A COMMUTATOR THEOREM FOR FRACTIONAL INTEGRALS IN SPACES OF HOMOGENEOUS TYPE

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ABSTRACT. We give a new proof of a commutator theorem for fractional integrals in spaces of homogeneous type.

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1. Introduction. Bramanti and Cerutti [3] and Bramanti [2] extended a classical commutator theorem for fractional integrals due to Chanillo [5] to the context of spaces of homogeneous type. In [3] Bramanti and Cerutti follow an idea contained in [7], based in holomorphic families of operators, used to study the $L^p$ boundedness of singular integrals in Euclidean spaces. In [2] Bramanti investigated the boundedness of the commutator of certain integral operators having positive kernels. A fractional integral appears as a particular case. Bramanti deduces the boundedness of the commutator from a suitable inequality that involves the maximal sharp function. In this paper, we give a different proof to the commutator theorem for fractional integrals in spaces of homogeneous type. We follow the original proof of Chanillo [5] and a good $\lambda$ inequality is essential.

We firstly recall the main definitions needed in the paper (see [8, 9, 11]). $(X, \delta, \mu)$ will be a space of homogeneous type. That is, $X$ is a nonvoid set, $\delta$ is a quasidestance on $X$, i.e., $\delta : X \times X \to [0, \infty)$ is a function satisfying the following properties:

(i) $\delta(x, y) = 0$ if and only if $x = y$,

(ii) $\delta(x, y) = \delta(y, x)$, for every $x, y \in X$, and

(iii) there exists a positive constant $k$ such that for every $x, y, z \in X$

$$\delta(x, y) \leq k(\delta(x, z) + \delta(z, y)), \quad (1.1)$$

and $\mu$ is a positive regular measure on $X$ defined on a $\sigma$-algebra of subsets of $X$ which contains the open sets (in the topology induced by the uniform structure associated to $\delta$) and the ball $B(x, r) = \{y \in X : \delta(x, y) < r\}$, for every $x \in X$ and $r > 0$, and that satisfies the doubling condition: there exists $A > 0$ for which

$$0 < \mu(B(x, 2r)) \leq A\mu(B(x, r)), \quad (1.2)$$

for each $x \in X$ and $r > 0$. Note that if $X$ has more than one element, then $k \geq 1$. The trivial case $k < 1$ is not considered in this paper.

There are many interesting examples of spaces of homogeneous type. For instance, any $C^\infty$ compact Riemannian manifold with the Riemanniann metric and volume and
the boundary of any bounded Lipschitz domain in $\mathbb{R}^n$ with the induced Euclidean metric and the Lebesgue measure are spaces of homogeneous type.

A space of homogeneous type is said to be normal if there exist positive constants $A_1$ and $A_2$ such that for every $x \in X$,

$$A_1 r \leq \mu(B(x,r)), \quad \text{when } 0 < r < R_x,$$

$$\mu(B(x,r)) \leq A_2 r, \quad \text{if } r \geq r_x,$$

where

$$R_x = \begin{cases} \infty, & \text{if } \mu(X) = \infty, \\ \inf\{r > 0 : B(x,r) = X\}, & \text{if } \mu(X) < \infty, \end{cases}$$

$$r_x = \begin{cases} 0, & \text{if } \mu(\{x\}) = 0, \\ \sup\{r > 0 : B(x,r) = \{x\}\}, & \text{if } \mu(\{x\}) > 0. \end{cases}$$

Sufficient conditions, in order that a space $(X, \delta, \mu)$ of homogeneous type admits a quasidistance $d$ that is equivalent to $\delta$ and such that $(X, d, \mu)$ is normal, are given in [14, Lemma 22].

A space of homogeneous type is of order $\rho$, $0 < \rho \leq 1$, if there is a positive constant $C$ such that for every $x, y, z \in X$

$$|\delta(x,z) - \delta(y,z)| \leq C \delta(x,y)^\rho \left( \max \{\delta(x,z), \delta(y,z)\}\right)^{1-\rho}.$$

For each $1 \leq p \leq \infty$, $L^p(X, \mu)$ and $\|\cdot\|_p$ have the usual meanings. We say that a complex valued measurable function $f$ on $X$ is in $L^p_{\text{loc}}(X, \mu)$, $1 \leq p < \infty$, if $\int_{B(x,r)} |f(x)|^p d\mu(x) < \infty$, for every $r > 0$ and for some (and then for all) $x \in X$.

Let $b \in L^1_{\text{loc}}(X, \mu)$. We define $b_\epsilon(x)$, with $x \in X$ and $\epsilon > 0$, as the mean value

$$b_\epsilon(x) = \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} b(y) d\mu(y).$$

If $1 \leq p < \infty$ we will say that a function $b \in L^p_{\text{loc}}(X, \mu)$ is in $\text{BMO}_p$ if and only if,

$$\|b\|_{*,p,\mu} = \left\| \sup_{\epsilon > 0} \left\{ \frac{1}{\mu(B(x,\epsilon))} \int_{B(x,\epsilon)} |b(y) - b_\epsilon(x)|^p d\mu(y) \right\}^{1/p} \right\|_{\infty} < \infty.$$

We define on $\text{BMO}_p$ a “norm” as follows:

$$\|b\|^{(p)} = \begin{cases} \|b\|_{*,p,\mu}, & \text{if } \mu(X) = \infty, \\ \|b\|_{*,p,\mu} + \left| \int_X b(x) d\mu(x) \right|, & \text{if } \mu(X) < \infty. \end{cases}$$

When $\mu(X) < \infty$, $(\text{BMO}_p, \|\cdot\|^{(p)})$ is a Banach space. If $\mu(X) = \infty$, then we introduce in $\text{BMO}_p$ the following relation: let $b_1$ and $b_2$ be in $\text{BMO}_p$,

$$b_1 \sim b_2 \iff \text{there exists } C \in \mathbb{C} \text{ such that } b_1 - b_2 = C.$$

It is clear that if $b_1, b_2 \in \text{BMO}_p$ and $b_1 \sim b_2$, then $\|b_1\|^{(p)} = \|b_2\|^{(p)}$. The quotient space $\text{BMO}_p / \sim$ will be denoted again by $\text{BMO}_p$ and by considering on it the norm
induced by \( \| \cdot \|^{(p)} \), \( \text{BMO}_p \) is a Banach space. As it was proved by Coifman and Weiss [9, page 594], if \( 1 \leq p, q < \infty \), the spaces \( \text{BMO}_p \) and \( \text{BMO}_q \) coincide and the norms \( \| \cdot \|^{(p)} \) and \( \| \cdot \|^{(q)} \) are equivalent. In the sequel, as usual, we will denote by \( \text{BMO} \) the space \( \text{BMO}_p, 1 \leq p < \infty \).

