

PAIRS OF PATHS AND CRITICAL POINTS

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ABSTRACT. Two sufficient conditions are presented, in terms of the values taken by a holomorphic function $f(z)$ on a pair of smooth paths intersecting at a point z_0 in its domain, implying that $f'(z_0) = 0$.

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In the present paper, we present two sufficient conditions expressed in terms of the values taken by a holomorphic function f on a pair of smooth paths intersecting at a point z_0 in the domain of f , with tangent vectors at z_0 linearly independent over \mathbb{R} , implying that $f'(z_0) = 0$.

THEOREM 1. *Let $f : D \subset \mathbb{C} \rightarrow \mathbb{C}$ be a holomorphic function, where $D \subset \mathbb{C}$ is a domain and let $\gamma, \Gamma : (0, 1) \rightarrow D$ be two smooth (C^1) paths. Assume the following:*

- (i) *for a certain $z_0 \in D$ and some $t_1, t_2 \in (0, 1)$ we have $z_0 = \gamma(t_1) = \Gamma(t_2)$;*
- (ii) *$\gamma'(t_1)$ and $\Gamma'(t_2)$ linearly independent over \mathbb{R} (i.e., non-collinear),*
- (iii) *$|f(z)|$ takes a constant value on the subset $\gamma((0, 1)) \cup \Gamma((0, 1))$ of D . Then $f'(z_0) = 0$.*

PROOF. Let $f = u + iv$, $\gamma = y_1 + iy_2$, and $\Gamma = \Gamma_1 + i\Gamma_2$, where u, v are real-valued functions while $y_1, y_2, \Gamma_1, \Gamma_2$ are real-valued smooth paths. The assumption (iii) can be written as

$$u^2(\gamma(t)) + v^2(\gamma(t)) = u^2(\Gamma(t)) + v^2(\Gamma(t)) = c \quad (1)$$

for any $t \in (0, 1)$, where c is some constant. Note first that if $c = 0$, from (1) together with the identity theorem of the holomorphic functions it follows that $f(z) = 0$ for any $z \in D$. This being the case, we assume $c \neq 0$ from now on. We differentiate (1) with respect to t . We then have, for any $t \in (0, 1)$,

$$\frac{d}{dt}(u^2(\gamma(t)) + v^2(\gamma(t))) = 0, \quad (2)$$

that is, by using the chain rule,

$$2u(\gamma(t))u_x(\gamma(t))\gamma'_1(t) + 2u(\gamma(t))u_y(\gamma(t))\gamma'_2(t) + 2v(\gamma(t))v_x(\gamma(t))\gamma'_1(t) + 2v(\gamma(t))v_y(\gamma(t))\gamma'_2(t) = 0 \quad (3)$$

together with the similar relation for Γ :

$$2u(\Gamma(t))u_x(\Gamma(t))\Gamma'_1(t) + 2u(\Gamma(t))u_y(\Gamma(t))\Gamma'_2(t) + 2v(\Gamma(t))v_x(\Gamma(t))\Gamma'_1(t) + 2v(\Gamma(t))v_y(\Gamma(t))\Gamma'_2(t) = 0 \quad (4)$$

holding also for any $t \in (0, 1)$. By using the Cauchy-Riemann equations in (3) and (4), respectively, we get, after a convenient grouping of terms,

$$u(y(t))[u_x(y(t))y_1'(t) - v_x(y(t))y_2'(t)] + v(y(t))[u_x(y(t))y_2'(t) + v_x(y(t))y_1'(t)] = 0, \quad (5)$$

$$u(\Gamma(t))[u_x(\Gamma(t))\Gamma_1'(t) - v_x(\Gamma(t))\Gamma_2'(t)] + v(\Gamma(t))[u_x(\Gamma(t))\Gamma_2'(t) + v_x(\Gamma(t))\Gamma_1'(t)] = 0, \quad (6)$$

for any $t \in (0, 1)$. By specializing $t = t_1$ in (5) and $t = t_2$ in (6), we obtain

$$u(z_0)[u_x(z_0)y_1'(t_1) - v_x(z_0)y_2'(t_1)] + v(z_0)[u_x(z_0)y_2'(t_1) + v_x(z_0)y_1'(t_1)] = 0, \quad (7)$$

$$u(z_0)[u_x(z_0)\Gamma_1'(t_2) - v_x(z_0)\Gamma_2'(t_2)] + v(z_0)[u_x(z_0)\Gamma_2'(t_2) + v_x(z_0)\Gamma_1'(t_2)] = 0.$$

