We establish the weak convergence of a sequence of Mann iterates of an $I$-nonexpansive map in a Banach space which satisfies Opial’s condition.

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1. Introduction and preliminaries

Let $K$ be a closed convex bounded subset of uniformly convex Banach space $X = (X, \| \cdot \|)$ and $T$ self-mappings of $X$. Then $T$ is called nonexpansive on $K$ if

$$\|Tx - Ty\| \leq \|x - y\| \quad (1.1)$$

for all $x, y \in K$. Let $F(T) = \{x \in K : Tx = x\}$ be denoted as the set of fixed points of a mapping $T$.

The first nonlinear ergodic theorem was proved by Baillon [1] for general nonexpansive mappings in Hilbert space $\mathcal{H}$: if $K$ is a closed and convex subset of $\mathcal{H}$ and $T$ has a fixed point, then for every $x \in K$, $\{T^nx\}$ is weakly almost convergent, as $n \to \infty$, to a fixed point of $T$. It was also shown by Pazy [7] that if $\mathcal{H}$ is a real Hilbert space and $(1/n)\sum_{i=0}^{n-1} T^ix$ converges weakly, as $n \to \infty$, to $y \in K$, then $y \in F(T)$.

The concept of a quasi-nonexpansive mapping was initiated by Tricomi in 1941 for real functions. Diaz and Metcalf [2] and Dotson [3] studied quasi-nonexpansive mappings in Banach spaces. Recently, this concept was given by Kirk [5] in metric spaces which we adapt to a normed space as follows: $T$ is called a quasi-nonexpansive mapping provided

$$\|Tx - f\| \leq \|x - f\| \quad (1.2)$$

for all $x \in K$ and $f \in F(T)$.

Remark 1.1. From the above definitions it is easy to see that if $F(T)$ is nonempty, a nonexpansive mapping must be quasi-nonexpansive, and linear quasi-nonexpansive mappings are nonexpansive. But it is easily seen that there exist nonlinear continuous quasi-nonexpansive mappings which are not nonexpansive.
2 Convergence theorems for \( I \)-nonexpansive mapping

There are many results on fixed points on nonexpansive and quasi-nonexpansive mappings in Banach spaces and metric spaces. For example, the strong and weak convergence of the sequence of certain iterates to a fixed point of quasi-nonexpansive maps was studied by Petryshyn and Williamson [8]. Their analysis was related to the convergence of Mann iterates studied by Dotson [3]. Subsequently, the convergence of Ishikawa iterates of quasi-nonexpansive mappings in Banach spaces was discussed by Ghosh and Debnath [4]. In [10], the weakly convergence theorem for \( I \)-asymptotically quasi-nonexpansive mapping defined in Hilbert space was proved. In [11], convergence theorems of iterative schemes for nonexpansive mappings have been presented and generalized.

In this paper, we consider \( T \) and \( I \) self-mappings of \( K \), where \( T \) is an \( I \)-nonexpansive mapping. We establish the weak convergence of the sequence of Mann iterates to a common fixed point of \( T \) and \( I \).

Let \( X \) be a normed linear space, let \( K \) be a nonempty convex subset of \( X \), and let \( T: K \to K \) be a given mapping. The Mann iterative scheme \( \{x_n\} \) is defined by \( x_0 = x \in K \) and

\[
x_{n+1} = (1 - k_n)x_n + k_nTx_n
\]

for every \( n \in \mathbb{N} \), where \( k_n \) is a sequence in \((0,1)\).

Recall that a Banach space \( X \) is said to satisfy Opial’s condition [6] if, for each sequence \( \{x_n\} \) in \( X \), the condition \( x_n \rightharpoonup x \) implies that

\[
\lim_{n \to \infty} \|x_n - x\| < \lim_{n \to \infty} \|x_n - y\|
\]

for all \( y \in X \) with \( y \neq x \). It is well known from [6] that all \( l_p \) spaces for \( 1 < p < \infty \) have this property. However, the \( L_p \) spaces do not, unless \( p = 2 \).

