ON \textit{n}-FOLD FUZZY IMPLICATIVE/COMMUTATIVE IDEALS OF BCK-ALGEBRAS

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ABSTRACT. We consider the fuzzification of the notion of an \textit{n}-fold implicative ideal, an \textit{n}-fold (weak) commutative ideal. We give characterizations of an \textit{n}-fold fuzzy implicative ideal. We establish an extension property for \textit{n}-fold fuzzy commutative ideals.

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1. Introduction. Huang and Chen [1] introduced the notion of \textit{n}-fold implicative ideals and \textit{n}-fold (weak) commutative ideals. The aim of this paper is to discuss the fuzzification of \textit{n}-fold implicative ideals, \textit{n}-fold commutative ideals and \textit{n}-fold weak commutative ideals. We show that every \textit{n}-fold fuzzy implicative ideal is an \textit{n}-fold fuzzy positive implicative ideal, and so a fuzzy ideal, and give a condition for a fuzzy ideal to be an \textit{n}-fold fuzzy implicative ideal. Using the level set, we provide a characterization of an \textit{n}-fold fuzzy implicative ideal. We also give a condition for a fuzzy ideal to be an \textit{n}-fold fuzzy (weak) commutative ideal. We show that every \textit{n}-fold fuzzy positive implicative ideal which is an \textit{n}-fold fuzzy weak commutative ideal is an \textit{n}-fold fuzzy implicative ideal. Finally, we establish an extension property for \textit{n}-fold fuzzy commutative ideals.

2. Preliminaries. We include some elementary aspects of BCK-algebras that are necessary for this paper, and for more details we refer to [1, 2, 4, 5]. By a BCK-algebra we mean an algebra $(X; \ast, 0)$ of type $(2, 0)$ satisfying the axioms:

$I)$ $((x \ast y) \ast (x \ast z)) \ast (z \ast y) = 0,$
$I)$ $(x \ast (x \ast y)) \ast y = 0,$
$III)$ $x \ast x = 0,$
$IV)$ $0 \ast x = 0,$
$V)$ $x \ast y = 0$ and $y \ast x = 0$ imply $x = y,$ for all $x, y, z \in X.$

We can define a partial ordering $\leq$ on $X$ by $x \leq y$ if and only if $x \ast y = 0.$ In any BCK-algebra $X$, the following hold:

(P1) $x \ast 0 = x,$
(P2) $x \ast y \leq x,$
(P3) $(x \ast y) \ast z = (x \ast z) \ast y,$
(P4) $(x \ast z) \ast (y \ast z) \leq x \ast y,$
(P5) $x \leq y$ implies $x \ast z \leq y \ast z$ and $z \ast y \leq z \ast x.$

Throughout, $X$ will always mean a BCK-algebra unless otherwise specified. A non-empty subset $I$ of $X$ is called an ideal of $X$ if it satisfies:

$I1$) $0 \in I,$
(I2) \( x \ast y \in I \) and \( y \in I \) imply \( x \in I \).

A nonempty subset \( I \) of \( X \) is said to be an implicative ideal of \( X \) if it satisfies:

(I1) \( 0 \in I \),
(I3) \((x \ast (y \ast x)) \ast z \in I \) and \( z \in I \) imply \( x \in I \).

A nonempty subset \( I \) of \( X \) is said to be a commutative ideal of \( X \) if it satisfies:

(I1) \( 0 \in I \),
(I4) \((x \ast y) \ast z \in I \) and \( z \in I \) imply \( x \ast (y \ast (y \ast x)) \in I \).

We now review some fuzzy logic concepts. A fuzzy set in a set \( X \) is a function \( \mu : X \rightarrow [0,1] \).

For a fuzzy set \( \mu \) in \( X \) and \( t \in [0,1] \) define \( U(\mu; t) \) to be the set

\[
U(\mu; t) = \{ x \in X \mid \mu(x) \geq t \}.
\]

A fuzzy set \( \mu \) in \( X \) is said to be a fuzzy ideal of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F2) \( \mu(x) \geq \min\{\mu((x \ast y) \ast z), \mu(z)\} \) for all \( x, y, z \in X \).