Since $u^2(z_0) + v^2(z_0) = c \neq 0$, it follows from (7) that

$$(u(z_0), v(z_0)) \neq (0, 0) \quad (8)$$

is a nontrivial solution of the linear homogeneous system

$$X[u_x(z_0)y_1'(t_1) - v_x(z_0)y_2'(t_1)] + Y[u_x(z_0)y_2'(t_1) + v_x(z_0)y_1'(t_1)] = 0, \quad (9)$$

$$X[u_x(z_0)\Gamma_1'(t_2) - v_x(z_0)\Gamma_2'(t_2)] + Y[u_x(z_0)\Gamma_2'(t_2) + v_x(z_0)\Gamma_1'(t_2)] = 0,$$

and so

$$\begin{vmatrix} u_x(z_0)y_1'(t_1) - v_x(z_0)y_2'(t_1) & u_x(z_0)y_2'(t_1) + v_x(z_0)y_1'(t_1) \\ u_x(z_0)\Gamma_1'(t_2) - v_x(z_0)\Gamma_2'(t_2) & u_x(z_0)\Gamma_2'(t_2) + v_x(z_0)\Gamma_1'(t_2) \end{vmatrix} = 0. \quad (10)$$

By expanding the determinant, equation (10) can be rewritten as

$$(u_x^2(z_0) + v_x^2(z_0))(y_1'(t_1)\Gamma_2'(t_2) - \Gamma_1'(t_2)y_2'(t_1)) = 0. \quad (11)$$

On the other hand, the assumption (iii) can be rewritten as

$$\begin{vmatrix} y_1'(t_1) & y_2'(t_1) \\ \Gamma_1'(t_2) & \Gamma_2'(t_2) \end{vmatrix} \neq 0. \quad (12)$$

Finally, from (11) and (12) it follows that

$$u_x^2(z_0) + v_x^2(z_0) = 0, \quad (13)$$

that is, $u_x(z_0) = v_x(z_0) = 0$. This, together with the Cauchy-Riemann relations [1] implies $u_y(z_0) = v_y(z_0) = 0$ and so $f'(z_0) = 0$. This concludes the proof of [Theorem 1](#). \square

The following exercise represents an interesting corollary of [Theorem 1](#).

COROLLARY 2. *Let $D \subset \mathbb{C}$ be a domain which contains the square $[-1, 1] \times [-1, 1]$. Assume that $f : D \rightarrow \mathbb{C}$ is a holomorphic function with the property that there exists $c \in \mathbb{R}_+^*$ such that*

$$|f(x + i0)| = c = \left| f\left(x + i \sin\left(\frac{1}{x}\right)\right) \right| \quad (14)$$

for any $x \in (0, 1)$. Then f is a constant function.

PROOF. Let $\gamma, \Gamma: (0, 1) \rightarrow \mathbb{C}$ defined by

$$\gamma(t) = (t, 0), \quad \Gamma(t) = \left(t, \sin\left(\frac{1}{t}\right)\right), \tag{15}$$

respectively. We have

$$\gamma'(t) = (1, 0), \quad \Gamma'(t) = \left(1, -\frac{1}{t^2} \cos\left(\frac{1}{t}\right)\right), \tag{16}$$

for any $t \in (0, 1)$. Consider the sequence

$$t_k = \frac{1}{k\pi} \in (0, 1) \tag{17}$$

convergent to 0. This choice of the sequence makes sure that

$$\gamma(t_k) = \Gamma(t_k) = (t_k, 0) \tag{18}$$

for any $k \geq 1$. We also have $\gamma'(t_k) = (1, 0)$ and $\Gamma'(t_k) = (1, -k^2(-1)^k\pi^2)$ which implies immediately that $\gamma(t_k)$ and $\Gamma(t_k)$ are linearly independent over \mathbb{R} for any $k \geq 1$. By [Theorem 1](#),

$$f'(t_k + i0) = 0 \tag{19}$$

holds true for any $k \geq 1$. Since f' is holomorphic and $t_k \rightarrow 0 \in D$ ($z = 0 \in D$ is an accumulation point for the zeros of f'), it follows that $f'(z) = 0$ for any $z \in D$, that is, f is a constant on D . □

Another result of similar flavour is the following theorem.

