ON THE MEAN VALUE PROPERTY OF SUPERHARMONIC AND SUBHARMONIC FUNCTIONS

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We prove a converse of the mean value property for superharmonic and subharmonic functions. The case of harmonic functions was treated by Epstein and Schiffer.

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Recall that a function $u$ is harmonic (superharmonic, subharmonic) in an open set $U \subset \mathbb{R}^n$ ($n \geq 1$) if $u \in C^2(U)$ and $\Delta u = 0$ ($\Delta u \leq 0$, $\Delta u \geq 0$) on $U$. Denote by $H(U)$ the space of harmonic functions in $U$ and $SH(U)$ ($sH(U)$) the subset of $C^2(U)$ consisting of superharmonic (subharmonic) functions in $U$. If $A \subset \mathbb{R}^n$ is Lebesgue measurable, $L^1(A)$ denotes the space of Lebesgue integrable functions on $A$ and $|A|$ denotes the Lebesgue measure of $A$ when $A$ is bounded.

We also recall the mean value property of harmonic, superharmonic, and subharmonic functions in $U$ ([2]): if $x \in U$ and $B(x, r) = \{ y \in \mathbb{R}^n; \| y - x \| < r \}, r > 0$, is such that $B(x, r) \subset U$, then for all $u \in H(U)$ ($SH(U), sH(U)$),

$$u(x) = (\geq, \leq) \frac{1}{|B(x, r)|} \int_{B(x, r)} u(y) dy. \quad (1)$$

Using the Lebesgue-dominated convergence theorem we see that the conclusion above holds whenever $B(x, r) \subset U$ if $u \in H(U) \cap L^1(B(x, r))$ ($SH(U) \cap L^1(B(x, r)), sH(U) \cap L^1(B(x, r))$). Epstein and Schiffer [1] proved the following converse.

**Theorem 1.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that

$$u(x_0) = \frac{1}{|\Omega|} \int_{\Omega} u(x) dx \quad (2)$$

for every $u \in H(\Omega) \cap L^1(\Omega)$. Then $\Omega$ is a ball with center $x_0$.

A more general result was obtained by Kuran [3]. In this note we give a proof of the following converse.
2 Mean value property of super (sub) harmonic functions

**Theorem 2.** Let $\Omega \subset \mathbb{R}^n$ ($n \geq 1$) be a bounded open set. Suppose that there exists $x_0 \in \Omega$ such that
\[
u(x_0) \geq (\leq) \frac{1}{|\Omega|} \int_\Omega u(x)dx
\]
for every $u \in \text{SH}(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$ ($\text{sH}(\Omega) \cap L^1(\Omega) \setminus H(\Omega)$). Then $\Omega$ is a ball with center $x_0$.

**Proof.** Clearly it is enough to consider the case of superharmonic functions. Since $\Omega$ is bounded, there exists a largest open ball $B$ centered at $x_0$ of radius $r_1$ which lies in $\Omega$. The compactness of $\partial \Omega$ implies that there is some $x_1 \in \partial \Omega$ such that $|x_1 - x_0| = r_1$. We will show that $\Omega = B$. Define
\[
h(x) = r_1^{n-2} \left(||x - x_0||^2 - r_1^2\right)||x - x_1||^{-n}
\]
for $x \in \mathbb{R}^n \setminus \{x_1\}$. Then $h \in H(\mathbb{R}^n \setminus \{x_1\})$ and $h(x_0) = -1$. Now let $R > r_1$ be such that $\Omega \subset B(x_0, R)$. For $k \in \mathbb{N}^*$ we set
\[
u_k(x) = 1 + h(x) + \frac{1}{2nk} \left(R^2 - ||x - x_0||^2\right), \quad x \in \Omega.
\]
Obviously $u_k \in C^2(\Omega)$ and $\Delta u_k = -1/k$ in $\Omega$, hence $u_k \in \text{SH}(\Omega) \setminus H(\Omega)$. Moreover $u_k \in L^1(\Omega)$ and $u_k(x) \geq 1$ for $x \in \Omega \setminus B$. Since $1 + h \in H(\Omega) \cap L^1(\Omega)$, we have
\[
0 = 1 + h(x_0) = \int_B (1 + h(x)) dx.
\]
Now using (6) we can write
\[
\frac{R^2}{2nk} = \frac{1}{|\Omega|} \int_\Omega \nu_k(x)dx = \frac{1}{|\Omega|} \int_{\Omega \setminus B} \nu_k(x)dx + \frac{1}{|\Omega|} \int_B \nu_k(x)dx
\]
\[
= \frac{1}{|\Omega|} \int_{\Omega \setminus B} \nu_k(x)dx + \frac{1}{2nk|\Omega|} \int_B \left(R^2 - ||x - x_0||^2\right)dx
\]
\[
\geq \frac{|\Omega \setminus B|}{|\Omega|} + \frac{\omega_n r_1^n}{2nk|\Omega|} \left(\frac{R^2}{n} - \frac{r_1^2}{n + 2}\right)
\]
\[
\geq \frac{|\Omega \setminus B|}{|\Omega|} + \frac{\omega_n r_1^n}{2nk|\Omega|} \left(\frac{R^2}{n} - \frac{r_1^2}{n + 2}\right)
\]
for all $k \in \mathbb{N}^*$, where $\omega_n$ denotes the measure of the unit sphere in $\mathbb{R}^n$. This implies that $|\Omega \setminus \overline{B}| = 0$. Then the open set $\Omega \setminus \overline{B}$ must be empty, hence $\Omega \subset \overline{B}$. Since $\Omega$ is open and $B \subset \Omega \subset \overline{B}$, we deduce that $\Omega = B$. $\square$

**References**


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