Let \( 0 \leq \alpha < 1 \). The fractional maximal function \( M_\alpha f \) of \( f \in L^1_{\text{loc}}(X, \mu) \) is defined by

\[
(M_\alpha f)(x) = \sup_{B: x \in B} \frac{1}{\mu(B)^{1-\alpha}} \int_B |f(y)| \, d\mu(y), \quad x \in X. \tag{1.10}
\]

Here, for each \( x \in X \), the supremum is taken over all those \( B \) balls in \( X \) containing to \( x \). As usual we denote by \( M \) the maximal operator \( M_0 \).

The fractional integral of order \( \alpha \) of \( f \), \( I_\alpha f \), is given by

\[
(I_\alpha f)(x) = \int_{X-\{x\}} \frac{f(y)}{\delta(x,y)^{1-\alpha}} \, d\mu(y). \tag{1.11}
\]

In this paper, we study the boundedness of the commutator \([I_\alpha, b] \) of the fractional integral \( I_\alpha \) and the multiplier operator associated to a measurable function \( b \) on \( X \) defined through

\[
[I_\alpha, b](f) = bI_\alpha(f) - I_\alpha(bf). \tag{1.12}
\]

Throughout this paper, for every \( 1 \leq p < \infty \), we will denote by \( p' \) the conjugate of \( p \). By \( C \) we will always represent a positive constant not necessarily the same in each occurrence.

The following theorem is the main result of the paper.

**Theorem 1.1.** Let \( 0 < \alpha < 1 \), \( 0 \leq \rho < 1 \), \( 1 < p < 1/\alpha \), and \( 1/q = 1/p - \alpha \). Assume that \( (X, \delta, \mu) \) is a normal space of homogeneous type of order \( \rho \) such that \( \mu(\{x\}) = 0 \), \( x \in X \). Then the commutator operator \([I_\alpha, b] \) is bounded from \( L^p(X, \mu) \) into \( L^q(X, \mu) \) provided that \( b \in \text{BMO} \).

Let now \( (X, \delta, \mu) \) be a normal space of homogeneous type and of order \( \rho \in (0,1) \), such that \( \mu(X) = \infty \) and \( \mu(\{x\}) = 0 \), for every \( x \in X \). Gatto, Segovia, and Vagi [10] defined, for every \( 0 < \alpha < 1 \), a function \( \delta_\alpha \) on \( X \times X \) as follows:

\[
\delta_\alpha(x,y) = \left( \int_0^\infty t^{\alpha-1} s(x,y,t) \, dt \right)^{1/\alpha-1}, \quad \text{for } x \neq y, \tag{1.13}
\]

where \( s \) represents a symmetric approximation to the identity in the sense of Coifman, and

\[
\delta_\alpha(x,y) = 0, \quad \text{for } x = y. \tag{1.14}
\]

In [10, Lemma 2] it is proved that, for every \( 0 < \alpha < 1 \), \( \delta_\alpha \) is a quasidistance equivalent to \( \delta \). Moreover, for each \( 0 < \alpha < 1 \), \( (X, \delta_\alpha, \mu) \) is a normal space of homogeneous type of order \( \rho \).

Also these authors introduced the fractional integral \( \tilde{I}_\alpha \) of order \( \alpha \in (0,1) \) through

\[
(\tilde{I}_\alpha f)(x) = \int_{X-\{x\}} \frac{f(y)}{\delta_\alpha(x,y)^{1-\alpha}} \, d\mu(y). \tag{1.15}
\]
If we represent by $BMO(\alpha)$ the $BMO$-space associated to the quasidistance $\delta_\alpha$, $0 < \alpha < 1$, it is immediately deduced from Theorem 1.1 the following commutator theorem for the fractional integral $I_\alpha$.

**Corollary 1.2.** Assume that $(X, \delta, \mu)$ is a normal space of homogeneous type and of order $\rho \in (0, 1)$, such that $\mu(X) = \infty$ and $\mu(\{x\}) = 0$, for every $x \in X$. Let $0 < \alpha < 1$. Then the commutator operator $[I_\alpha, b]$ defined by

$$
[I_\alpha, b](f) = bI_\alpha(f) - I_\alpha(bf),
$$

is a bounded operator from $L^p(X, \mu)$ into $L^q(X, \mu)$ provided that $1 < p < 1/\alpha$, $1/q = 1/p - \alpha$ and $b \in BMO(\alpha)$.

**2. The proof of the commutator theorem.** In this section, we will prove Theorem 1.1. To see that result we previously establish six lemmas.

Boundedness of the fractional integral $I_\alpha$ was studied in [11, Theorem 1] and [12, Theorems 2.2 and 2.4].

**Lemma 2.1** (see [11, Theorem 1]). Let $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. If $(X, \delta, \mu)$ is a normal space of homogeneous type, then

(i) $I_\alpha$ maps continuously $L^p(X, \mu)$ into $L^q(X, \mu)$.

(ii) There exists $C_1 > 0$ such that

$$
\mu(\{x \in X : |I_\alpha(f)(x)| > \lambda\}) \leq C_1 \left( \frac{\|f\|_1}{\lambda} \right)^{1/1-\alpha},
$$

for every $f \in L^1(X, \mu)$ and $\lambda > 0$.

Kokilashvili and Kufner [12, Theorem 3.2] proved a weighted version of [11, Theorem 1].


The following result can be easily inferred from [15, Theorem 4] (also from [12, Proposition A]).

**Lemma 2.2.** Let $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Then $M_\alpha$ is a bounded operator from $L^p(X, \mu)$ into $L^q(X, \mu)$.

We now define the auxiliary operator $C(b, f)$ on $X$ as follows:

$$
C(b, f)(x) = \sup_{\epsilon > 0} \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y) \right|, \quad x \in X,
$$

where $b$ and $f$ are measurable complex functions on $X$.