THEOREM 3. *Let $f: \mathbb{C} \rightarrow \mathbb{C}$ be holomorphic on an open neighborhood V of z_0 , and let $\gamma_1, \gamma_2: (0, 1) \rightarrow V$ be a pair of C^1 paths such that for some $t_1, t_2 \in (0, 1)$, we have $\gamma_1(t_1) = \gamma_2(t_2) = z_0$ and $\gamma_1'(t_1), \gamma_2'(t_2)$ are linearly independent over \mathbb{R} . We also assume that $f(\gamma_k(t)) \in \mathbb{R}, k = 1, 2$ for any $t \in (0, 1)$. Then, under the above assumptions, $f'(z_0) = 0$. If, in addition, $\arg(\gamma_1'), \arg(\gamma_2')$ are constant functions, then there exists a nonnegative integer n and a holomorphic function h defined on some open neighborhood of 0 such that $f(z) = h((z - z_0)^n)$ for $z \in V$.*

PROOF. Let ϕ be the angle between $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$. Consider two sequences $\{x_n\}, \{y_n\}$ of numbers from $(0, 1)$ such that $\lim_{n \rightarrow \infty} x_n = t_1$ while $\lim_{n \rightarrow \infty} y_n = t_2$. Then

$$\begin{aligned} f'(z_0) &= \lim_{n \rightarrow \infty} \frac{f(\gamma_1(x_n)) - f(\gamma_1(t_1))}{\gamma_1(x_n) - \gamma_1(t_1)} \\ &= \lim_{n \rightarrow \infty} \frac{(f(\gamma_1(x_n)) - f(\gamma_1(t_1))) / (x_n - t_1)}{(\gamma_1(x_n) - \gamma_1(t_1)) / (x_n - t_1)} \in \mathbb{R} e^{-i \arg(\gamma_1'(t_1))}. \end{aligned} \tag{20}$$

In a similar way, it is shown that

$$f'(z_0) \in \mathbb{R} e^{-i \arg(\gamma_2'(t_2))}. \tag{21}$$

From (20) and (21), together with the assumption that $\gamma_1'(t_1)$ and $\gamma_2'(t_2)$ are linearly independent over \mathbb{R} , it follows that $f'(z_0)$ has to be zero. This concludes the proof of the

first part of the theorem. We assume now that $\arg(y'_1), \arg(y'_2)$ are constant functions, say $\arg(y'_k) = c_k, k = 1, 2$, where $c_1 \neq c_2$. Then, keeping in mind that $f(y_k(t)) \in \mathbb{R}, k = 1, 2$ for any $t \in (0, 1)$, we see that

$$f'(y_k(t)) \in \mathbb{R}e^{-ic_k} \quad (22)$$

for $k = 1, 2$ and $t \in (0, 1)$. By induction on r , we can show that

$$f^{(r)}(y_k(t)) \in \mathbb{R}e^{-irc_k} \quad (23)$$

holds true for any nonnegative integer r where $k = 1, 2$ and $t \in (0, 1)$. Indeed, for $r = 0$ and $r = 1$, equation (23) is already shown. Assuming that (23) is true, by differentiation we get

$$f^{(r+1)}(y_k(t))y'_k(t) \in \mathbb{R}e^{-irc_k}. \quad (24)$$

From (24) and the fact that $\arg(y'_k(t)) = c_k$, it follows that

$$f^{(r+1)}(y_k(t)) \in \mathbb{R}e^{-i(r+1)c_k} \quad (25)$$

which concludes the inductive proof of (23). By specializing $t = t_1$ and then $t = t_2$ in (23), it follows that

$$f^{(r)}(z_0) \in \mathbb{R}e^{-irc_1} \cap \mathbb{R}e^{-irc_2} \quad (26)$$

for any $r = 0, 1, 2, \dots$. From (26) it follows that, for any given r , either $f^{(r)}(z_0) = 0$ or $e^{ir\phi} \in \mathbb{R}$ (i.e., $r\phi \in 2\pi\mathbb{Z}$). At this moment we distinguish two cases. First, if $\phi/\pi \in \mathbb{R} \setminus \mathbb{Q}$, it follows that $f^{(r)}(z_0) = 0$ for any $r = 0, 1, 2, \dots$ which implies that $f(z)$ is constant on a neighborhood of z_0 and this being the case the choice $h = \text{constant} = c$ would work. We consider now the second case, when $\phi = m\pi/n$, where $0 < m < n, m, n \in \mathbb{Z}_{>0}, (m, n) = 1$. From (26) it follows that $f^{(r)}(z_0) = 0$ for any r which is not divisible by n , since in this case $e^{ir\phi} = e^{irm\pi/n} \notin \mathbb{R}$. Therefore, on some neighborhood of z_0 the power series expansion of f has the form

$$f(z) = \sum_{l \leq 0} a_{ln}(z - z_0)^{ln} = \sum_{l \geq 0} a_{ln}[(z - z_0)^n]^l. \quad (27)$$

If we denote

$$h(z) := \sum_{l \geq 0} a_{ln}z^l, \quad (28)$$

it follows that h is holomorphic on some neighborhood of 0 and satisfies $f(z) = h((z - z_0)^n)$. This concludes the proof of [Theorem 3](#). \square

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