It is shown that Fefferman’s mapping theorem extends to the case of manifolds, that is a biholomorphic map between two strictly pseudoconvex manifolds extends smoothly to their boundaries.

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1. Introduction. A central question in complex analysis is “does every proper holomorphic mapping \( f : D \rightarrow D' \) of bounded domains \( D, D' \) with smooth boundaries in \( \mathbb{C}^n \) extend smoothly to the boundary of \( D' \)?”

The answer has been known to be “yes” in dimension one for a long time. In higher dimensions, in case \( D \) and \( D' \) are strictly pseudoconvex and \( f \) is biholomorphic, Fefferman’s famous mapping theorem [8] answers the question in the affirmative.

Bell and Ligocka [4] simplified the proof of Fefferman’s mapping theorem and extended the theorem to a wide class of pseudoconvex domains.

In [3], Fefferman’s mapping theorem was extended to smoothly bounded pseudoconvex subdomains of Stein manifolds that satisfy condition \( R \). Thereafter, the question was asked whether all smoothly bounded pseudoconvex domains satisfy condition \( R \). Recently, Barrett [2] and Christ [5] have shown that this question has an answer in the negative. But that was not the end of condition \( R \), because the case of strictly pseudoconvex manifolds that are not Stein had not been determined. At first it was thought (because of the work of Barrett [1]) that one could not do without the assumption of Steinness.

In this note, we show that a strictly pseudoconvex manifold need not be Stein before it satisfies condition \( R \); and following the work of Bedford et al. [3], we extend Fefferman’s mapping theorem to all strictly pseudoconvex manifolds.

2. Preliminaries. Let \( \Omega \) be a relatively compact domain in an \( n \)-dimensional complex manifold \( X \). The space \( L^2_{(n,0)}(\Omega) \) is defined to be the set of \((n,0)\) forms \( \omega \) such that

\[
\|\omega\|^2 = (\sqrt{-1})^{n^2} \int_{\Omega} \omega \wedge \bar{\omega} \tag{2.1}
\]

is finite. The space \( L^2_{(n,0)}(\Omega) \) is a Hilbert space with inner product given by

\[
(\omega, \eta) = (\sqrt{-1})^{n^2} \int_{\Omega} \omega \wedge \bar{\eta}. \tag{2.2}
\]
The Bergman-Kobayashi projection $P_\Omega$ associated to $\Omega$ is the orthogonal projection of $L^2_{(n,0)}(\Omega)$ onto $H_{(n,0)}(\Omega)$, the closed subspace of $L^2_{(n,0)}(\Omega)$ consisting of holomorphic $(n,0)$ forms. If $\Omega$ has a smooth boundary, $\Omega$ satisfies condition $R$ if the Bergman-Kobayashi projection associated to $\Omega$ maps $C^\infty_{(n,0)}(\overline{\Omega})$ into $C^\infty_{(n,0)}(\overline{\Omega})$.

To make use of the proof in [3], we show that if $\Omega$ above has smooth boundary and it is strictly pseudoconvex, then $\Omega$ satisfies condition $R$; and, in addition, if $p_0$ is a point in $X$ near the boundary $\partial X$ of $\Omega$, then there are $n$ functions $g_1, \ldots, g_n$ that are holomorphic in a neighborhood of $\overline{\Omega}$ and that form a coordinate system at $p_0$.

Our main result is the following theorem.

**Theorem 2.1.** Let $X_1$ and $X_2$ be $n$-dimensional complex manifolds and $\Omega_1 \subset X_1$, $\Omega_2 \subset X_2$ strictly pseudoconvex subdomains with smooth boundaries. Let $f : \Omega_1 \to \Omega_2$ be a biholomorphic mapping between $\Omega_1$ and $\Omega_2$. Then $f$ extends smoothly to a $C^\infty$ diffeomorphism of $\overline{\Omega}_1$ and $\overline{\Omega}_2$.

3. **Condition $R$.** To establish condition $R$ for smoothly bounded strictly pseudoconvex subdomains of complex manifolds, we need a result of Gunning and Rossi [9] which we met on the way to proving theorems in [6, 7]. Their result is the following theorem.

**Theorem 3.1.** Let $\Omega$ be a strictly pseudoconvex domain in a complex manifold $Y$. There are a Stein manifold $X$ and a proper holomorphic mapping $\pi : \Omega \to X$ with the following properties:

(i) $\pi : C_X \cong C_\Omega$;

(ii) there are finitely many points $x_1, \ldots, x_2$ in $X$ such that $\pi^{-1}(x_j)$ is a compact subvariety of $\Omega$ of positive dimension, and $\pi : \Omega \setminus \bigcup \pi^{-1}(x_j) \cong X \setminus \{x_1, \ldots, x_2\}$.

The first statement means that the rings of holomorphic functions $C_X$ and $C_\Omega$ on $X$ and $\Omega$, respectively, are isomorphic under the map induced by $\pi$. The second means that $\Omega \setminus \bigcup \pi^{-1}(x_j)$ and $X \setminus \{x_1, \ldots, x_2\}$ are biholomorphic.