Next a useful weak type inequality for the operator $C(b, f)$ is established.
Lemma 2.3. Assume that \((X, \delta, \mu)\) is a normal space of homogeneous type. Let \(1 < p < 1/\alpha\). If \(f \in L^p(X, \mu)\) and \(b \in L^{p'}(X, \mu)\), then

\[
\mu\left(\{x \in X : C(b, f)(x) > \lambda\}\right) \leq C_0 \left( \frac{\|b\|_{p'} \|f\|_p}{\lambda} \right)^{1/1-\alpha}, \quad \text{for every } \lambda > 0. \tag{2.3}
\]

Proof. It is not hard to see that

\[
C(b, f)(x) \leq \sup_{c>0} \int_{X \setminus B(x, c)} \frac{|b(y)||f(y)|}{\delta(x, y)^{1-\alpha}} d\mu(y) + \sup_{c>0} |b_c(x)| \int_{X \setminus B(x, c)} \frac{|f(y)|}{\delta(x, y)^{1-\alpha}} d\mu(y) \tag{2.4}
\]

Moreover, Holder inequality and Lemmas 2.1 and 2.2 lead to

\[
\int_X M(b)(x)^{1/1-\alpha} I^\alpha(|f|)(x)^{1/1-\alpha} d\mu(x) \leq \left( \int_X M(b)(x)^{p'} d\mu(x) \right)^{1/1-\alpha} \left( \int_X I^\alpha(|f|)(x)^{p'/p'} d\mu(x) \right)^{1-1/p'(1-\alpha)} \leq C \|b\|_{p'}^{1/1-\alpha} \|f\|_{p}^{1/1-\alpha}. \tag{2.5}
\]

Hence, if \(\lambda > 0\), then

\[
\mu\left(\{x \in X : M(b)(x) I^\alpha(|f|)(x) > \lambda\}\right) \leq C \left( \frac{\|b\|_{p'} \|f\|_p}{\lambda} \right)^{1/1-\alpha}. \tag{2.6}
\]

Also by taking into account Lemma 2.1 we have

\[
\mu\left(\{x \in X : I^\alpha(|bf|)(x) > \lambda\}\right) \leq C \left( \frac{\|bf\|_1}{\lambda} \right)^{1/1-\alpha}, \quad \lambda > 0. \tag{2.7}
\]

Now to finish the proof of this lemma it is sufficient to combine (2.4), (2.6), and (2.7).

\[\square\]

Lemma 2.4. Assume that \((X, \delta, \mu)\) is a normal space of homogeneous type such that \(\mu(\{x\}) = 0, \ x \in X\). Let \(0 < \alpha < 1, \ 1 < p < 1/\alpha, \ 0 < \beta < 1/k, \text{ and } d, \ y > 0\). Let \(b \in BMO\) and \(f\) be a measurable function. Then

\[
d^y \int_{X \setminus B(x_0, d)} \frac{|b(y) - b_d(x)|}{\delta(x, y)^{1+y-\alpha}} |f(y)| d\mu(y) \leq C \left( M_{\alpha p}(|f|)(x_0) \right)^{1/p} \|b\|_{*,p'}, \tag{2.8}
\]

provided that \(\delta(x, x_0) \leq \beta d\). Here \(C\) is a constant that does not depend on \(d\).
Proof. Suppose that $\mu(X) = \infty$. If $\mu(X) < \infty$ we can proceed in a similar way. Hölder inequality implies that

$$d^\gamma \int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|}{\delta(x,y)^{1+y-\alpha}} |f(y)| d\mu(y)$$

$$\leq \left( d^\gamma \int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|}{\delta(x,y)^{1+y}} d\mu(y) \right)^{1/p'}$$

$$\times \left( d^\gamma \int_{X \setminus B(x,d)} \frac{|f(y)|^p}{\delta(x,y)^{1+y-p\alpha}} d\mu(y) \right)^{1/p}.$$  \hspace{1cm} (2.9)

Since $\mu$ is doubling we can write for every $x \in X$ and $j \in \mathbb{N}$,

$$|b_{2j-1,d}(x) - b_{2j,d}(x)| \leq \frac{1}{\mu(B(x,2^{j-1}d))} \int_{B(x,2^{j-1}d)} |b(y) - b_{2j,d}(x)| d\mu(y)$$

$$\leq C \frac{1}{\mu(B(x,2^j d))} \int_{B(x,2^j d)} |b(y) - b_{2j,d}(x)| d\mu(y)$$

$$\leq C \|b\|_{s,1}. \hspace{1cm} (2.10)$$

Hence, it concludes that

$$|b_d(x) - b_{2j,d}(x)| \leq C j \|b\|_{s,1}, \hspace{1cm} j \in \mathbb{N}, x \in X. \hspace{1cm} (2.11)$$

Then, since $(X,\delta,\mu)$ is normal, it follows

$$d^\gamma \int_{X \setminus B(x,d)} \frac{|b(y) - b_d(x)|}{\delta(x,y)^{1+y}} d\mu(y)$$

$$\leq d^\gamma \sum_{j=0}^\infty \int_{2^{j+1}d \leq \delta(x,y) < 2^j d} \frac{|b(y) - b_d(x)|}{\delta(x,y)^{1+y}} d\mu(y)$$

$$\leq d^\gamma \sum_{j=0}^\infty \frac{2^{-j} (2^j d)^{1-y}}{2^{j+1} d} \int_{2^{j+1}d \leq \delta(x,y) < 2^j d} |b(y) - b_d(x)|^{p'} d\mu(y)$$

$$\leq C \sum_{j=0}^\infty \frac{2^{-j} (2^j d)^{1-y}}{2^{j+1} d} \left( \int_{B(x,2^{j+1}d)} |b(y) - b_{2j+1,d}(x)|^{p'} d\mu(y) \right)^{1/p'}$$

$$\leq C \sum_{j=0}^\infty \frac{1}{\mu(B(x,2^{j+1}d))} \int_{B(x,2^{j+1}d)} |b(y) - b_{2j+1,d}(x)|^{p'} d\mu(y)$$

$$\leq C \sum_{j=0}^\infty \frac{2^{-j} (2^j d)^{1-y}}{2^{j+1} d} \left( \int_{B(x,2^{j+1}d)} |b(y) - b_d(x)|^{p'} d\mu(y) \right)^{1/p'}$$

$$\leq \|b\|_{s,1} \|b\|_{s,p'}.$$  \hspace{1cm} (2.12)

