A REMARK ON GWINNER’S EXISTENCE THEOREM ON VARIATIONAL INEQUALITY PROBLEM

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Abstract. Gwinner (1981) proved an existence theorem for a variational inequality problem involving an upper semicontinuous multifunction with compact convex values. The aim of this paper is to solve this problem for a multifunction with open inverse values.

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1. Introduction. In 1981, Gwinner [1] proved an existence theorem for a variational inequality problem, which is an infinite dimensional version of Walras excess demand theorem (see also Zeidler [5]).

Theorem 1.1. Let $P$ and $Q$ be nonempty compact convex subsets of locally convex Hausdorff topological vector spaces $X$ and $Y$, respectively. Let $f : P \times Q \rightarrow \mathbb{R}$ be continuous. Let $S : P \rightarrow Q$ be a multifunction. Suppose that

(i) for each $y \in Q$, \{ $x \in P : f( x, y) < t$ \} is convex for all $t \in \mathbb{R}$,

(ii) $S$ is an upper semicontinuous multifunction with nonempty compact convex values. Then there exist $x_0 \in P$ and $y_0 \in S(x_0)$ such that $f( x_0, y_0) \leq f( x, y_0)$ for all $x \in P$.

In this paper, our aim is to obtain the above variational inequality for a multifunction with open inverse values. We follow the method of Tarafdar and Yuan [4].

2. Preliminaries. In $N \in \mathbb{N}$, let $\langle N \rangle$ be the set of all nonempty subsets of \{0, 1, 2, ..., $N$\}, $\Delta_N = \text{co}\{e_0, e_1, ..., e_N\}$ be the standard simplex of dimension $N$, where \{e_0, e_1, ..., e_N\} is the canonical basis of $\mathbb{R}^{N+1}$, and for $J \in \langle N \rangle$, let $\Delta_J = \text{co}\{e_j : j \in J\}$. Horvath [2] proved the following result.

Lemma 2.1. Let $X$ be a topological space and $F : \langle N \rangle \rightarrow X$. For each $J \in \langle N \rangle$, let $F(J)$ be a nonempty contractible subset of $X$ and for all $J, J' \in \langle N \rangle$ such that $J \subseteq J'$, suppose that $F(J) \subseteq F(J')$. Then there exists a continuous function $f : \Delta_N \rightarrow X$ such that $f(\Delta_J) \subset F(J)$ for all $J \in \langle N \rangle$.

Also, we need the following fixed point theorem due to Lassonde [4].

Lemma 2.2. Let $F : \Delta_N \rightarrow \Delta_N$ be a multifunction such that $F = F_n \circ F_{n-1} \circ \cdots \circ F_1 \circ F_0$, $\Delta_N \xrightarrow{F_0} X_1 \xrightarrow{F_1} X_2 \xrightarrow{F_2} \cdots \xrightarrow{F_n} X_{n+1} = \Delta_N$, where each $F_i$ is either a single-valued continuous function (in which case $X_{i+1}$ is assumed to be a Hausdorff topological space)
or an upper semicontinuous multifunction with \( F_i(x) \), a nonempty compact convex subset of \( X_{i+1} \) (in which case \( X_{i+1} \) is a convex subset of a Hausdorff topological vector space). Then \( F \) has a fixed point.

3. Main theorem

**Theorem 3.1.** Let \( P \) as in Theorem 1.1 and \( Q \) be an arbitrary subset of a locally convex Hausdorff topological vector space \( Y \). Let \( f : P \times Q \to \mathbb{R} \) be continuous and satisfy condition (i) of Theorem 1.1. Let \( S : P \to Q \) be a multifunction such that

(i) \( S^{-1}(X) \) is open for all \( x \in Q \);

(ii) for each open set \( F \subset P \), the set \( \bigcap_{y \in F} S(y) \) is empty or contractible;

(iii) \( S(P) \) is compact and contractible. Then the conclusion of Theorem 1.1 holds.

**Proof.** Since \( P \) is compact, there exists a finite subset \( \{x_0,x_1,x_2,\ldots,x_N\} \) of \( S(P) \) such that \( P = \bigcup_{i=0}^{N} S^{-1}(x_i) \). Define \( F : \langle N \rangle \to S(P) \) by

\[
F(J) = \begin{cases} \bigcap_{i \in J} S(y) : y \in \bigcap_{j \in J} S^{-1}(x_j) & \text{if } \bigcap_{j \in J} S^{-1}(x_j) \neq \emptyset, \\ S(P) & \text{otherwise.} \end{cases}
\]  

(3.3.1)

It is clear that if \( y \in \bigcap_{j \in J} S^{-1}(x_j) \), then \( x_j \in S(y) \) for all \( j \in J \). Thus, \( F(J) \) is nonempty and contractible. Further, \( F(J) \subseteq F(J') \) whenever \( J \subseteq J' \). By Lemma 2.1, there exists a continuous function \( f : \Delta_N \to S(K) \) such that \( f(\Delta_j) \subseteq F(J) \) for all \( J \in \langle N \rangle \). Let \( \{g_i : i \in \{0,1,2,\ldots,N\} \} \) be a continuous partition of unity subordinated to the covering \( \{S^{-1}(x_i) : i \in \{0,1,\ldots,N\}\} \), that is, for each \( i \), \( g_i : P \to [0,1] \) is continuous, \( \{y \in P : g_i(y) \neq 0\} \subseteq S^{-1}(x_i) \), and \( \sum_{i=0}^{N} g_i(y) = 1 \) for all \( y \in P \). Now, define \( g : P \to \Delta_N \) by \( g(y) = (g_0(y),g_1(y),\ldots,g_N(y)) \) for all \( y \in P \). Then \( g \) is continuous. Further, \( g(y) \in \Delta_{\langle y \rangle} \) for all \( y \in P \), where \( f(y) = \{i : g_i(y) = 0\} \). Therefore, \( f \circ g(y) \in f(\Delta_{\langle y \rangle}) \subseteq F_{\langle y \rangle} \subseteq S(y) \).

Consider \( T : S(P) \to \Delta \) defined by \( T(y) = \{z \in P : f(z,y) \leq f(w,y) \text{ for all } w \in P\} \). For each \( y \in S(P) \), \( T(y) \) is nonempty since \( f \) assumes its minimum on the compact set \( P \). Also, it is closed and hence compact. Further, \( T(y) \) is convex. Indeed, let \( z_1 \) and \( z_2 \) be such that \( f(z_1,y) \leq f(w,y) \) for all \( w \in P \) and \( i = 1,2 \). By the assumption on \( f \), \( f(\lambda z_1 + (1-\lambda)z_2,y) \leq f(w,y) \) for all \( w \in P \). Since \( f \) is continuous, the graph of \( T, Gr(T) = \{(y,z) : y \in S(P) \} \) is a closed subset of the compact set \( S(P) \times P \). Then it follows that \( T \) is upper semicontinuous.

Consider \( G := g \circ T \circ f : \Delta_N \to \Delta_N \). Now, by Lemma 2.2, there exists \( z_0 \in \Delta_N \) such that \( z_0 \in G(z_0) \). Let \( y_0 = f(z_0) \). Then \( y_0 \in f \circ g \circ T \circ f(z_0) \), that is, there exists \( x_0 \in T(y_0) \) so that \( y_0 \in f \circ g(x_0) \in S(x_0) \). This completes the proof. \( \square \)

**References**


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As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

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