CONVEX ISOMETRIC FOLDING

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ABSTRACT. We introduce a new type of isometric folding called “convex isometric folding.” We prove that the infimum of the ratio $\text{Vol}_N/\text{Vol}_\varphi(N)$ over all convex isometric foldings $\varphi : N \to N$, where $N$ is a compact 2-manifold (orientable or not), is $1/4$.

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1. Introduction. A map $\varphi : M \to N$, where $M$ and $N$ are $C^\infty$ Riemannian manifolds of dimensions $m$ and $n$, respectively, is said to be an isometric folding of $M$ into $N$ if and only if for any piecewise geodesic path $\gamma : J \to M$, the induced path $\varphi \circ \gamma : J \to N$ is a piecewise geodesic and of the same length. The definition is given by Robertson [4]. The set of all isometric foldings $\varphi : M \to N$ is denoted by $\mathcal{F}(M,N)$.

Let $p : M \to N$ be a regular locally isometric covering and let $G$ be the group of covering transformations of $p$. An isometric folding $\varphi \in \mathcal{F}(M)$ is said to be $p$-invariant if and only if for all $g \in G$ and all $x \in X$, $p(\varphi(x)) = p(\varphi(g,x))$. See Robertson and Elkholy [5]. The set of $p$-invariant isometric foldings is denoted by $\mathcal{F}_i(M,p)$.

**DEFINITION 1.1.** Let $\varphi \in \mathcal{F}(M,N)$, where $M$ and $N$ are $C^\infty$ Riemannian manifolds of dimensions $m$ and $n$, respectively. We say that $\varphi$ is a convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^n$.

We denote the set of all convex isometric foldings of $M$ into $N$ by $C(M,N)$, and if $C(M,N) \neq \emptyset$, then it forms a subsemigroup of $\mathcal{F}(M,N)$.

**DEFINITION 1.2.** We say that $\varphi \in \mathcal{F}_i(M,p)$ is a $p$-invariant convex isometric folding if and only if $\varphi(M)$ can be embedded as a convex set in $\mathbb{R}^m$.

We denote the set of $p$-invariant convex isometric foldings of $M$ by $C_i(M,p)$. If $C_i(M,p) \neq \emptyset$, then for any covering map, $p : M \to N$, $C_i(M,p)$ is a subsemigroup of $C(M)$.

To solve our main problem we need the following:

1. Robertson and Elkholy [5] proved that if $N$ is an $n$-smooth Riemannian manifold, $p : M \to N$ is its universal covering, and $G$ is the group of covering transformations of $p$, then $\mathcal{F}(N)$ is isomorphic as a semigroup to $\mathcal{F}_i(M,p)/G$.

2. Elkholy [1] proved that if $N$ is an $n$-smooth Riemannian manifold, $p : M \to N$ is its universal covering, and $\varphi \in \mathcal{F}(N)$ such that $\varphi_\pi : \pi_1(N) \to \pi_1(N)$ is trivial, then the
corresponding folding $\psi \in \mathcal{F}_i(M, p)$ maps each fiber of $p$ to a single point.

(3) Elkholy and Al-Ahmady [3] proved that under the same conditions of (2), if $N$ is a compact 2-manifold, then

$$\frac{\text{Vol} N}{\text{Vol} \varphi(N)} = \frac{\text{Vol} F}{\text{Vol} \psi(F)},$$

where $F$ is a fundamental region of $G$ in $M$.

2. Convex isometric folding and covering spaces. The next theorem establishes the relation between the set of convex isometric folding of a manifold, $C(N)$, and the set of $p$-invariant convex isometric folding of its universal covering space, $C_i(M, p)$.

**Theorem 2.1.** Let $N$ be a manifold and $p : M \to N$ its universal covering. Let $G$ be the group of covering transformations of $p$. If $C(N) \neq \emptyset$, then $C(N)$ is isometric as a semigroup to $C_i(M, p)/G$.