On the other hand, if $\delta(x_0,y) \leq \beta d$ and $\delta(x,y) \leq d$, then $\delta(x_0,y) \geq (1 - k\beta)/k) d$ and $\delta(x_0,y) \leq k(\beta + 1) \delta(x,y)$. Hence, by invoking again the normality of $(X,\delta,\mu)$ we can write
\[
\begin{align*}
    \int_{X \setminus B(x,d)} \frac{|f(y)|^p}{\delta(x,y)^{1+y-p\alpha}} \, d\mu(y) \\
    \leq C \int_{X \setminus B(x,((1-k\beta)/k)d)} \frac{|f(y)|^p}{\delta(x,y)^{1+y-p\alpha}} \, d\mu(y) \\
    \leq C \sum_{j=0}^{\infty} \int_{B(x,2^{j+1}((1-k\beta)/k)d) \setminus B(x,2^j((1-k\beta)/k)d)} \frac{|f(y)|^p}{\delta(x,y)^{1+y-p\alpha}} \, d\mu(y) \\
    \leq C \sum_{j=0}^{\infty} 2^{-jy} \int_{B(x,2^{j+1}((1-k\beta)/k)d)} \frac{1}{\mu(B(x,2^{j+1}((1-k\beta)/k)d))^{1-p\alpha}} \int_{B(x,2^{j+1}((1-k\beta)/k)d)} |f(y)|^p \, d\mu(y) \\
    \leq C M_{p\alpha}(|f|^p)(x_0).
\end{align*}
\]

(2.13)

Thus the result is proved.

The following Whitney type covering lemma will be useful in the sequel.

**Lemma 2.5** (see [4, Lemma 1] and [13, Lemma 2]). Let \( \Omega \) be a proper open subset of \( X \) and let \( B \) be a ball in \( X \) such that \( B \cap \Omega \neq \emptyset \) and \( B \cap (X \setminus \Omega) \neq \emptyset \). Then there exists a sequence \( (B_j)_{j \in \mathbb{N}} \) of balls in \( X \) satisfying the following three properties:

(i) \( \Omega \cap B \subset \bigcup_{j \in \mathbb{N}} B_j \subset \Omega \cap (B^*)^* \),
(ii) \( B_j^* \cap (X \setminus \Omega) = \emptyset, j \in \mathbb{N} \), and
(iii) \( \mu(\Omega \cap B) \leq \sum_{j=1}^{\infty} \mu(B_j) \leq C \mu(\Omega \cap (B^*)^*) \).

Here if \( B = B(x,r) \), with \( x \in X \) and \( r > 0 \), \( B^* \) denotes the ball \( B(x,r k(2k+1)) \).

Next we will prove in the main lemma a good-\( \lambda \) inequality.

**Lemma 2.6.** Let \( 0 \leq p < 1 \) and \( 1 < p < 1/\alpha \). Assume that \( (X, \delta, \mu) \) is a normal space of homogeneous type that is of order \( p \) and such that \( \mu(\{x\}) = 0, \ x \in X \). Let \( b \in \text{BMO} \) and \( f \) be a measurable function on \( X \). Then there exists \( \beta_0 \) such that for every \( \beta \geq \beta_0 \) and \( \gamma > 1 \)

\[
    \mu \left( \left\{ x \in X : C(b,f)(x) > \beta \lambda, \|b\|_{*,p'} \left( I_{\alpha}(|f|)(x) + (M_{p\alpha}(|f|^p)(x))^{1/p} \right) \leq y \lambda \right\} \right) \\
    \leq C \gamma \mu \left( \left\{ x \in X : C(b,f)(x) > \lambda \right\} \right),
\]

provided that one of the following two conditions holds:

(i) \( \lambda > 0 \) and \( \mu(X) = \infty \),
(ii) \( \lambda > \left( \frac{C_0}{\mu(X)} \right)^{1-\alpha} \|b\|_{p'} \|f\|_p \) and \( \mu(X) < \infty \), where \( C_0 \) is the positive constant appearing in Lemma 2.3.

**Proof.** Let \( \beta, \gamma > 0 \) and \( \lambda \) satisfying the imposed conditions. We define the following sets:

\[
E_\lambda(\beta,y) = \left\{ x \in X : C(b,f)(x) > \beta y, \|b\|_{*,p'} \left( I_{\alpha}(|f|)(x) + (M_{p\alpha}(|f|^p)(x))^{1/p} \right) \leq y \lambda \right\},
\]

\[
W_\lambda = \left\{ x \in X : C(b,f)(x) > \lambda \right\}.
\]

(2.15)
Note that we can assume, without loss of generality, that \( W_\lambda \neq \emptyset \) and \( W_\lambda \neq X \). Indeed, suppose firstly that \( \mu(X) = \infty \). If \( W_\lambda = X \), then (2.14) is clear for every \( \beta > 0 \) and \( \gamma > 0 \). On the other hand, if \( \mu(X) < \infty \), then Lemma 2.3 implies that \( \mu(W_\lambda) < \mu(X) \) when \( \lambda > (C_0/\mu(X))^{1-\alpha} \| b \|_{p^*} \| f \|_p \) and where \( C_0 \) is the positive constant that appears in Lemma 2.3. Also if \( \mu(X) \leq \infty \) and \( W_\lambda = \emptyset \), then (2.14) holds for every \( \beta > 1 \) and \( \gamma > 0 \).

Let \( B \) be a ball in \( X \) such that \( B \cap W_\lambda \neq \emptyset \) and \( B \cap (X \setminus W_\lambda) \neq \emptyset \). Then there exists a sequence \( \{B_j\}_{j=1}^\infty \) of balls in \( X \) satisfying conditions (i), (ii), and (iii) in Lemma 2.5 by replacing \( \Omega \) by \( W_\lambda \).

Let \( j \in \mathbb{N} \). Write \( B_j = B(a, \alpha_j d) \), with \( a \in X \) and \( d > 0 \). We define \( B_j^1 = B(a, \alpha_1 d) \) and \( B_j^2 = B(a, \alpha_2 d) \), where \( \alpha_1 \leq k(2k^2(1+k(2k+1)) + 1) \) and \( \alpha_2 > k(1+k(\alpha_1 + 1)) \).

Assume that \( B_j \cap E_\lambda(\beta, y) \neq \emptyset \) and choose \( x_1 \in B_j \cap E_\lambda(\beta, y) \). We write \( f = f_1 + f_2 \), where \( f_1 = f_{X_{B_j^1}} \), and \( b = b_1 + b_2 \), being \( b_1 = (b - b_{B_j^2})_X b_{B_j^2} \) and \( b_{B_j^2} = 1/\mu(B_j^2) \times \int_{B_j^2} b(y) d\mu(y) \).

