FUNCTIONAL EVOLUTION EQUATIONS WITH NONCONVEX LOWER SEMICONTINUOUS MULTIVALUED PERTURBATIONS

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ABSTRACT. In this paper we prove some existence theorems concerning the solutions and integral solution for functional (delay) evolution equations with nonconvex lower semicontinuous multivalued perturbations

KEY WORDS AND PHRASES: Functional evolution equations, m-accretive operators, integral solutions

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1. INTRODUCTION

Let $E$ be a Banach space, $r, T \in \mathbb{R}^+$ and $I = [a, b]$ Let us denote

$C_E([-r,T])$ the vector space of all continuous functions from $[-r,T]$ to $E$ endowed with the uniform topology

For all $\theta \geq 0$, $s \in C([-r,0])$, $C([-r,0])$,

$s(t,f)(\theta) = f(t + \theta), \ \forall \theta \in [-r,0].$

$A : I \times E \to 2^E$ such that $A(t,.)$ is an m-accretive multivalued operator

$P_{wc}(E)$ the family of nonempty weakly compact subsets of $E$

In this paper we are concerned with the following problems

(1) Existence of solutions of the perturbated evolution equation with delay

\[ \begin{align*}
  u'(t) &\in -A(t,u(t)) + F(t,s_t u) \text{ a.e. on } I, \\
  u &\equiv \psi \text{ on } [-r,0]
\end{align*} \]

where $F : I \times C_E([-r,0]) \to P_{wc}(E)$ is a multivalued function such that $F(t,.)$ is lower semicontinuous and $\psi \in C_E([-r,0])$ is arbitrary but fixed.

(2) Existence of solutions of the perturbated evolution equation with delay

\[ \begin{align*}
  u'(t) &\in -N_{\Gamma(t)}(u(t)) + F(t,s_t u) \text{ a.e. on } I, \\
  u &\equiv \psi \text{ on } [-r,0]
\end{align*} \]

where $N_{\Gamma(t)}(x)$ is the normal cone of the convex set $\Gamma(t)$ at the point $x \in E$; $t \in I$. It should be noticed that the problem $(Q)$ is not a special case of the problem $(P)$

(3) Existence of integral solutions of $(P)$, when the operator $A$ is independent of $t$, under conditions that are weaker than those imposed in $(P)$

The results obtained in the present paper generalized the following interesting known cases

Problem $(P)$ for which the dual of $E$ is uniformly convex, $A(t,.)$ is an m-accretive single-valued operator and $F$ is a Lipschitz single-valued function cf Kartsatos and Parrott [1]
Problem \((P)\) for which \(E\) is reflexive, \(A(t,.)\) is an \(m\)-accretive multivalued operator and \(F\) is a Lipschitz single-valued function cf Tanaka [2]

Problems \((P)\) and \((Q)\) without delay cf Cichon [3], [4], Ibrahim [5] and the references therein

2. NOTATIONS AND DEFINITIONS

Let \(E^*\) be the dual of \(E\), \(E_\circ\) the Banach space \(E\) endowed with the weak topology \(\sigma(E, E^*)\). If \(B\) is a multivalued operator from \(E\) to \(2^E\), then \(B\) is said to be accretive if for each \(\lambda > 0\), \(x_1, x_2 \in D(B)\), \(y_1 \in B(x_1)\) and \(y_2 \in B(x_2)\) we have

\[
\|x_1 - x_2\| \leq \|x_1 - x_2 + \lambda(y_1 - y_2)\|.
\]

We say that \(B\) is \(m\)-accretive if \(B\) is accretive and if there exists \(\lambda > 0\) such that \(R(I + \lambda B) = E\), where \(I\) is the identity map. It is known that if \(B\) is \(m\)-accretive, then for every \(\lambda > 0\) the resolvent \(J_\lambda B = (I + \lambda B)^{-1}\) and the Yosida approximation of \(B\), \(B_\lambda = (I - J_\lambda B)/\lambda\), are defined everywhere. The generalized domain of \(B\) is defined by

\[
D'(B) = \left\{ x \in E : |B(x)| = \lim_{\lambda \to \infty} \|B_\lambda x\| < \infty \right\}.
\]

For the properties of \(m\)-accretive multivalued operators refer to [6] and [7].

