We introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given left module. We also introduce the notion of SSRS-modules. It is shown that (1) if $M$ is an amply supplemented module and $0 \to N' \to N \to N'' \to 0$ an exact sequence, then $M$ is $N$-lifting if and only if it is $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

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1. Introduction and preliminaries

Extending modules and their generalizations have been studied by many authors (see [2, 3, 8, 7]). The motivation of the present discussion is from [2, 8], where the concepts of extending modules and (quasi-)continuous modules with respect to a given module and CESS-modules were studied, respectively. In this paper, we introduce the concepts of lifting modules and (quasi-)discrete modules relative to a given module and SSRS-modules. It is shown that (1) if $0 \to N' \to N \to N'' \to 0$ is an exact sequence and $M$ an amply supplemented module, then $M$ is $N$-lifting if and only if it is both $N'$-lifting and $N''$-lifting; (2) if $M$ is a Noetherian module, then $M$ is lifting if and only if $M$ is $R$-lifting if and only if $M$ is an amply supplemented SSRS-module; and (3) let $M$ be an amply supplemented SSRS-module such that $\text{Rad}(M)$ is finitely generated, then $M = K \oplus K'$, where $K$ is a radical module and $K'$ is a lifting module.

Throughout this paper, $R$ is an associative ring with identity and all modules are unital left $R$-modules. We use $N \leq M$ to indicate that $N$ is a submodule of $M$. As usual, $\text{Rad}(M)$ and $\text{Soc}(M)$ stand for the Jacobson radical and the socle of a module $M$, respectively.

Let $M$ be a module and $S \leq M$. $S$ is called small in $M$ (notation $S \ll M$) if $M \neq S + T$ for any proper submodule $T$ of $M$. Let $N$ and $L$ be submodules of $M$, $N$ is called a supplement of $L$ in $M$ if $N + L = M$, and $N$ is minimal with respect to this property. Equivalently,
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\( M = N + L \) and \( N \cap L \ll N \). \( N \) is called a supplement submodule if \( N \) is a supplement of some submodule of \( M \). \( M \) is called an amply supplemented module if for any two submodules \( A \) and \( B \) of \( M \) with \( A + B = M \), \( B \) contains a supplement of \( A \). \( M \) is called a weakly supplemented module (see [5]) if for each submodule \( A \) of \( M \) there exists a submodule \( B \) of \( M \) such that \( M = A + B \) and \( A \cap B \ll M \). Let \( B \leq A \leq M \). If \( A/B \ll M/B \), then \( B \) is called a coessential submodule of \( A \) and \( A \) is called a coessential extension of \( B \) in \( M \). A submodule \( A \) of \( M \) is called coclosed if \( A \) has no proper coessential submodules in \( M \). Following [5], \( B \) is called an s-closure of \( A \) in \( M \) if \( B \) is a coessential submodule of \( A \) and \( B \) is coclosed in \( M \).

Let \( M \) be a module. \( M \) is called a lifting module (or satisfies \((D_1)\)) (see [9]) if for every submodule \( A \) of \( M \), there exists a direct summand \( K \) of \( M \) such that \( K \leq A \) and \( A/K \ll M/K \), equivalently, \( M \) is amply supplemented and every supplement submodule of \( M \) is a direct summand. \( M \) is called discrete if \( M \) is lifting and has the following condition:

\((D_2)\) If \( A \leq M \) such that \( M/A \) is isomorphic to a direct summand of \( M \), then \( A \) is a summand of \( M \).

\( M \) is called quasidiscrete if \( M \) is lifting and has the following condition:

\((D_3)\) For each pair of direct summands \( A \) and \( B \) of \( M \) with \( A + B = M \), \( A \cap B \) is a direct summand of \( M \). For more details on these concepts, see [9].

Lemma 1.1 (see [12, 19.3]). Let \( M \) be a module and \( K \leq L \leq M \).

(1) \( L \ll M \) if and only if \( K \ll M \) and \( L/K \ll M/K \).

(2) If \( M' \) is a module and \( \phi : M \to M' \) a homomorphism, then \( \phi(L) \ll M' \) whenever \( L \ll M \).

Lemma 1.2 (see Lemma 1.1 in [5]). Let \( M \) be a weakly supplemented module and \( N \leq M \). Then the following statements are equivalent.

(1) \( N \) is a supplement submodule of \( M \).

(2) \( N \) is coclosed in \( M \).

(3) For all \( X \leq N \), \( X \ll M \) implies \( X \ll N \).

Lemma 1.3 (see Proposition 1.5 in [5]). Let \( M \) be an amply supplemented module. Then every submodule of \( M \) has an s-closure.

Lemma 1.4 (see [12, 41.7]). Let \( M \) be an amply supplemented module. Then every coclosed submodule of \( M \) is amply supplemented.

2. Relative lifting modules

To define the concepts of relative lifting and (quasi-)discrete modules, we dualize the concepts of relative extending and (quasi-)continuous modules introduced in [8] in this section. We start with the following.