**Proof.** Let $C(N) \neq \emptyset$. Then by using (1), there exists an isomorphism $f$ from $\mathcal{F}_i(M, p)/G$ into $\mathcal{F}(N)$. Since $C_i(M, p)$ is a subsemigroup of $\mathcal{F}_i(M, p)$, $C_i(M, p)/G$ is a subsemigroup of $\mathcal{F}_i(M, p)/G$.

Let $h = f \mid (C_i(M, p)/G)$. Since $C_i(M, p)/G$ is a semigroup, $h$ is a homeomorphism and also it is one-one. To show that $h$ is an onto map, we suppose that $\varphi \in C(N)$. Hence, $\varphi \in \mathcal{F}(N)$ and, consequently, there exists $\psi \in \mathcal{F}_i(M, p)/G$. Since $\varphi \in C(N)$, $\varphi_\psi$ is trivial and hence for all $x \in M$, $\psi(G, x) = \psi(x)$, and therefore $\psi \in C_i(M, p)/G$. \qed

**Theorem 2.2.** Let $N$ be a compact orientable 2-manifold and consider the universal covering space $(\mathbb{R}^2, P)$ of $N$. Let $\varphi \in C(N)$ and $\psi \in C_i(\mathbb{R}^2, p)$. Then for all $x, y \in \mathbb{R}^2$, $d(\psi(x), \psi(y)) \leq \Delta$, where $\Delta$ is the radius of a fundamental region for the covering space.

**Proof.** Elkholy [1] proved the truth of the theorem for $N = S^2$. So, we have to prove it for the connected sum of $n$-tori. First, let $N = T$ be a torus homomorphic to the quotient space obtained by identifying opposite sides of a square of length “$a$” as shown in Figure 1(a)

\[\text{Figure 1.}\]
Suppose that \( \varphi : T \to T \) is a convex isometric folding. Then \( \varphi \) is trivial. By Theorem 2.1, there exists a convex isometric folding \( \psi : \mathbb{R}^2 \to \mathbb{R}^2 \) such that for all \( x, y \in \mathbb{R}^2 \) and for all \( g \in G \), \( p(\psi(x)) = p(\psi(g \cdot x)) \). Equivalently, for all \( (P, Q) \in \mathbb{R}^2 \) and for all \( g \in \mathbb{Z} \times \mathbb{Z} \), there exists a unique \( h \in \mathbb{Z} \times \mathbb{Z} \) such that \( h \circ \psi(P, Q) = \psi(g(P, Q)) \), i.e.,

\[
\psi(P, Q) + (\sqrt{2} \Delta m, \sqrt{2} \Delta n') = \psi(P + \sqrt{2} \Delta m, Q + \sqrt{2} \Delta n), \quad \text{where } m, n, m', n' \in \mathbb{Z}.
\]

(2.1)

Consider any fundamental region \( F \) of the covering space \((\mathbb{R}^2, p)\) of \( T \), i.e., a closed square of length “\( a \)” with sides identified as shown in Figure 1(b). Since \( \varphi \) is trivial, by (2), for all \( x \in \mathbb{R}^2 \), \( \psi(G \cdot x) = \psi(x) \). Now, let \( x \) and \( y \) be distinct points of \( \mathbb{R}^2 \) such that \( x = g \cdot y \) for all \( g \in G \) and let \( d(x, y) = \alpha_1 \). Then there exists a point \( x^* = g \cdot x \) such that

\[
d(y, x^*) = \min(\alpha_i), \quad \alpha_i = d(y, g_i \cdot x), \quad i = 1, \ldots, 4.
\]

(2.2)

Thus, there are always four equivalent points \( g_i \cdot x \), \( i = 1, \ldots, 4 \) which form the vertices of a square of length “\( a \)” and such that \( d(g_i \cdot x, y) \leq 2 \Delta \). From Figure 1(b), it is clear that \( \max d(x^*, y) \leq \Delta \) and since \( \psi \) is an isometric folding, by Robertson [4],

\[
d(\psi(x), \psi(y)) \leq d(g_i \cdot x, y), \quad i = 1, \ldots, 4,
\]

and this proves the theorem for \( N = T \).