If \(C\) is a convex subset of \(E\) and \(x \in C\), then the normal cone of \(C\) at \(x\) is defined by

\[
N_C(x) = \{ y \in E^* : \langle y, z - x \rangle \leq 0, \forall z \in C \}.
\]

Now we recall some concepts concerning multivalued functions. Let \(Y\) be a locally convex space and let \(G : E \to 2^Y - \{\emptyset\}\). We say that \(G\) is lower semicontinuous (resp. upper semicontinuous) if for every open \(V\) in \(Y\) the set \(\{ x \in E : G(x) \cap V \neq \emptyset \}\) (resp \(\{ x \in E : G(x) \subseteq V \}\)) is open in \(E\). We say that \(G\) is lower semicontinuous (resp. upper semicontinuous) in the Kuratowski sense iff for all \(v_n \to v\) in \(E\), \(G(v) \subseteq \lim_{n \to \infty} \inf G(v_n)\) (resp \(\lim_{n \to \infty} \sup G(v_n) \subseteq G(v)\)), where

\[
\lim_{n \to \infty} \inf G(v_n) = \left\{ z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall n \geq 1 \right\},
\]

\[
\lim_{n \to \infty} \sup G(v_n) = \left\{ z \in Y : z = \lim_{n \to \infty} z_n, z_n \in G(v_n), \forall k \geq 1 \right\}.
\]

If \(E\) is metrizable then lower semicontinuity and lower semicontinuity in the Kuratowski sense are equivalent (cf [8], [9]).

The following known result will be used in the sequel

**LEMMA 2.1** [6]. For every \(t \in I\), let \(A(t,.)\) be an \(m\)-accretive multivalued operator from \(E\) to \(2^E - \{\emptyset\}\) satisfying the following condition:

\((C_1)\) There exist \(\lambda_0 > 0\), a continuous function \(h : I \to E\) and a nondecreasing continuous function \(L : [0, \infty) \to [0, \infty)\) such that for all \(\lambda \in (0, \lambda_0)\) and for almost \(t, s \in I\),

\[
\|A_{\lambda}(t, x) - A_{\lambda}(s, x)\| \leq \|h(s) - h(t)\|L(\|x\|), \quad \forall x \in E.
\]

Then \(D^*(A(t, .))\) and \(\overline{D}(A(t, .))\) are independent of \(t\).

So if \(A\) is as in Lemma 2.1 we may write \(D^*(A) := D^*(A(t, .))\) and \(\overline{D}(A) := \overline{D}(A(t, .)); t \in I\) respectively.

**LEMMA 2.2** [10]. Let \(E\) be a Banach space and \(M\) a compact metric space. If \(A\) is a lower semicontinuous multivalued function on \(M\) and with nonempty closed decomposable values in \(L^1_{\infty}(I)\), then \(A\) has a continuous selection.

3. EXISTENCE OF SOLUTIONS FOR THE PROBLEMS \((P)\) AND \((Q)\)

To prove our results we need the following lemmas

**LEMMA 3.1.** Let \(\psi\) be an element of \(C_E([-r, 0])\) and \(\beta\) be a positive real number. The set
\[
\chi = \left\{ u \in C_E([-r,0]) : u \equiv \psi \text{ on } [-r,0] \text{ and } u(t) = \psi(0) + \int_0^t f(s)ds ; f \in K_\beta \right\},
\]
is nonempty and convex, where \( K_\beta = \{ f \in L_E^2(I) : ||f(t)|| \leq \beta \text{ a.e. on } I \}. \) If \( E \) is reflexive then \( \chi \) is compact subset of \( C_{E_0}([-r,T]) \). If, in addition, \( E \) is separable then \( \chi \) is metrizable.