The following definitions and statements will be needed for the proof of our theorem.

Let \( K \) be a subset of a normed space \( X = (X, \| \cdot \|) \) and \( T \) and \( I \) self-mappings of \( K \). Then \( T \) is called \( I \)-nonexpansive on \( K \) if

\[
\|Tx - Ty\| \leq \|Ix - Iy\|
\]

for all \( x, y \in K \) [9].

\( T \) is called \( I \)-quasi-nonexpansive on \( K \) if

\[
\|Tx - f\| \leq \|Ix - f\|
\]

for all \( x \in K \) and \( f \in F(T) \cap F(I) \).

2. The main result

**Theorem 2.1.** Let \( K \) be a closed convex bounded subset of uniformly convex Banach space \( X \), which satisfies Opial’s condition, and let \( T, I \) self-mappings of \( K \) with \( T \) be an \( I \)-nonexpansive mapping, \( I \) a nonexpansive on \( K \). Then, for \( x_0 \in K \), the sequence \( \{x_n\} \) of Mann iterates converges weakly to common fixed point of \( F(T) \cap F(I) \).
Proof. If \( F(T) \cap F(I) \) is nonempty and a singleton, then the proof is complete. We will assume that \( F(T) \cap F(I) \) is nonempty and that \( F(T) \cap F(I) \) is not a singleton.

\[
\|x_{n+1} - f\| = \|(1 - k_n)x_n + k_nTx_n - (1 - k_n + k_n)f\|
\]
\[
= \|(1 - k_n)(x_n - f) + k_n(Tx_n - f)\|
\]
\[
\leq (1 - k_n)\|x_n - f\| + k_n\|Tx_n - f\|
\]
\[
\leq (1 - k_n)\|x_n - f\| + k_n\|I_{\text{set}} - f\|
\]
\[
= (1 - k_n)\|x_n - f\| + k_n\|x_n - f\|
\]
\[
= \|x_n - f\|
\]
(2.1)

where \( \{k_n\} \) is a sequence in \((0,1)\).

Thus, for \( k_n \neq 0 \), \( \|x_n - f\| \) is a nonincreasing sequence. Then, \( \lim_{n \to \infty} \|x_n - f\| \) exists.

Now we show that \( \{x_n\} \) converges weakly to a common fixed point of \( T \) and \( I \). The sequence \( \{x_n\} \) contains a subsequence which converges weakly to a point in \( K \). Let \( \{x_{n_k}\} \) and \( \{x_{m_j}\} \) be two subsequences of \( \{x_n\} \) which converge weakly to \( f \) and \( q \), respectively. We will show that \( f = q \). Suppose that \( X \) satisfies Opial’s condition and that \( f \neq q \) is in weak limit set of the sequence \( \{x_n\} \). Then \( \{x_{n_k}\} \to f \) and \( \{x_{m_j}\} \to q \), respectively. Since \( \lim_{n \to \infty} \|x_n - f\| \) exists for any \( f \in F(T) \cap F(I) \), by Opial’s condition, we conclude that

\[
\lim_{n \to \infty} \|x_n - f\| = \lim_{k \to \infty} \|x_{n_k} - f\| < \lim_{k \to \infty} \|x_{n_k} - q\|
\]
\[
= \lim_{n \to \infty} \|x_n - q\| = \lim_{j \to \infty} \|x_{m_j} - q\|
\]
\[
< \lim_{j \to \infty} \|x_{m_j} - f\| = \lim_{n \to \infty} \|x_n - f\|.
\]
(2.2)

This is a contradiction. Thus \( \{x_n\} \) converges weakly to an element of \( F(T) \cap F(I) \). \( \square \)

References

4 Convergence theorems for $I$-nonexpansive mapping


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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<th>Event</th>
<th>Date</th>
</tr>
</thead>
<tbody>
<tr>
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<td>December 1, 2008</td>
</tr>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
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<tr>
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<td>June 1, 2009</td>
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