We have that \( C(b, f_1)(x) \leq C(b_1, f_1)(x) \), for every \( x \in B_j \). Indeed, let \( x \in B_j \) and \( \epsilon > 0 \). Since \( \alpha_2 > k(1+k(\alpha_1+1)) \), if \( B(x, \epsilon) \cap (X \setminus B_j) \neq \emptyset \), then \( B_j \subset B(x, \epsilon) \). Hence we can write

\[
(b_1)_\epsilon(x) = \frac{1}{\mu(B(x, \epsilon))} \int_{B(x, \epsilon)} b_1(y) d\mu(y)
\]

\[
= \frac{1}{\mu(B(x, \epsilon))} \int_{B(x, \epsilon) \cap B_j^2} (b(y) - b_{B_j^2}) d\mu(y)
\]

\[
= \frac{1}{\mu(B(x, \epsilon))} \int_{B(x, \epsilon) \cap B_j^2} b(y) d\mu(y) - b_{B_j^2} = b_\epsilon(x) - b_{B_j^2},
\]

provided that \( B_j^1 \cap (X \setminus B(x, \epsilon)) = \emptyset \).

Then, since \( B_j^1 \subset B_j^2 \), one has

\[
C(b, f_1)(x) = \sup_{\epsilon > 0} \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f_1(y) d\mu(y) \right|
\]

\[
= \sup_{\epsilon > 0} \left| \int_{X \setminus B(x, \epsilon) \cap B_j^1} \frac{b_1(y) + b_{B_j^2} - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f_1(y) d\mu(y) \right|
\]

\[
\leq \sup_{\epsilon > 0} \left| \int_{X \setminus B(x, \epsilon) \cap B_j^1} \frac{b_1(y) - (b_1)_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f_1(y) d\mu(y) \right| = C(b_1, f_1)(x).
\]

Moreover from Lemma 2.3 we deduce that for every \( \beta > 1 \),

\[
\mu(\{x \in B_j : C(b_1, f_1)(x) > \beta \lambda \}) \leq C \left( \frac{\| b_1 \|_{p^*} \| f_1 \|_p}{\beta \lambda} \right)^{1-\alpha}
\]

\[
= C \left( \int_{B_j^1} |b(y) - b_{B_j^2}|^{p'} d\mu(y) \right)^{1/p'(1-\alpha)} \left( \int_{B_j^1} |f(y)|^p d\mu(y) \right)^{1/p(1-\alpha)}
\]

\[
\leq C \lambda^{1/\alpha - 1} \mu(B_j) \left( \| b \|_{*, p'} (M_p \alpha(\| f \|^p)(x_1))^{1/p} \right)^{1/1-\alpha},
\]

because \( \mu \) is doubling.
Hence, since $x_1 \in B_j \cap E_1(\beta, \gamma)$ if $\gamma < 1$, then
\[
\mu(\{x \in B_j : C(b, f_1)(x) > \beta \lambda\}) \leq C \gamma \mu(B_j),
\] (2.19)

By virtue of (ii) in Lemma 2.5, $B_j^* \cap (X \setminus W_\lambda) \neq \emptyset$. Choose $x_0 \in B_j^* \cap (X \setminus W_\lambda)$, that is, $x_0 \in B_j^*$ and $C(b, f)(x_0) \leq \lambda$.

Now our objective is to estimate
\[
\mu(\{x \in B_j : C(b, f_2)(x) > \beta \lambda\}).
\] (2.20)

We consider two cases.

Assume firstly that $\epsilon > \sigma d$, where $\alpha_2/k - 1 > \sigma > (\alpha_1 + 1)/k$. Since $\sigma > (\alpha_1 + 1)/k$, for every $x \in B_j$, $B_j^* \subset B(x, \epsilon)$. Let $x \in B_j$.

We have
\[
\int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y)
\] (2.21)

where
\[
I_1 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) - \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x_0)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y),
\]
\[
I_2 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0, y)^{1-\alpha}} f(y) d\mu(y) - \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0, y)^{1-\alpha}} f(y) d\mu(y),
\] (2.22)
\[
I_3 = \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0, y)^{1-\alpha}} f_2(y) d\mu(y).
\]

We are going to estimate $I_i$, $i = 1, 2, 3$.

As mentioned above if $\delta(x, y) > \epsilon$, then $y \notin B_j^*$. Hence $\delta(x, y) > \epsilon$ implies that $
\delta(x, y) \geq ((\alpha_1 - k)/k)d > 0$. Then we can write
\[
\frac{\delta(x_1, y)}{\delta(x, y)} \leq \frac{k(\delta(x, x_1) + \delta(x, y))}{\delta(x, y)} \leq \frac{2k^3}{\alpha_1 - k} + k,
\] (2.23)

provided that $\delta(x, y) > \epsilon$.

Therefore it follows
\[
|I_1| = \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y) - \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_\epsilon(x_0)}{\delta(x, y)^{1-\alpha}} f(y) d\mu(y) \right|
\]
\[
\leq \int_{X \setminus B(x, \epsilon)} \left| \frac{b_\epsilon(x_0) - b_\epsilon(x)}{\delta(x, y)^{1-\alpha}} \right| |f(y)| d\mu(y)
\]
\[
\leq C |b_\epsilon(x_0) - b_\epsilon(x)| \int_{X \setminus B} \frac{|f(y)|}{\delta(x_1, y)^{1-\alpha}} d\mu(y)
\]
\[
\leq C |b_\epsilon(x_0) - b_\epsilon(x)| \int_{X \setminus B} |f(y)| d\mu(y)
\]
\[
\leq C |b_\epsilon(x_0) - b_\epsilon(x)| I_{\alpha}(|f|)(x_1).
\] (2.24)

Moreover if $y \in B(x_0, \epsilon)$, then
\[
\delta(x, y) \leq k(\delta(y, x_0) + \delta(x_0, x)) \leq k(\epsilon + kd(1 + k(2k + 1))) < 2^m \epsilon,
\] (2.25)
where \( m \in \mathbb{N} \) is large enough and \( m \) is not depending on \( d \) and \( \epsilon \).