**PROOF.** It is obvious that \( \chi \) is nonempty, convex and equicontinuous and that the set \( \{ u(t) : u \in \chi \}; t \in I, \) is bounded. So, if \( E \) is reflexive then, \( \chi \) is relatively compact in \( C_{E_0}([-r,T]) \) by Ascoli’s theorem. Let us verify that \( \chi \) is closed in \( C_{E_0}([-r,T]) \). Let \( (u_n) \) be a sequence in \( \chi \) converging to \( u \in C_{E_0}([-r,T]) \). Then \( u \equiv \psi \) on \( [-r,0] \) and for each \( n \geq 1 \) there exists \( f_n \in K_\beta \) such that \( u_n(t) = \psi(0) + \int_0^t f_n(s)ds ; t \in I \). Since \( E \) is reflexive, \( K_\beta \) is weakly compact in \( L_E^2(I) \). Hence, the sequence \( (f_n) \) has a subsequence, denoted again by \( (f_n) \), converging weakly to \( f \in K_\beta \). Then \( u(t) = \psi(0) + \int_0^t f(s)ds ; t \in I \) This proves that \( \chi \) is closed in \( C_{E_0}([-r,T]) \). Now if \( E \) is separable then so is \( L_E^2(I) \). Consequently, \( K_\beta \) is metrizable. Since \( \chi \) is isomorphic to \( \{ \psi(0) \} \times K_\beta \), then \( \chi \) is metrizable.

**LEMMA 3.2.** Let \( G \) be a multivalued function from \( E_0 \) to the nonempty closed subsets of \( E \) such that \( G \) is lower semicontinuous in the Kuratowski sense. If \( (x_n) \) is a sequence converging to \( x \) in \( E_0 \), then for every \( z \in E \),

\[
\lim_{n \to \infty} \sup_{z \in L_E^2(I)} d(z, G(x_n)) \leq d(z, \lim_{n \to \infty} \inf G(x_n)) \leq d(z, G(x)).
\]

**PROOF.** Let \( y \in \lim_{n \to \infty} \inf G(x_n) \). Then there exists a sequence \( (y_n) \) such that \( y_n \in G(x_n); n \geq 1 \) and \( y_n \to y \) as \( n \to \infty \). For any \( z \in E \) we have

\[
\lim_{n \to \infty} \sup_{z \in L_E^2(I)} d(z, G(x_n)) \leq \lim_{n \to \infty} \sup_{z \in L_E^2(I)} \| z - y_n \| = \| z - y \|,
\]
which proves the first inequality. The second inequality follows from the lower semicontinuity of \( G \).

**THEOREM 3.1.** Let \( E \) be a reflexive separable Banach space. Let \( A(t, \cdot); t \in I \) be an m-accretive multivalued operator from \( E \) to \( 2^E - \{ \phi \} \) satisfying condition \((C1)\) together with the following conditions

\( (C2) \) There exist \( \mu > 0 \) such that for all \( x \in E \), the function \( \omega_z : t \to (I + \mu A(t, \cdot))^{-1} \) belongs to \( L_E^2(I) \).

\( (C3) \) For all \( r > 0 \) there exists \( \delta(r) > 0 \) such that for all \( \lambda > 0 \) and all \( x \in D^* \) with \( \| x \| < r \),

\[
\| J_x A(0,x) - x \| \leq \lambda \delta(r).
\]

Let \( F \) be a measurable multivalued function from \( I \times C_E([-r,0]) \) to \( P_{ac}(E) \) satisfying the following conditions

\( (F1) \) There exists \( \alpha > 0 \) such that

\[
\sup \{ ||y|| : y \in F(t,u) \} \leq \alpha, \forall (t,u) \in I \times C_E([-r,0]).
\]

\( (F2) \) For all \( t \in I \), \( F(t, \cdot) \) is lower semicontinuous in the sense of Kuratowski from \( C_E([-r,0]) \) to \( E \).

\( (F3) \) For all \( u \in C_E([-r,0]) \) the multivalued function \( t \to F(t, u(t)) \) admits a measurable selection.

Then for every \( \psi \in C_E([-r,0]) \) with \( \psi(0) \in D^*(A) \), the problem \( (P) \) has a solution.

**PROOF.** We split the proof into the following three steps

(1) Let \( f \in K_\alpha = \{ g \in L_E^2(I) : ||g(t)|| \leq \alpha \text{ a.e. on } I \}. \) Since \( A \) satisfies conditions \((C1)\), \((C2)\) and \((C3)\), then by Theorem 4 of [5], there exists a unique absolutely continuous function \( u_f : I \to E \) such that:

\( i) \ u_f(t) \in - A(t, u(t)) + f(t) \text{ a.e. on } I, u_f(0) = \psi(0), \)

\( ii) \ ||u_f(t)|| \leq \beta \alpha (1 + L(r) \sup_{t \in I} ||h(t)|| + \delta(r), \forall t \in I, \) where \( r = \alpha(1 + L(||\psi(0)||)) + |A(0,x_0)|, \)
(iii) the function $f \to u_f$ is continuous from $K_{\alpha}$ to $C_{E_0}(I)$