We consider the fuzzification of the concept of \( n \)-fold implicative ideals.

**Definition 3.1** (see [1]). A subset \( A \) of \( X \) is called an \( n \)-fold implicative ideal of \( X \) if

(I1) \( 0 \in A \),
(I5) \((x \ast (y \ast x^n)) \ast z \in A \) and \( z \in A \) imply \( x \in A \) for every \( x, y, z \in X \).

We consider the fuzzification of the concept of \( n \)-fold implicative ideal.

**Definition 3.2.** A fuzzy set \( \mu \) in \( X \) is called an \( n \)-fold fuzzy implicative ideal of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F5) \( \mu(x) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\} \) for every \( x, y, z \in X \).

Notice that the 1-fold fuzzy implicative ideal is a fuzzy implicative ideal.

**Theorem 3.3.** Every \( n \)-fold fuzzy implicative ideal is a fuzzy ideal.

**Proof.** The condition (F2) follows from taking \( y = 0 \) in (F5). □

The following example shows that the converse of Theorem 3.3 may not be true.
**Example 3.4.** Let \( X = \mathbb{N} \cup \{0\} \), where \( \mathbb{N} \) is the set of natural numbers, in which the operation \( \ast \) is defined by \( x \ast y = \max\{0, x - y\} \) for all \( x, y \in X \). Then \( X \) is a BCK-algebra (see [1, Example 1.3]). Let \( \mu \) be a fuzzy set in \( X \) given by \( \mu(0) = t_0 > t_1 = \mu(x) \) for all \( x (\neq 0) \in X \). Then \( \mu \) is a fuzzy ideal of \( X \). But \( \mu \) is not a 2-fold fuzzy implicative ideal of \( X \) because

\[
\mu(3) = t_1 < t_0 = \mu(0) = \min\{\mu((3 \ast (14 \ast 3^2)) \ast 0), \mu(0)\}. \tag{3.2}
\]

We give a condition for a fuzzy ideal to be an \( n \)-fold fuzzy implicative ideal.

**Theorem 3.5.** A fuzzy ideal \( \mu \) of \( X \) is \( n \)-fold fuzzy implicative if and only if \( \mu(x) \geq \mu(x \ast (y \ast x^n)) \) for all \( x, y \in X \).

**Proof.** Necessity is by taking \( z = 0 \) in (F5). Suppose that a fuzzy ideal \( \mu \) satisfies the inequality \( \mu(x) \geq \mu(x \ast (y \ast x^n)) \) for all \( x, y \in X \). Then

\[
\mu(x) \geq \mu(x \ast (y \ast x^n)) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\}. \tag{3.3}
\]

Hence \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \).

**Theorem 3.6.** A fuzzy set \( \mu \) in \( X \) is an \( n \)-fold fuzzy implicative ideal of \( X \) if and only if the nonempty level set \( U(\mu; t) \) of \( \mu \) is an \( n \)-fold implicative ideal of \( X \) for every \( t \in [0, 1] \).

**Proof.** Assume that \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \) and \( U(\mu; t) \neq \emptyset \) for every \( t \in [0, 1] \). Then there exists \( x \in U(\mu; t) \). It follows from (F1) that \( \mu(0) \geq \mu(x) \geq t \) so that \( 0 \in U(\mu; t) \). Let \( x, y, z \in X \) be such that \((x \ast (y \ast x^n)) \ast z \in U(\mu; t) \) and \( z \in U(\mu; t) \). Then \( \mu((x \ast (y \ast x^n)) \ast z) \geq t \) and \( \mu(z) \geq t \), which imply from (F5) that

\[
\mu(x) \geq \min\{\mu((x \ast (y \ast x^n)) \ast z), \mu(z)\} \geq t \tag{3.4}
\]