Hence, since \((X, \delta, \mu)\) is normal we have that

\[
| b_\epsilon(x_0) - b_\epsilon(x) |
\leq \frac{1}{\mu(B(x_0, \epsilon))} \int_{B(x_0, \epsilon)} |b(y) - b_\epsilon(x)| d\mu(y)
\leq C \frac{1}{\mu(B(x, 2^m \epsilon))} \int_{B(x, 2^m \epsilon)} |b(y) - b_\epsilon(x)| d\mu(y)
\leq C \left( \frac{1}{\mu(B(x, 2^m \epsilon))} \int_{B(x, 2^m \epsilon)} |b(y) - b_{2^m \epsilon}(x)| d\mu(y) + |b_{2^m \epsilon}(x) - b_\epsilon(x)| \right)
\leq C\|b\|_{*,p'}.
\]

(2.26)

Thus we conclude that

\[
|I_1| \leq C\|b\|_{*,p'} I_\alpha(|f|)(x_1) \leq C\gamma\lambda.
\]

(2.27)

On the other hand, to estimate \( I_2 \) we will use that \((X, \delta, \mu)\) is a space of homogeneous type which is of order \( \rho \in (0, 1) \). It is clear that

\[
|I_2| \leq \int_{\delta(x,y) \geq \epsilon} |b(y) - b_\epsilon(x_0)| |f_2(y)| |\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}| d\mu(y)
+ \int_{\delta(x,y) \geq \epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y)
- \int_{\delta(x,y) \geq \epsilon} \frac{b(y) - b_\epsilon(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y).
\]

(2.28)

Note that, since \( \sigma > 2k^2(1 + k(2k+1)) \), \( \delta(x_0,y) \leq 2k\delta(x_0,x) \) provided that \( \delta(x_0,y) > \epsilon \). Hence, according to [11, Lemma II.3] and Lemma 2.4, since \( \delta(x_0,x_1) < (1/2k)\epsilon \), we have,

\[
\int_{\delta(x,y) \geq \epsilon} |b(y) - b_\epsilon(x_0)| |f_2(y)| |\delta(x,y)^{\alpha-1} - \delta(x_0,y)^{\alpha-1}| d\mu(y)
\leq C\delta(x,x_0)^{\rho} \int_{X \setminus B(x_0, \epsilon)} |b(y) - b_\epsilon(x_0)| |f_2(y)| |\delta(x_0,y)^{\alpha-p-1}| d\mu(y)
\leq Ce^{\rho} \int_{X \setminus B(x_0, \epsilon)} |b(y) - b_\epsilon(x_0)| |f_2(y)| |\delta(x_0,y)^{\alpha-p-1}| d\mu(y)
\leq C\|b\|_{*,p'} (M_{p\alpha}(|f|^p)(x_1))^{1/p} \leq C\gamma\lambda.
\]

(2.29)

Moreover, \( \delta(x,y) < \epsilon \) implies that \( \delta(x_0,y) \leq \epsilon(k + (1/2)) \) and this inequality implies that \( \delta(x_1,y) \leq \epsilon(k(k + (1/2)) + (1/2)) \). Then, by taking into account the normality of
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\[(X, \delta, \mu)\], Holder inequality leads to
\[
\left| \int_{\delta(x,y) \geq \epsilon} \frac{b(y) - b_e(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \right|
\]
\[
- \int_{\delta(x_0,y) \geq \epsilon \text{ and } \delta(x,y) < \epsilon} \frac{b(y) - b_e(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y)
\]
\[
\leq C \epsilon^{\alpha - 1} \int_{B(x_0, \epsilon(k+(1/2)))} |b(y) - b_e(x_0)| |f_2(y)| d\mu(y) \quad (2.30)
\]
\[
\leq C \frac{1}{\mu(B(x_0, \epsilon(k+(1/2))))^{1-\alpha}} \int_{B(x_0, \epsilon(k+(1/2)))} |b(y) - b_e(x_0)| |f_2(y)| d\mu(y)
\]
\[
\leq \|b\|_{p'} (M_p \alpha(|f|^p)(x_1))^{1/p} \leq C \gamma \lambda.
\]

Finally, since \(x_0 \notin W_\lambda\), we have
\[
|I_3| \leq \lambda. \quad (2.31)
\]

By combining (2.21), (2.27), and (2.31) we conclude that
\[
\sup_{\epsilon > d \sigma} \left| \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_e(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \right| \leq C \gamma \lambda + \lambda. \quad (2.32)
\]

Assume now \(0 < \epsilon \leq d \sigma\). Let \(x \in B_j\). It is not hard to see that if \(y\) is in the support of \(f_2\) then \(\delta(x,y) \gtrsim ((\alpha_1 - k)/k)d\) and \(\delta(x_0,y) \gtrsim ((\alpha_1 - k^2(2k+1))/k)d\). We choose \(\omega > 0\) such that \(\omega < (\alpha_1 - k)/k\).

We can write
\[
\int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_e(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) = J_1 + J_2 + J_3, \quad (2.33)
\]

where
\[
J_1 = \int_{X \setminus B(x, \epsilon)} \frac{b_{\text{odd}}(x_0) - b_e(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y),
\]
\[
J_2 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_{\text{odd}}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) - \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_{\text{odd}}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y),
\]
\[
J_3 = \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_{\text{odd}}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y).
\]

(2.34)

We will estimate \(J_i\), \(i = 1, 2, 3\).

By proceeding as in the study of \(I_1\), since \(k(\sigma + 1) < \alpha_2\), we obtain
\[
|J_1| \leq C |b_{\text{odd}}(x_0) - b_e(x)| |I_\alpha(|f|)(x_1)
\]
\[
\leq C \frac{1}{\mu(B(x, \epsilon))} \int_{B(x, \epsilon)} |b(y) - b_{\text{odd}}(x_0)| |\chi_{B_j^2}(y)| d\mu(y) |I_\alpha(|f|)(x_1)
\]
\[
\leq CM \left( (b - b_{\text{odd}}(x_0)) \chi_{B_j^2}(x) \right) I_\alpha(|f|)(x_1). \quad (2.35)
\]
On the other hand, we have that

\[
J_2 = \int_{X \setminus B(x, \epsilon)} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y) - \int_{X \setminus B(x_0, \epsilon)} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y)
\]

\[
= \int_{\delta(x, y) \geq \epsilon} \left( \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} \right) f_2(y) \left( \delta(x, y)^{\alpha-1} - \delta(x_0, y)^{\alpha-1} \right) d\mu(y) + \int_{\delta(x, y) \geq \epsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} f_2(y) d\mu(y)
\]

\[
- \int_{\delta(x, y) \leq \epsilon} \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_0, y)^{1-\alpha}} f_2(y) d\mu(y).
\]