(2) Set $\chi_1 = \{ u \in C_E([-r, T]) : u \equiv \psi$ on $[-r, 0]$ and $u(t) = \psi(0) + \int_0^t f(s)ds, f \in K_{\alpha} \}$. By Lemma 3.1, $\chi_1$ is a compact subset of $C_0([-r, T])$ and is metrizable. Define a multivalued function $T_1$ on $\chi_1$ by $T_1(u) = \{ f \in K_{\alpha} : f(t) \in F(t, s, u) \text{ a.e. on } I \}$. In this step we prove that $T_1$ has a continuous selection $V_1 : \chi_1 \to K_{\alpha}$. For this purpose, we show that $T_1$ satisfies the conditions of Lemma 2.2. Condition $(F_3)$ assures that the values of $T_1$ are nonempty. Moreover, if $D$ is a measurable subset of $I$ and $g_1, g_2 \in T_1(u)$ for some $u \in \chi_1$, then the function $g = N_Dg_1 + N_{I-D}g_2$ belongs to $T_1(u)$, where $N$ is the characteristic function. Then the values of $T_1$ are decomposable. It remains to prove that $T_1$ is lower semicontinuous. Since $\chi_1$ is compact metrizable in $C_{E_0}([-r, T])$, it suffices to show that $T_1$ is lower semicontinuous in the Kuratowski sense. So, let $(u_n)$ be a sequence in $\chi_1$ converging to $u \in \chi_1$, with respect to the topology on $C_{E_0}([-r, T])$ and let $g \in T_1(u)$. Since $F$ is measurable, then for all $n \geq 1$ the multivalued function

\[ B_n(t) = \{ z \in F(t, s, u_n) : \|g(t) - z\| \leq d(g(t), F(t, s, u_n)) \} \]

has a measurable selection $g_n : I \to E$. Thus, by Lemma 3.2, for all $t \in I$,

\[ \lim_{n \to \infty} \|g(t) - g_n(t)\| \leq \limsup_{n \to \infty} d(g(t), F(t, s, u_n)) \leq d(g(t), \liminf_{n \to \infty} F(t, s, u_n)) = d(g(t), F(t, s, u)) = 0. \]

This means that $T_1$ is lower semicontinuous and hence there exists a continuous function $V_1 : \chi_1 \to K_{\alpha}$ such that $V_1(x) \in T(x), \forall x \in \chi_1$.

(3) Define a function $\theta : \chi_1 \to \chi_1$ by $\theta(x) = u_f, f = V_1(x)$. By (iii) of the first step, $\theta$ is continuous. Hence, by Tichonoff's fixed point theorem, there exists $u \in \chi_1$ such that $u = u_f, f = V_1(u) \in T_1(u)$. This means that $u'(t) \in -A(t, u(t)) + f(t)$ and $f(t) \in F(t, s, u)$ a.e. on $I$. The theorem is thus proved.

**THEOREM 3.2.** Let $H$ be a Hilbert space and $F$ be a measurable multivalued function from $I \times C_H([-r, 0])$ to $P_{ac}(H)$ satisfying conditions $(F_1)$, $(F_2)$ and $(F_3)$. Let $\Gamma$ be a multivalued function from $I$ to the family of nonempty closed convex subsets of $H$, with compact graph $G$ and satisfies the following conditions.

$(\Gamma_1)$ There exists $\gamma > 0$ such that $\|x - \text{proj}_{\Gamma(t)}x\| \leq \gamma(\tau - t)$ for all $(t, x) \in G$ and all $\tau \in I, (t < \tau)$

$(\Gamma_2)$ The function $(t, x) \to \delta^\gamma(t, x, \Gamma(t)) = \sup\{ (x, y) : y \in \Gamma(t) \}$ is lower semicontinuous on $I \times B_\gamma$, where $B_\gamma$ is the relative weak topology.

Then for all $\psi \in C_E([-r, 0])$ with $\psi(0) = \Gamma(0)$, the problem $(Q)$ has a solution.