so that \( x \in U(\mu; t) \). Therefore \( U(\mu; t) \) is an \( n \)-fold implicative ideal of \( X \). Conversely, suppose that \( U(\mu; t)(\neq \emptyset) \) is an \( n \)-fold implicative ideal of \( X \) for every \( t \in [0, 1] \). For any \( x \in X \), let \( \mu(x) = t \). Then \( x \in U(\mu; t) \). Since \( 0 \in U(\mu; t) \), we get \( \mu(0) \geq t = \mu(x) \) and so \( \mu(0) \geq \mu(x) \) for all \( x \in X \). Now assume that there exist \( a, b, c \in X \) such that

\[
\mu(a) < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \tag{3.5}
\]

Selecting \( s_0 = (1/2)\mu(a) + \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\} \), then

\[
\mu(a) < s_0 < \min\{\mu((a \ast (b \ast a^n)) \ast c), \mu(c)\}. \tag{3.6}
\]

It follows that \((a \ast (b \ast a^n)) \ast c \in U(\mu; s_0), c \in U(\mu; s_0)\), and \( a \notin U(\mu; s_0) \). This is a contradiction. Hence \( \mu \) is an \( n \)-fold fuzzy implicative ideal of \( X \).

**Definition 3.7** (see [3]). A fuzzy set \( \mu \) in \( X \) is called an \( n \)-fold fuzzy positive implicative ideal of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),

(F6) \( \mu(x \ast y^n) \geq \min\{\mu((x \ast y^{n+1}) \ast z), \mu(z)\} \) for all \( x, y, z \in X \).
Lemma 3.8 (see [3, Theorem 3.13]). Let \( \mu \) be a fuzzy set in \( X \). Then \( \mu \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if the nonempty level set \( U(\mu; t) \) of \( \mu \) is an \( n \)-fold positive implicative ideal of \( X \) for every \( t \in [0, 1] \).

Lemma 3.9 (see [1, Theorem 2.5]). Every \( n \)-fold implicative ideal is an \( n \)-fold positive implicative ideal.

Using Theorem 3.6 and Lemmas 3.8 and 3.9, we have the following theorem.

Theorem 3.10. Every \( n \)-fold fuzzy implicative ideal is an \( n \)-fold fuzzy positive implicative ideal.

4. \( n \)-fold fuzzy commutative ideals

Definition 4.1 (see [1]). A subset \( A \) of \( X \) is called an \( n \)-fold commutative ideal of \( X \) if

(1) \( 0 \in A \),
(6) \( (x * y) * z \in A \) and \( z \in A \) imply \( x * (y * (y * x^n)) \in A \) for all \( x, y, z \in X \).

A subset \( A \) of \( X \) is called an \( n \)-fold weak commutative ideal of \( X \) if

(1) \( 0 \in A \),
(7) \( x * (x * y^n)) * z \in A \) and \( z \in A \) imply \( y * (y * x) \in A \) for all \( x, y, z \in X \).

We consider the fuzzification of \( n \)-fold (weak) commutative ideals as follows.

Definition 4.2. A fuzzy set \( \mu \) in \( X \) is called an \( n \)-fold fuzzy commutative ideal of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F7) \( \mu(x * (y * (y * x^n))) \geq \min\{\mu((x * y) * z), \mu(z)\} \) for all \( x, y, z \in X \).

A fuzzy set \( \mu \) in \( X \) is called an \( n \)-fold fuzzy weak commutative ideal of \( X \) if

(F1) \( \mu(0) \geq \mu(x) \) for all \( x \in X \),
(F8) \( \mu(y * (y * x)) \geq \min\{\mu((x * (x * y^n)) * z), \mu(z)\} \) for all \( x, y, z \in X \).

Note that the 1-fold fuzzy commutative ideal is a fuzzy commutative ideal. Putting \( y = 0 \) and \( y = x \) in (F7) and (F8), respectively, we know that every \( n \)-fold fuzzy commutative (or fuzzy weak commutative) ideal is a fuzzy ideal.