Now, consider the connected sum of two tori, obtained as a quotient space of an octagon with sides identified as shown in Figure 2(a). The group of covering transformations \( G \) is isometric to \( \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z} \). Using the same previous technique, we can obtain four equivalent points as the vertices of a square of diameter \( 2 \Delta \) such that \( \max d(y, x^*) \leq \Delta \), and the result follows. This theorem, by using the above method, is true for the connected sum of \( n \)-tori.

\[\square\]
**Theorem 2.3.** Let $N$ be a compact nonorientable 2-manifold and consider the universal covering space $(M, p)$ of $N$. Let $\phi \in C(N)$ and $\psi \in C_i(M, p)$. Then for all $x, y \in M$, $d(\psi(x), \psi(y)) \leq \Delta$, where $\Delta$ is the radius of a fundamental region for the covering space.

**Proof.** By Elkholy [2], the theorem is true for $N = p^2$ and $M = S^2$. Now, consider the connected sum of two projective planes, the Klein bottle $K$, homeomorphic to the quotient space obtained by identifying the opposite sides of a square as shown in Figure 3(a).

![Figure 3](image)

Suppose that $\varphi : K \to K$ is a convex isometric folding. Then there exists a convex isometric folding $\psi : \mathbb{R}^2 \to \mathbb{R}^2$ such that for all $x \in \mathbb{R}^2$ and $g \in G$, $p(\varphi(x)) = p(\psi(g \cdot x))$. Equivalently, for all $(P, Q) \in \mathbb{R}^2$ and for all $g \in \mathbb{Z} \times \mathbb{Z}$, there exists a unique $h \in \mathbb{Z} \times \mathbb{Z}$ such that $h \circ \psi(P, Q) = \psi(g(P, Q))$, i.e.,

$$
\psi(P, Q) + (\sqrt{2}\Delta m', \sqrt{2}\Delta n') = \psi(P + \sqrt{2}\Delta m, \sqrt{2}\Delta n + (-)^m Q), \quad \text{where } m, n, m', n' \in \mathbb{Z}.
$$

Any fundamental region $F$ of the covering space $(\mathbb{R}^2, p)$ of $K$ is a closed square of diameter $2\Delta$ with the boundary identified as shown in Figure 3(b). Since $\varphi_*$ is trivial, for all $x \in \mathbb{R}^2$, $\psi(G \cdot x) = \psi(x)$.

Now, let $x$ and $y$ be distinct points of $\mathbb{R}^2$ such that $y \neq g \cdot x$ for all $g \in G$, and let $d(x, y) = \alpha_1$. Thus, there exists a point $x^* = g \cdot x$ such that

$$
d(y, x^*) = \min(\alpha_i), \quad \alpha_i = d(y, g_i \cdot x), \quad i = 1, \ldots, 4.
$$

Thus, there are always four equivalent points $g_i \cdot x$ which form the vertices of a parallelogram such that the shortest diameter is of length less than $2\Delta$.

Now, the point $y$ is either inside or on the boundary of a triangle of vertices $g_1 \cdot x = x, g_2 \cdot x, g_3 \cdot x$. Let $y'$ be a point equidistant from the vertices of this triangle, i.e.,

$$
d(y', x) = d(y', g_2 \cdot x) = d(y', g_3 \cdot x).
$$

![Figure 3](image)
From Figure 3(b), it is clear that $d(y', x) < \Delta$ and, hence, $d(x^*, y) < \Delta$. Therefore,

$$
\begin{equation}
\begin{aligned}
d(\psi(x), \psi(y)) &= d(\psi(g_i \cdot x), \psi(y)) \\
&\leq d(g \cdot x_i, y) = d(x^*, y) < \Delta
\end{aligned}
\end{equation}
$$

and the result follows.

