SUBORDINATION PROPERTIES OF $p$-VALENT FUNCTIONS DEFINED BY INTEGRAL OPERATORS

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Received 20 June 2005; Revised 14 November 2005; Accepted 28 November 2005

By applying certain integral operators to $p$-valent functions we define a comprehensive family of analytic functions. The subordinations properties of this family is studied, which in certain special cases yield some of the previously obtained results.

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1. Introduction

For the natural numbers $p$ let $A(p)$ denote the class of functions of the form $f(z) = z^p + a_{p+1}z^{p+1} + a_{p+2}z^{p+2} + \cdots$, which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$. For $f(z) \in A(p)$ we define

$$I^\sigma f(z) = \frac{(p+1)^\sigma}{z^\Gamma(\sigma)} \int_0^z \left( \log \frac{z}{t} \right)^{\sigma-1} f(t) dt = z^p + \sum_{n=p+1}^{\infty} \frac{(p+1)^{\sigma}}{n+1} a_n z^n, \quad \sigma > 0. \quad (1.1)$$

Also, for $-1 \leq B < A \leq 1$ and $\lambda \geq 0$, let $\Omega^\sigma_p(A,B,\lambda)$ be the class of functions $f \in A(p)$ so that

$$\frac{\lambda}{p} \frac{I^{\sigma-1} f(z)}{z^p} + \frac{p - \lambda}{p} \frac{I^\sigma f(z)}{z^p} < \frac{1}{1+Bz}, \quad \lambda \geq 0, \quad (1.2)$$

where “≺” denotes the usual subordination. See [2].

The family $\Omega^\sigma_p(A,B,\lambda)$ is a comprehensive family containing various well-known as well as new classes of analytic functions. For example, for $\sigma = 0$ and $\lambda = p + 1$ we obtain the class $\Omega^0_p(A,B,p+1)$ studied by Patel and Mohanty [3] or for nonzero $\sigma$ see Liu [1].

2. Main results

Our first theorem examines the containment properties of the family $\Omega^\sigma_p(A,B,\lambda)$. 
2  \( p \)-valent functions

**Theorem 2.1.** For \( f \in A(p) \) suppose that \( f \in \Omega^p_\sigma(A,B,\lambda) \) and \( 0 \leq \lambda \leq p(p+1) \). Then \( f \in \Omega^\sigma_p(A,B,0) \).

To prove our theorem we will need the following lemma which is due to Miller and Mocanu [2].

**Lemma 2.2.** Let \( g(z) \) be analytic and convex univalent in \( U \) and \( g(0) = 1 \). Also let \( p(z) \) be analytic in \( U \) with \( p(0) = 1 \). If \( p(z) + (zp'(z))/\gamma < g(z) \), where \( \gamma \neq 0 \) and Re \( \gamma \geq 0 \), then \( p(z) < \gamma z^{-\gamma} \int_0^z t^{\gamma-1} g(t) dt \).

**Proof of Theorem 2.1.** First, we note that

\[
(z(I^\sigma f(z)))' = (p+1)I^{\sigma-1}f(z) - I^\sigma f(z).
\] (2.1)

Setting \( p(z) = (I^\sigma f(z))/zp \) we also observe that

\[
\frac{(I^\sigma f(z))'}{pz^{p-1}} = p(z) + \frac{zp'(z)}{p},
\]

\[
\frac{I^{\sigma-1}f(z)}{z^p} = p(z) + \frac{zp'(z)}{p+1}.
\] (2.2)

Therefore, for \( f \in \Omega^\sigma_p(A,B,\lambda) \), we conclude that

\[
p(z) + \frac{\lambda}{p(p+1)}zp'(z) < \frac{1+Az}{1+Bz}.
\] (2.3)

Now from Lemma 2.2 for \( \gamma = p(p+1)/\lambda \) it follows that

\[
\frac{I^\sigma f(z)}{z^p} < \frac{p(p+1)}{\lambda}z^{-p(p+1)/\lambda} \int_0^z t^{p(p+1)/\lambda-1} \frac{1+At}{1+Bt} dt = q(z) < \frac{1+Az}{1+Bz}.
\] (2.4)

Thus \( f \in \Omega^\sigma_p(A,B,0) \). □

As a special case to Theorem 2.1, we obtain the following.

**Corollary 2.3.** Let \( f \in A(p) \). Then \( (1/(p+1))[(zf'(z) + f(z))/z^p] < (1+Az)/(1+Bz) \), implies \( f(z)/z^p < (1+Az)/(1+Bz) \).

**Theorem 2.4.** For \( f \in A(p) \) suppose that \( f \in \Omega^\sigma_p(A,B,\lambda) \). If \( 0 \leq \lambda \leq p(p+1) \), then

\[
\text{Re} \left( \frac{I^\sigma f(z)}{z^p} \right) \geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)/\lambda-1} \frac{1-Au}{1-Bu} du.
\] (2.5)

The result is sharp.

**Proof.** Set \( p(z) = I^\sigma f(z)/z^p \). Then, by Theorem 2.1, we have

\[
p(z) < \frac{p(p+1)}{\lambda}z^{-p(p+1)/\lambda} \int_0^z t^{p(p+1)/\lambda-1} \frac{1+At}{1+Bt} dt < \frac{1+Az}{1+Bz}.
\] (2.6)
This is equivalent to

\[
I_\sigma f(z) = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)\lambda - 1} \frac{1 + uAw(z)}{1 + uBw(z)} du,
\]

(2.7)

where \( w(z) \) is analytic in \( U \) with \( w(0) = 0 \) and \( |w(z)| < 1 \) in \( U \). Therefore

\[
\text{Re} \left( \frac{I_\sigma f(z)}{zp} \right) = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)\lambda - 1} \text{Re} \left\{ \frac{1 + uAw(z)}{1 + uBw(z)} \right\} du
\]

\[
\geq \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)\lambda - 1} \frac{1 - Au}{1 - Bu} du.
\]

(2.8)

Therefore

\[
I_\sigma f(z) = \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)\lambda - 1} \frac{1 + Au}{1 + Bu} du,
\]

(2.9)

such that for this function we have

\[
\frac{\lambda}{p} I_\sigma^{-1} f(z) = \frac{p - \lambda}{p} I_\sigma f(z) = \frac{1 + Az}{1 + Bz}.
\]

(2.10)

Letting \( z \to -1 \) yields

\[
I_\sigma f(z) \rightarrow \frac{p(p+1)}{\lambda} \int_0^1 u^{p(p+1)\lambda - 1} \frac{1 - Au}{1 - Bu} du.
\]

(2.11)

\[ \square \]

References


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