Theorem 4.3. Let \( \mu \) be a fuzzy ideal of \( X \). Then

(i) \( \mu \) is an \( n \)-fold fuzzy commutative ideal of \( X \) if and only if

\[
\mu(x * (y * (y * x^n))) \geq \mu(x * y) \quad \forall x, y \in X. \tag{4.1}
\]

(ii) \( \mu \) is an \( n \)-fold fuzzy weak commutative ideal of \( X \) if and only if

\[
\mu(y * (y * x)) \geq \mu(x * (x * y^n)) \quad \forall x, y \in X. \tag{4.2}
\]

Proof. The proof is straightforward.

Lemma 4.4 (see [3, Theorem 3.12]). A fuzzy set \( \mu \) in \( X \) is an \( n \)-fold fuzzy positive implicative ideal of \( X \) if and only if \( \mu \) is a fuzzy ideal of \( X \) in which the following inequality holds:

(F9) \( \mu((x * z^n) * (y * z^n)) \geq \mu((x * y) * z^n) \) \( \forall x, y, z \in X \).
**Theorem 4.5.** If $\mu$ is both an $n$-fold fuzzy positive implicative ideal and an $n$-fold fuzzy weak commutative ideal of $X$, then it is an $n$-fold fuzzy implicative ideal of $X$.

**Proof.** Let $x, y \in X$. Using Theorem 4.3(ii), Lemma 4.4, (P3), and (III), we have
\[
\mu(x \ast (x \ast (y \ast x^n))) \geq \mu((y \ast x^n) \ast ((y \ast x^n) \ast x^n))
\]
\[
\geq \mu((y \ast (y \ast x^n)) \ast x^n)
\]
\[
= \mu((y \ast x^n) \ast (y \ast x^n))
\]
\[
= \mu(0). \tag{4.3}
\]
It follows from (F1) and (F2) that
\[
\mu(x) \geq \min \{\mu(x \ast (y \ast x^n)), \mu(x \ast (y \ast x^n))\}
\]
\[
\geq \min \{\mu(0), \mu(x \ast (y \ast x^n))\}
\]
\[
= \mu(x \ast (y \ast x^n)) \tag{4.4}
\]
so from Theorem 3.5, $\mu$ is an $n$-fold fuzzy implicative ideal of $X$. \qed

**Theorem 4.6** (extension property for $n$-fold fuzzy commutative ideals). Let $\mu$ and $\nu$ be fuzzy ideals of $X$ such that $\mu(0) = \nu(0)$ and $\mu \subseteq \nu$, that is, $\mu(x) \leq \nu(x)$ for all $x \in X$. If $\mu$ is an $n$-fold fuzzy commutative ideal of $X$, then so is $\nu$.

**Proof.** Let $x, y \in X$. Taking $u = x \ast (x \ast y)$, we have
\[
\nu(0) = \mu(0) = \mu(u \ast y)
\]
\[
\leq \mu(u \ast (y \ast (y \ast u^n)))
\]
\[
\leq \nu(u \ast (y \ast (y \ast u^n))) \tag{4.5}
\]
\[
= \nu((x \ast (x \ast y)) \ast (y \ast (y \ast u^n)))
\]
\[
= \nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)).
\]
Since $x \ast (y \ast (y \ast x^n)) \leq x \ast (y \ast (y \ast u^n))$ and since $\nu$ is order reversing, it follows that
\[
\nu(x \ast (y \ast (y \ast x^n))) \geq \nu(x \ast (y \ast (y \ast u^n)))
\]
\[
\geq \min \{\nu((x \ast (y \ast (y \ast u^n))) \ast (x \ast y)), \nu(x \ast y)\} \tag{4.6}
\]
\[
= \nu(x \ast y).
\]
Hence, by Theorem 4.3(i), $\nu$ is an $n$-fold fuzzy commutative ideal of $X$. \qed

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**References**


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