Now, let $N$ be the connected sum of three projective planes obtained as the quotient space of a hexagon with the sides identified in pairs as indicated in Figure 4(a). In this case, $(\mathbb{R}^2, p)$ is the universal cover of $N$ and $G \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2$. Using the same method as that used above, we can always have equivalent points $g_i \cdot x, i = 1, \ldots, 4$ which form the vertices of a parallelogram whose shortest diameter is of length less than $2\Delta$. From Figure 4(b), we can see that $\max d(y, x^*) < \Delta$ and the theorem is proved.

In general and by using the same technique, the theorem is also true for the connected sum of $n$-projective planes.

\[ P \# P \# P \]

\[ \gamma \]

\[ g_1 \cdot x \quad g_2 \cdot x \quad g_3 \cdot x \quad g_4 \cdot x \]

\[ \text{(a)} \quad \text{(b)} \]

\textbf{Figure 4.}

3. Volume and convex folding. The following theorem succeeds in estimating the maximum volume we may have if we convexly folded a compact 2-manifold into itself.

\textbf{Theorem 3.1.} The infimum of the ratio

$$
\begin{equation}
\begin{aligned}
e_N = \frac{\text{Vol } N}{\text{Vol } \phi(N)}
\end{aligned}
\end{equation}
$$

where $N$ is a compact 2-manifold over all convex isometric foldings $\phi \in C(N)$ of degree zero, is 4.

\textbf{Proof.} Robertson [4] has shown that if $N$ is a compact 2-manifold, and $\phi : N \to N$ is a convex isometric folding, any convex isometric folding is an isometric folding, then $\deg \phi$ is $\pm 1$ or 0. We consider only the case for which $\deg \phi$ is zero otherwise $\phi(N)$ cannot be embedded as a convex subset of $\mathbb{R}^2$ unless $N$ is. In this case, the set of singularities of $\phi$ decomposes $N$ into an even number of strata, say $k$, each of which is homeomorphic to $\phi(N)$ and, hence,

$$
\begin{equation}
\begin{aligned}
\text{Vol } N = k \text{Vol } \phi(N)
\end{aligned}
\end{equation}
$$
that is, $e_N$ should be an even number. To calculate the exact value of $e_N$, consider first an orientable 2-compact manifold $N$. By using (1.1)

$$e_N = \frac{\text{Vol} F}{\text{Vol} \varphi(F)}$$

and this means that $e_N$ can be calculated by calculating the volume of $F$ and of its image $\varphi(F)$, but $F$ is a closed square of diameter $2\Delta$ and $\varphi(F)$ is a closed subset of $F$ such that the distance $d(x,x')$ between any two points $x, x' \in \varphi(F)$ is at most $\Delta$. The supremum of 2-dimensional volume of such set is $\phi(\Delta/2)^2$ and, hence, $2 < e_N$. But $e_N$ is an even number. Hence, $e_N = 4$.

Now, let $N$ be a nonorientable 2-compact manifold, i.e., a connected sum of $n$-projective planes. Elkholy [2] proved the theorem for $n = 1$.

The fundamental region in this case is a square or a rectangle of diameter $2\Delta$ according to whether $n$ is even or odd. If $n$ is an even number, then

$$\text{Vol} F = 2\Delta^2$$

and the result follows. Now, let $n$ be an odd number. Then $F$ is a rectangle of lengths $((n+1)/2)a$, $((n-1)/2)a$ and hence

$$\text{Vol} F = 4\Delta^2 \sin \theta \cos \theta = 4\Delta^2 \frac{a(n+1)/2}{a\sqrt{(n+1)/2}} \frac{a(n-1)/2}{a\sqrt{(n-1)/2}} = \frac{n^2-1}{n^2+1} 2\Delta^2.$$

(3.5)

Therefore, $e_N > 2$ for all $n > 1$. Since $e_N$ is an even number, $e_N = 4$. \hfill $\sq$

**References**


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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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