(2.36)

Since \((X, \delta, \mu)\) is a space of homogeneous type of order \(\rho \in (0, 1)\), by virtue of [11, Lemma 2.3], we have

\[
\left| \int_{\delta(x, y) \geq \epsilon} \left( \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x, y)^{1-\alpha}} \right) f_2(y) \left( \delta(x, y)^{\alpha-1} - \delta(x_0, y)^{\alpha-1} \right) d\mu(y) \right| \leq C \delta(x_0)^{\rho} \int_{\delta(x, y) \geq \epsilon} |b(y) - b_{\text{wod}}(x_0)| \left| f_2(y) \right| \delta(x, y)^{\alpha-\rho-1} d\mu(y),
\]

(2.37)

because if \(y\) is in the support of \(f_2\), then \(\delta(x, y) \geq ((\alpha_1 - k)/k)d \geq 2k^2(k(2k+1) + 1)d \geq 2k\delta(x_0, x).\) Hence, since \(y \in \text{supp} \ f_2\) implies that \(\delta(x_1, y) > \omega d\), by proceeding as in the proof of Lemma 2.4 we conclude

\[
\left| \int_{\delta(x, y) \geq \epsilon} \left( \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_1, y)^{1-\alpha}} \right) f_2(y) \left( \delta(x_1, y)^{\alpha-1} - \delta(x_0, y)^{\alpha-1} \right) d\mu(y) \right| \leq C \delta(x_0)^{\rho} \int_{\delta(x, y) > \omega d} \left\| f(y) \right\|_{L_1} \delta(x_1, y)^{\alpha-\rho-1} d\mu(y)
\]

\[
\leq C \|b\|_{L_p} (|f|^p(x_1))^{1/p} \leq Cy\lambda.
\]

(2.38)

Also, since if \(\delta(x, y) < \epsilon\), then \(\delta(x_0, y) < dk(k(2k+1) + \sigma)\) and since \(\omega \leq \alpha_1\), we have

\[
\left| \int_{\delta(x, y) \geq \epsilon} \left( \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_0, y)^{1-\alpha}} \right) f_2(y) d\mu(y) \right| - \int_{\delta(x, y) < \epsilon} \left( \frac{b(y) - b_{\text{wod}}(x_0)}{\delta(x_0, y)^{1-\alpha}} \right) f_2(y) d\mu(y)
\]

\[
\leq \int_{\omega d < \delta(x_0, y) < k(k(2k+1) + \sigma)d} \left| b(y) - b_{\text{wod}}(x_0) \right| \left| f_2(y) \right| d\mu(y)
\]

\[
\times \left( \delta(x, y)^{\alpha-1} + \delta(x_0, y)^{\alpha-1} \right) d\mu(y)
\]

\[
\leq C \int_{\omega d < \delta(x_0, y) < k(k(2k+1) + \sigma)d} \left| b(y) - b_{\text{wod}}(x_0) \right| \delta(x_0, y)^{\alpha-\alpha-1} f_2(y) d\mu(y).
\]

(2.39)
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Now by proceeding as in the proof of Lemma 2.4 we obtain that
\[
\left| \int_{\delta(x,y) \geq \epsilon} \frac{b(y) - b_{\omega d}(x_0)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \right| - \int_{\delta(x,y) < \epsilon} \frac{b(y) - b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}} f_2(y) d\mu(y) \leq C \|b\|_{*,p'} (M_{p\alpha}(|f|^p)(x_1))^{1/p} \leq C \gamma \lambda. \tag{2.40}
\]

In a similar way we can see that
\[
|J_3| \leq \int_{X \setminus B(x_0,\omega d)} \left| \frac{b(y) - b_{\omega d}(x_0)}{\delta(x_0,y)^{1-\alpha}} \right| f_2(y) d\mu(y) \leq C \|b\|_{*,p'} (M_{p\alpha}(|f|^p)(x_1))^{1/p} \leq C \gamma \lambda. \tag{2.41}
\]

By combining the above estimates we can conclude
\[
\sup_{0 < \epsilon < \omega d} \left| \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_{\epsilon}(x)}{\delta(x,y)^{1-\alpha}} f_2(y) d\mu(y) \right| \leq C \left( \lambda \gamma + I_{\alpha}(|f|)(x_1) M \left( (b - b_{\omega d}(x_0)) \chi_{B_j^2}(x) \right) \right). \tag{2.42}
\]

From (2.32) and (2.42) follows that for every \( x \in B_j \)
\[
C(b, f_2)(x) \leq C \left( \lambda \gamma + \lambda + I_{\alpha}(|f|)(x_1) M \left( (b - b_{\omega d}(x_0)) \chi_{B_j^2}(x) \right) \right). \tag{2.43}
\]

Hence if \( \beta \) is large enough, then according to Lemma 2.2 and since \( \mu \) is doubling
\[
\mu \left( \{ x \in B_j : C(b, f_2)(x) > \beta \lambda \} \right) \leq \mu \left( \{ x \in B_j : I_{\alpha}(|f|)(x_1) M \left( (b - b_{\omega d}(x_0)) \chi_{B_j^2}(x) \right) \lambda \} \right) \leq C \lambda \mu \left( \{ x \in B_j^2 : |b(y) - b_{\omega d}(x_0)| d\mu(y) \right) \leq C \lambda \mu \left( B_j \right). \tag{2.44}
\]

Thus we obtain that for \( \beta \geq \beta_0 \) and \( \gamma < 1 \), where \( \beta_0 \) is large enough,
\[
\mu(B_j \cap E_\lambda(\beta, \gamma)) \leq C \gamma \mu(B_j). \tag{2.45}
\]

Hence
\[
\mu(B \cap E_\lambda(\beta, \gamma)) \leq C \gamma \sum_{j=1}^{\infty} \mu(B_j) \leq C \gamma \mu(W_\lambda), \quad \beta \geq \beta_0, \ \gamma < 1. \tag{2.46}
\]

Arbitrariness of \( B \) allows to conclude that
\[
\mu(E_\lambda(\beta, \gamma)) \leq C \gamma \mu(W_\lambda), \quad \beta \geq \beta_0, \ \gamma < 1, \tag{2.47}
\]
and the proof is finished.
**Proof of Theorem 1.1.** To prove Theorem 1.1 we proceed as in the proof of [6, Theorem III]. We start proving that the operator $C(b,f)$ is bounded from $L^p(X,\mu)$ into $L^q(X,\mu)$, when $1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Assume that $b \in L^\infty(X,\mu)$.