Now from the proof of Theorem 3.1 as given in [9], it is clear that there is a strictly pseudoconvex neighborhood $\Omega'$ of $\Omega$ such that $\Omega'$ can replace $\Omega$ in Theorem 3.1 so that the compact set $\bigcup \pi^{-1}(x_j)$ corresponding to $\Omega'$ is contained in $\Omega$.

If $X'$ corresponds to $\Omega'$ in Theorem 3.1 and $X = \pi(\Omega)$, then clearly if $\Omega$ has a smooth boundary then $X$ is a Stein strictly pseudoconvex manifold with a smooth boundary, and therefore, as is well-known, $X$ satisfies condition $R$.

We can regard $\Omega$ as imbedded in $X$. Then it is clear that $L^2_{(n,0)}(\Omega) = L^2_{(n,0)}(X)$ and $H_{(n,0)}(\Omega) = H_{(n,0)}(X)$. Therefore the Bergman-Kobayashi projections $P_X$ and $P_\Omega$ are equal, and it is not difficult to see (using Sobolev spaces) that $\Omega$ satisfies condition $R$.

4. **Local coordinates near the boundary.** Again from Theorem 3.1 we get the last theorem that we need in the proof of Theorem 2.1.

**Theorem 4.1.** Let $\Omega$ be a strictly pseudoconvex subdomain of a complex manifold $Y$. Then near the boundary $\partial \Omega$ of $\Omega$, local coordinates are given by holomorphic functions in a neighborhood of $\overline{\Omega}$. 
PROOF. As indicated in Section 3, from the proof of Theorem 3.1 as given in [9] it is clear that there is a strictly pseudoconvex neighborhood $\Omega'$ of $\Omega$ such that $\Omega'$ can replace $\Omega$ in Theorem 3.1 so that the compact set $\cup^{-1}(x_j)$ corresponding to $\Omega'$ is contained in $\Omega$. Now if $p_0$ is a point in $\Omega'$ near the boundary $\partial \Omega$ of $\Omega$, let $\pi(p_0)$ have holomorphic functions $g_1, \ldots, g_n$ on the Stein manifold $X$ that form local coordinates at $\pi(p_0)$. Then $g_1 \circ \pi, \ldots, g_n \circ \pi$ form local coordinates at $p_0$, which are holomorphic in a neighborhood of $\bar{\Omega}$.

5. Proof of Theorem 2.1. The proof of Theorem 2.1 relies on the following two lemmas whose proofs are in [3].

**Lemma 5.1.** If $\omega$ is a holomorphic $(n,0)$ form in $C^\infty_{(n,0)}(\tilde{\Omega}_2)$, then $f^* \omega$ is in $C^\infty_{(n,0)}(\tilde{\Omega}_1)$.

**Lemma 5.2.** If $\omega$ is a holomorphic $(n,0)$ form in $C^\infty_{(n,0)}(\tilde{\Omega}_2)$ that vanishes to at most finite order at any boundary point of $\Omega_2$, then $f^* \omega$ vanishes to at most finite order at any boundary point of $\Omega_1$.

Now to prove Theorem 2.1, we initiate the proof of Theorem 2.1 in [3]:

Let $p_0$ be a boundary point of $\Omega_1$ and let $z_1, \ldots, z_n$ be holomorphic coordinates near $p_0$. We show that $f$ extends smoothly to $\partial \Omega_1$ near $p_0$. Let $\{p_i\}$ be a sequence of points in $\Omega_1$ that converges to $p_0$. Then $\{f(p_i)\}$ converges to a point $q_0$ in $\partial \Omega_2$. Let $g_1, \ldots, g_n$ be $n$ functions on $\Omega_2$ that extend to be holomorphic in a neighborhood of $\Omega_2$ in $X_2$ and that form a coordinate chart at $q_0$. Define a holomorphic function $u$ near $p_0$ via

$$
udz_1 \wedge dz_2 \wedge \cdots \wedge dz_n = f^*(dg_1 \wedge dg_2 \wedge \cdots \wedge dg_n).
$$

(5.1)

By Lemmas 5.1 and 5.2 $u$ extends smoothly to $\partial \Omega_1$ near $p_0$ and $u$ vanishes to a finite order near $p_0$.

If $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, then we define $g^\alpha = \prod_{i=1}^n \alpha_i g_i$. Lemma 5.1 implies that the form $f^*(g^\alpha dg_1 \wedge \cdots \wedge dg_n)$ extends smoothly to $\partial \Omega_1$ near $p_0$ for each $\alpha$. Hence, $u$ and $(g^\alpha \circ f)$ extend smoothly to $\partial \Omega_1$ near $p_0$ for each $\alpha$, and $u$ vanishes to at most finite order at $p_0$. By the division theorem cited in [3], $g_i \circ f$ extends smoothly to $\partial \Omega_1$ near $p_0$ for each $i$. Hence $f$ extends smoothly to $\partial \Omega_1$ near $p_0$. Since $p_0$ was arbitrarily chosen, we conclude that $f$ extends smoothly to all of $\partial \Omega_1$. Now we can replace $f$ by $f^{-1}$ and then the theorem follows.

**References**


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