**PROOF.** We split the proof into the following three steps:

(1) Let $f \in K_{\alpha}$. Since $\Gamma$ has a compact graph and satisfies conditions $(\Gamma_1)$ and $(\Gamma_2)$ then by Theorem 3.1 [11], there exists a unique absolutely continuous function $u_f : I \to H$ such that

(i) $u'_f(t) \in -N_{\Gamma(t)}(u(t)) + f(t)$ a.e. on $I$,

(ii) $u_f(0) = \psi(0), u_f(t) \in \Gamma(t), \forall t \in I$,

(iii) $\|u_f(t)\| \leq \beta_2 = T(\gamma + \alpha), \forall t \in I$ and the function $f \to u_f$ is continuous from $K_{\alpha}$ to $C_{H_0}$.

(2) Set $\chi_2 = \{ u \in C_H([-r, T]) : u = \psi$ on $[-r, 0]$ and $u(t) = \psi(0) + \int_0^t f(s)ds, f \in K_{\beta_2} \}$ and define a multivalued function $T_2$ on $\chi_2$ by $T_2(u) = \{ f \in K_{\alpha} : f(t) \in F(t, s, u) \text{ a.e. on } I \}$. As in the second step of the proof of Theorem 3.1 we can show that $T_2$ has a continuous selection $V_2 : \chi_2 \to K_{\alpha}$.

(3) Define the function $\theta : \chi_2 \to \chi_2$ by $\theta(x) = u_f, f = V_2(x)$. As in the third step of the proof of Theorem 3.1, we can show that there exists a unique $u \in \chi_2$ such that $u = u_f, f \in T_2(u)$. Clearly $u$ is a solution of $(Q)$.
4. EXISTENCE OF INTEGRAL SOLUTIONS FOR THE PROBLEM (P) 
WHEN THE OPERATOR A IS INDEPENDENT OF TIME

In this section A denotes a multivalued operator from $E$ to $2^E - \{\phi\}$. Consider the evolution equation

$$
(P') \begin{cases}
  u'(t) \in -A(u(t)) + f(t) & \text{a.e. on } I \\
  u(0) = x_0 \in \overline{D(A)},
\end{cases}
$$

where $f \in L^1_b(I)$. By an integral solution of $(P')$ we mean a continuous function $u : I \rightarrow \overline{D(A)}$ with $u(0) = x_0$ such that

$$
\|u(t) - z\| \leq \|u(s) - z\| + \int_s^t \|u(r) - z, f(r) - y\| \, dr,
$$

for each $z \in D(A), y \in A(z)$ and $0 \leq s \leq t < T$, where

$$
[x_1, x_2]_\| = \lim_{h \downarrow 0} \left( \frac{\|x_1 + hx_2\| - \|x_1\|}{h} \right), \forall x_1, x_2 \in E.
$$

It is known that [7] if $A$ is an $m$-accretive operator then for each $(x_0, f) \in D(A) \times L^1_b(I)$, the problem $(P')$ has a unique integral solution $u_f$, such that the function $f \rightarrow u_f$ is continuous. In this section we are concerned with the existence of integral solutions of the functional evolution equation

$$
(P^{**}) \begin{cases}
  u'(t) \in -A(u(t)) + F(t, u, u) & \text{a.e. on } I \\
  u \equiv \psi
\end{cases}
$$

on $[-r, 0]$, where

$$
F : I \times C_b([-r, 0]) \rightarrow 2^E - \{\phi\}
$$

is a multivalued function from $I \times C_b([-r, 0])$ to the non-empty closed subsets of $E$. We say that the operator $A : E \rightarrow 2^E - \{\phi\}$ has the $(M)$-property ([7], [12]) if for each $x_0 \in D(A)$ and each uniformly integrable subset $Q$ of $L^1_b(I)$, the set $\{u_s \in Q\}$ is a relatively compact subset of $C_b(I)$ where $u_s$ is the unique integral solution of the evolution equation $u'(t) = -A(u(t)) + g(t)$ a.e. on $I$; $u(0) = x_0$. It is well known that ([7], [12]) if the proper operator $-A$ generates a compact semigroup (via Crandall-Liggett’s exponential formula [3], [13]), then $A$ has the property $(M)$.