Let \( N \) and \( M \) be modules. We define the family

\[
\$ (N, M) = \left\{ A \leq M \mid \exists X \leq N, \exists f \in \text{Hom}(X, M), \exists \frac{A}{f(X)} \ll \frac{M}{f(X)} \right\}.
\] (2.1)
Proposition 2.1. $(N, M)$ is closed under small submodules, isomorphic images, and coessential extensions.

Proof. We only show that $(N, M)$ is closed under coessential extensions. Let $A \in (N, M)$, $A \leq A' \leq M$, and $A'/A \ll M/A$. There exist $X \leq N$ and $f \in \text{Hom}(X, M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ since $A \in (N, M)$. Note that $A'/A \ll M/A$, so $A'/f(X) \ll M/f(X)$ by Lemma 1.1(1). Thus $A' \in (N, M)$. □

Lemma 2.2. Let $A \in (N, M)$ and $A$ be coclosed in $M$. Then $B \in (N, M)$ for any submodule $B$ of $A$.

Proof. There exist $X \leq N$ and $f \in \text{Hom}(X, M)$ such that $f(X) \leq A$ and $A/f(X) \ll M/f(X)$ by hypothesis. Since $A$ is coclosed in $M$, $f(X) = A$. Let $B$ be any submodule of $A$ and $Y = f^{-1}(B) \leq X \leq N$. Then $f|_Y : Y \to M$ is a homomorphism such that $f|_Y(Y) = B$ for $f(X) = A$. Clearly $B/f|_Y(Y) \ll M/f|_Y(Y)$. Therefore $B \in (N, M)$. □

Lemma 2.3. Let $C \leq A \leq B \leq M$ and $A$ be a coessential submodule of $B$. If $C$ is an $s$-closure of $A$, then it is also an $s$-closure of $B$.

Proof. It is clear by Lemma 1.1(1). □

Proposition 2.4. Let $M$ be an amply supplemented module. Then every $A$ in $(N, M)$ has an $s$-closure $\overline{A}$ in $(N, M)$.

Proof. Since $A \in (N, M)$, there exist $X \leq N$ and $f \in \text{Hom}(X, M)$ such that $A/f(X) \ll M/f(X)$. Note that $M$ is amply supplemented, and so $f(X)$ has an $s$-closure $\overline{A}$ in $M$ by Lemma 1.3. Thus $\overline{A}$ is also an $s$-closure of $A$ by Lemma 2.3. The verification for $\overline{A} \in (N, M)$ is analogous to that for $B \in (N, M)$ in Lemma 2.2. □

Let $N$ be a module. Consider the following conditions for a module $M$.

$(N, M)$-$D_1$ For every submodule $A \in (N, M)$, there exists a direct summand $K$ of $M$ such that $K \leq A$ and $A/K \ll M/K$.

$(N, M)$-$D_2$ If $A \in (N, M)$ such that $M/A$ is isomorphic to a direct summand of $M$, then $A$ is a direct summand of $M$.

$(N, M)$-$D_3$ If $A$ and $L$ are direct summands of $M$ with $A \in (N, M)$ and $A + L = M$, then $A \cap L$ is a direct summand of $M$.

Definition 2.5. Let $N$ be a module. A module $M$ is said to be $N$-lifting, $N$-discrete, or $N$-quasidiscrete if $M$ satisfies $(N, M)$-$D_1$, $(N, M)$-$D_1$ and $(N, M)$-$D_2$ or $(N, M)$-$D_1$ and $(N, M)$-$D_3$, respectively.

One easily obtains the hierarchy: $M$ is $N$-discrete $\Rightarrow$ $M$ is $N$-quasidiscrete $\Rightarrow$ $M$ is $N$-lifting. Clearly, the notion of relative discreteness generalizes the concept of discreteness. For any module $N$, lifting modules are $N$-lifting. But the converse is not true as shown in the following examples.

Example 2.6. Since, for any module $M$, $(0, M) = \{A \mid A \ll M\}$ and $0$ is a direct summand of $M$ such that $A/0 \ll M/0$ for any $A \in (0, M)$, all modules are $0$-lifting. However, the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ is not lifting since the supplement submodule $\langle(1,2)\rangle$
Example 2.7. Let $M$ be a module with zero socle and $S$ a simple module. Then $M$ is $S$-lifting since $\mathcal{S}(S,M)$ is a family only containing all small submodules of $M$. So all torsion-free $\mathbb{Z}$-modules are $S$-lifting for any simple $\mathbb{Z}$-module $S$ (see [12, Exercise 21.17]). In particular, $\mathbb{Z}\mathbb{Z}$ and $\mathbb{Z}\mathbb{Q}$ are $S$-lifting for any simple $\mathbb{Z}$-module, but each one is not a lifting module.

Lemma 2.8. Let $M$ be a module. Then $\mathcal{S}(M,M) = \{A \mid A \subseteq M\} = \bigcup_{N \in \text{R-Mod}} \mathcal{S}(N,M)$, where $\text{R-Mod}$ denotes the category of left $\text{R}$-module.

Proof. It is straightforward.

Proposition 2.9. