x$ belong to the branch determined by $x$, that is, $B(x_0)$. Hence $x, y$ and $x \ast (y \ast x) \in B(x_0)$. Since $X$ is branchwise commutative, therefore,

$$(x \ast (x \ast (y \ast x))) \ast (0 \ast x)$$

$$= [[(y \ast x) \ast ((y \ast x) \ast x \ast (x \ast (y \ast x))))] \ast (0 \ast x)$$

$$= [[(y \ast x) \ast (0 \ast x)] \ast [(y \ast x) \ast (x \ast (x \ast (y \ast x)))] \quad \text{(using (2.6))}$$

$$= [[[((y \ast x) \ast x) \ast (0 \ast x)] \ast (0 \ast x)] \ast [(y \ast x) \ast (x \ast (x \ast (y \ast x)))] \quad \text{(using (3.13))}. \quad (3.15)$$

Now by using (2.6) three times, we get

$$(x \ast (x \ast (y \ast x))) \ast (0 \ast x) \text{ for some } 0 \ast x \ast 0 \ast x \ast 0 \ast x \ast 0 \ast x \ast 0 \ast x \ast 0 \ast x. \quad (3.16)$$

Since $x, y$ and $x \ast (y \ast x) \in B(x_0)$, therefore $x \ast y, y \ast x, x \ast (x \ast (y \ast x)) \in M = B(0)$. Since $X$ is branchwise commutative, therefore,

$$(x \ast (x \ast (y \ast x))) \ast (0 \ast x)$$

$$= [[[((x \ast (x \ast (y \ast x))) \ast ((x \ast (x \ast (y \ast x)) \ast (y \ast x))] \ast (y \ast x))] \ast (0 \ast x)] \ast (0 \ast x)$$

$$= (((x \ast (x \ast (y \ast x))) \ast (0 \ast x)) \ast (0 \ast x)] \ast (0 \ast x)$$

$$= (((x \ast (x \ast (y \ast x))) \ast (0 \ast x)) \ast (0 \ast x) \ast (0 \ast x)$$

$$= (0 \ast (0 \ast (0 \ast x))) \ast (0 \ast x)$$

$$= (0 \ast (0 \ast (0 \ast x))) \ast (0 \ast x)$$

$$= (0 \ast (0 \ast (0 \ast x))) \ast (0 \ast x) = 0 \ast (0 \ast x). \quad (3.17)$$

Hence

$$(x \ast (x \ast (y \ast x))) \ast (0 \ast x) = 0 \ast (0 \ast x) \in \text{Med}(X). \quad (3.18)$$
But (2.10) implies \( 0 \ast (0 \ast (0 \ast x)) = 0 \ast x \). So \( 0 \ast x \in Med(X) \). Since \( Med(X) \) is an ideal of \( X \), therefore, \( x \ast (x \ast (y \ast x)) \in Med(X) \). Hence

\[
x \ast (x \ast (y \ast x)) = 0 \ast (0 \ast (x \ast (x \ast (y \ast x)))).
\]  

(3.19)

Since \( x \ast (x \ast (y \ast x)) \in M = B(0) \), therefore, \( 0 \ast (x \ast (x \ast (y \ast x))) = 0 \). Thus \( x \ast (x \ast (y \ast x)) = 0 \), which gives

\[
x \leq x \ast (y \ast x).
\]  

(3.20)

Using (3.14) and (3.20), we get

\[
x = x \ast (y \ast x) \quad \forall x, y \in B(x_0).
\]  

(3.21)

Hence \( X \) is branchwise implicative. This completes the proof. \( \square \)

**Remark 3.12.** Since in a BCK-algebra \( X \), \( Med(X) = \{0\} \) is always an ideal of \( X \), therefore the following well-known result regarding BCK-algebra follows as a corollary from Theorem 3.11.

**Corollary 3.13.** A BCK-algebra is implicative if and only if it is positive implicative and commutative.

**Remark 3.14.** The following example shows that there exist proper BCI-algebras in which \( Med(X) \) is an ideal. Thus the condition, \( Med(X) \) is an ideal of \( X \), in Theorem 3.11 is not unnatural.

**Example 3.15** (see [12, Example 2]). The set \( X = \{0, 1, 2, 3\} \) with the operation \( \ast \) defined as

\[
\begin{array}{cccc}
\ast & 0 & 1 & 2 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 \\
3 & 0 & 0 & 0 \\
\end{array}
\]

is a proper BCI-algebra. Here \( Med(X) = \{0, 2\} \) is an ideal of \( X \). Further, \( X \) is branchwise implicative but is not medial.

**Definition 3.16.** Let \( X \) be a BCI-algebra. Two elements \( x, y \) of \( X \) are said to be comparable if and only if either \( x \ast y = 0 \) or \( y \ast x = 0 \), that is, either \( x \leq y \) or \( y \leq x \).

**Definition 3.17.** Let \( X \) be a BCI-algebra. If \( x_0 \in Med(X) \) and \( x_0 \neq 0 \), then \( B(x_0) \), the branch of \( X \) determined by \( x_0 \), is called a proper BCI-branch of \( X \).

**Theorem 3.18.** Let \( X \) be a BCI-algebra such that any two elements of a proper BCI-branch of \( X \) are comparable. Then \( X \) is branchwise implicative if and only if \( X \) is branchwise commutative and satisfies

\[
(x \ast y) \ast (0 \ast y) = (((x \ast y) \ast y) \ast (0 \ast y)) \ast (0 \ast y) \quad \forall x, y \in X.
\]  

(3.22)
**Proof.** \((\Rightarrow)\) Sufficiency follows from Theorems 3.8 and 3.9.

\((\Leftarrow)\) For necessity we consider the following two cases.

**Case 1.** Let \(x, y \in B(0) = M\). Then \(0 \ast y = 0 \ast x = 0\) and hence (3.22) becomes \(x \ast y = (x \ast y) \ast y\). Further, \((x \ast (y \ast x)) \ast x = (x \ast x) \ast (y \ast x) = 0 \ast (y \ast x) = 0\). Hence

\[x \ast (y \ast x) \leq x.\] (3.23)

Since \(x \ast y \in M = B(0)\) and \(X\) is branchwise commutative, therefore,

\[x \ast (x \ast (y \ast x)) = (y \ast x) \ast ((y \ast y) \ast x) = (y \ast x) \ast (y \ast x) = 0.\] (3.24)

Thus

\[x \ast \leq x \ast (y \ast x).\] (3.25)

From (3.23) and (3.25), we get \(x = x \ast (y \ast x)\) for all \(x, y \in B(0)\).

**Case 2.** Let \(x, y \in B(x_0)\), where \(x_0 \in \text{Med}(X)\) and \(x_0 \neq 0\). Thus \(x \ast y \in M\) and \(y \ast x \in M\). So \(0 \ast (x \ast y) = 0\) and \(0 \ast (y \ast x) = 0\). Further, taking \(y = x \ast y\) in (3.22), we get

\[x \ast (x \ast y) = (x \ast (x \ast y)) \ast (x \ast y) \quad \forall x, y \in B(x_0).\] (3.26)

Interchanging \(x\) and \(y\) in (3.26), we get

\[y \ast (y \ast x) = (y \ast (y \ast x)) \ast (y \ast x) \quad \forall x, y \in B(x_0).\] (3.27)

Since \(x, y\) are comparable, therefore, either \(y \ast x = 0\) or \(x \ast y = 0\). If \(y \ast x = 0\), then

\[x \ast (y \ast x) = x \ast 0 = x.\] (3.28)

If \(x \ast y = 0\), then branchwise commutativity of \(X\) gives

\[y \ast (y \ast x) = x \ast (x \ast y) = x \ast 0 = x.\] (3.29)

Using (3.27) and (3.29), we get

\[x = x \ast (y \ast x).\] (3.30)

Thus \(X\) is branchwise implicative. \(\square\)

**Remark 3.19.** The following example shows that the conditions \(\text{Med}(X)\) is an ideal of \(X\) and any two elements of a proper BCI-branch of \(X\) are comparable cannot be removed from Theorems 3.11 and 3.18, respectively.

**Example 3.20.** Let \(X = \{0, 1, 2, 3, 4, 5\}\) in which \(\ast\) is defined by

\[
\begin{array}{ccccccc}
\ast & 0 & 1 & 2 & 3 & 4 & 5 \\
0 & 0 & 0 & 3 & 2 & 3 & 3 \\
1 & 1 & 0 & 3 & 2 & 3 & 3 \\
2 & 2 & 2 & 0 & 3 & 0 & 0 \\
3 & 3 & 3 & 2 & 0 & 2 & 2 \\
4 & 4 & 2 & 1 & 3 & 0 & 1 \\
5 & 5 & 2 & 1 & 3 & 1 & 0 \\
\end{array}
\]

\[
\begin{array}{c}
\bullet 0 \\
\bullet 1 \\
\bullet 2 \\
\bullet 3 \\
\bullet 4 \\
\bullet 5 \\
\end{array}
\]

\[
\begin{array}{c}
0 \\
1 \\
4 \\
5 \\
\end{array}
\]

\[
\begin{array}{c}
2 \\
3 \\
\end{array}
\]
Routine calculations give that $X$ is a BCI-algebra, which is branchwise commutative and satisfies (3.22). But we note that

1. $\text{Med}(X) = \{0, 2, 3\}$ is not an ideal of $X$ because $4 \ast 3 = 3 \in \text{Med}(X)$, $3 \in \text{Med}(X)$ but $4 \notin \text{Med}(X)$. Further, $X$ is not branchwise implicative because $4, 5 \in B(2)$ and $4 \ast (5 \ast 4) = 4 \ast 1 = 2 \neq 4$;

2. the elements 4 and 5 of $B(2)$ are not comparable and also $X$ is not branchwise implicative.

Combining Theorems 3.11 and 3.18, we get the following theorem.

**Theorem 3.21.** Let $X$ be a BCI-algebra such that either $\text{Med}(X)$ is an ideal of $X$ or every pair of elements of a proper BCI-branch of $X$ are comparable, then $X$ is branchwise implicative if and only if $X$ is branchwise commutative and satisfies (3.22).

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**References**


**Muhammad Anwar Chaudhry:** Department of Mathematical Sciences, King Fahd University of Petroleum and Minerals, Dhahran 31261, Saudi Arabia

E-mail address: chaudhry@kfupm.edu.sa
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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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<table>
<thead>
<tr>
<th>Due Date</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
<td>Manuscript Due</td>
<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk

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