Let $1 < p_1 < p < 1/\alpha$ and $1/q = 1/p - \alpha$. Assume firstly that $\mu(X) = \infty$. According to Lemma 2.6, $f \in L^p(X,\mu)$ we have

$$
\int_X (C(b,f)(x))^q d\mu(x)
= \beta q \int_0^\infty \lambda^{q-1} \mu(\{|x : C(b,f)(x) > \beta \lambda\}) d\lambda
\leq C \beta (\int_X \lambda^{q-1} \mu(\{|x : C(b,f)(x) > \lambda\}) d\lambda
+ \int_0^\infty \lambda^{q-1} \mu(\{|x : C(b,f)(x) > \lambda\}) d\lambda
\leq C \beta (\int_0^{\int_X (C(b,f)(x))^q d\mu(x)} \lambda^{q-1} d\lambda + \gamma \int_X (C(b,f)(x))^q d\mu(x)
+ \gamma^{-q} \|\|_s,\|_p \int_X (I_\alpha(|f|)(x) + (M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}) d\mu(x)

(2.48)
$$

provided that $\beta \geq \beta_0$ and $0 < \gamma < 1$, where $\beta_0$ is given in Lemma 2.6.

Hence by (2.4) and Lemma 2.1 and by taking $\gamma$ so small we can conclude that

$$
\|C(b,f)\|_q \leq C \|b\|_{s,p'} (\|I_\alpha(|f|)\|_q + \|M_{p_1\alpha}(|f|^{p_1})\|_{q/p_1}^{1/p_1}).

(2.49)
$$

According to Lemmas 2.1 and 2.2 it follows

$$
\|C(b,f)\|_q \leq C \|b\|_{s,p'} \|f\|_p.

(2.50)
$$

Suppose now that $\mu(X) < \infty$. Since $C(b,f) = C(b-a,f)$, for every $a \in C$, we can assume, without loss of generality, that $\int_X b d\mu = 0$. Then Lemma 2.6 leads, for every $f \in L^p(X,\mu)$, to

$$
\int_X (C(b,f)(x))^q d\mu(x)
= \beta q \int_0^\infty \lambda^{q-1} \mu(\{|x : C(b,f)(x) > \beta \lambda\}) d\lambda
\leq C \beta (\int_0^{\int_X (C(b,f)(x))^q d\mu(x)} \lambda^{q-1} d\lambda + \gamma \int_X (C(b,f)(x))^q d\mu(x)
+ \gamma^{-q} \|\|_s,\|_p \int_X (I_\alpha(|f|)(x) + (M_{p_1\alpha}(|f|^{p_1})(x))^{1/p_1}) d\mu(x)

(2.51)
$$

when $\beta \geq \beta_0$ and $0 < \gamma < 1$, $\beta_0$ being as in Lemma 2.6.

Thus we deduce from Lemmas 2.1 and 2.2 that

$$
\|C(b,f)\|_q \leq C \|b\|_{s,p'} \|f\|_p.

(2.52)
$$
Now we note that
\[
[b,I_\alpha](f)(x) = \lim_{\epsilon \to 0^+} \left( b(x) \int_{X \setminus B(x,\epsilon)} \frac{f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) - \int_{X \setminus B(x,\epsilon)} \frac{b(y)f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) \right)
= \lim_{\epsilon \to 0^+} \left( (b(x) - b_\epsilon(x)) \int_{X \setminus B(x,\epsilon)} \frac{f(y)}{\delta(x,y)^{1-\alpha}} d\mu(y) \right.
\left. - \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y) d\mu(y) \right)
= \lim_{\epsilon \to 0^+} \int_{X \setminus B(x,\epsilon)} \frac{b(y) - b_\epsilon(x)}{\delta(x,y)^{1-\alpha}} f(y) d\mu(y),
\tag{2.53}
\]
for every \( f \in L^p(X,\mu) \), and a.e. \( x \in X \).

Then
\[
\| [b,I_\alpha] \|_q \leq \| C(b,f) \|_q,
\tag{2.54}
\]
for each \( f \in L^p(X,\mu) \).

To finish the proof it is sufficient to take into account [3, Lemma 2.5] and Fatou’s lemma.

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**References**


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Call for Papers

As a multidisciplinary field, financial engineering is becoming increasingly important in today's economic and financial world, especially in areas such as portfolio management, asset valuation and prediction, fraud detection, and credit risk management. For example, in a credit risk context, the recently approved Basel II guidelines advise financial institutions to build comprehensible credit risk models in order to optimize their capital allocation policy. Computational methods are being intensively studied and applied to improve the quality of the financial decisions that need to be made. Until now, computational methods and models are central to the analysis of economic and financial decisions.

However, more and more researchers have found that the financial environment is not ruled by mathematical distributions or statistical models. In such situations, some attempts have also been made to develop financial engineering models using intelligent computing approaches. For example, an artificial neural network (ANN) is a nonparametric estimation technique which does not make any distributional assumptions regarding the underlying asset. Instead, ANN approach develops a model using sets of unknown parameters and lets the optimization routine seek the best fitting parameters to obtain the desired results. The main aim of this special issue is not to merely illustrate the superior performance of a new intelligent computational method, but also to demonstrate how it can be used effectively in a financial engineering environment to improve and facilitate financial decision making. In this sense, the submissions should especially address how the results of estimated computational models (e.g., ANN, support vector machines, evolutionary algorithm, and fuzzy models) can be used to develop intelligent, easy-to-use, and/or comprehensible computational systems (e.g., decision support systems, agent-based system, and web-based systems).

This special issue will include (but not be limited to) the following topics:

• Computational methods: artificial intelligence, neural networks, evolutionary algorithms, fuzzy inference, hybrid learning, ensemble learning, cooperative learning, multiagent learning

• Application fields: asset valuation and prediction, asset allocation and portfolio selection, bankruptcy prediction, fraud detection, credit risk management

• Implementation aspects: decision support systems, expert systems, information systems, intelligent agents, web service, monitoring, deployment, implementation

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