**THEOREM 4.1.** Let $E$ be a Banach space and $A$ an $m$-accretive multivalued operator from $E$ to $2^E - \{\phi\}$ having the $(M)$-property. Let $F$ be a measurable multivalued function from $I \times C_b([-r, 0])$ to the non-empty closed subsets of $E$ satisfying the condition $(F_3)$ together with the following conditions

$(F_4)$ There exists a function $h \in L^1_b(I)$ such that

$$
\sup \{\|z\| : z \in F(t, u)\} \leq h(t), \forall (t, u) \in I \times C_b([-r, 0]).
$$

$(F_5)$ For all $t \in I$, $F(t, \cdot) : C_b([-r, 0]) \rightarrow E$ is lower semicontinuous in the Kuratowski sense.

Then for all $\psi \in C_b([-r, 0])$ with $\psi(0) \in \overline{D(A)}$, the problem $(P^{**})$ has an integral solution.

**PROOF.** Consider the set $Q = \{f \in L^1_b(I) : \|f(t)\| \leq h(t) \text{ a.e. on } I\}$ One can easily show that $Q$ is nonempty and uniformly integrable subset of $L^1_b(I)$. As mentioned above, for each $f \in Q$ there exists a unique continuous function $u_f : I \rightarrow \overline{D(A)}$ such that $u_f$ is the unique integral solution of the evolution equation $u'(t) = A(u(t)) + f(t), u(0) = \psi(0)$. Then for all $\psi \in C_b([-r, 0])$ with $\psi(0) \in \overline{D(A)}$, the problem $(P^{**})$ has an integral solution.

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Also, define a function \( \Phi : \mathcal{X}^* \to \mathcal{X}^* \), \( \Phi(x) = u^*_f \), \( f = V(x) \). The function \( \Phi \) is clearly continuous and hence has a fixed point \( x \in \mathcal{X}^* \). It is obvious that \( x \) is the desired solution.

5. EXAMPLES

In this section we give some examples illustrating the scope of the results developed in sections 3 and 4.

**EXAMPLE 1.** Let for all \( t \in I \), \( A(t) = B - h(t) \) where \( h : I \to E \) is integrable and \( B \) is an \( m \)-accretive operator on \( E \). Clearly \( A(t) \) is \( m \)-accretive for all \( t \in I \). Let \( \lambda > 0 \), \( s, t \in I \) and \( x \in E \). Then
\[
\|A_x(t,x) - A_x(s,x)\| \leq \frac{1}{\lambda} \|J_\lambda A(t,x) - J_\lambda A(s,x)\| \leq \|h(t) - h(s)\|.
\]
Hence condition \((C_1)\) of Lemma 2.1 holds.

**EXAMPLE 2.** In [6] there are several examples for operators \( A \) such that for every \( t \in I \), \( A(t) \) is \( m \)-accretive and satisfies condition \((C_1)\).

**EXAMPLE 3.** Let \( H \) be a real Hilbert space with inner product \((\cdot, \cdot)\) and let \( \Phi : H \to H \) be a proper lower semicontinuous convex function. The set \( \partial \Phi(x) = \{z \in H : \Phi(x) \leq \Phi(y) + (z - y, z)\} \) for each \( y \in H \) is called the subdifferential of \( \Phi \) at the point \( x \). We recall that \( D(\partial \Phi) = \{x \in H : \partial \Phi(x) \) is nonempty\}.

Now if we define an operator \( A : D(A) = D(\partial \Phi) \to 2^H \) by \( A(x) = \partial \Phi(x) \), then \( A \) is \( m \)-accretive and the following conditions are equivalent [7]:

(i) For each \( \lambda > 0 \), the resolvent \( J_\lambda A \) is a compact operator.

(ii) The function \( \Phi \) is of compact type.

(iii) The semigroup generated by the operator \(-A\) is compact.

**EXAMPLE 4.** Take \( E = L^p_2([0, \pi]) \) and let us define \( A : D(A) \subseteq E \to E \) by \( Au = -u''(t) \) for each \( u \in D(A) \) where \( D(a) = \{u \in E : u''(t) \in E, u(0) = u(\pi) = 0\} \). The operator \( A \) is \( m \)-accretive and the semigroup \( \{S(t) : t > 0\} \) generated by \(-A\) (Simon [7]) is compact [7].

REFERENCES


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Time-Dependent Billiards

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This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

To be acceptable for publication in the special issue of Mathematical Problems in Engineering, papers must make significant, original, and correct contributions to one or more of the topics above mentioned. Mathematical papers regarding the topics above are also welcome.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

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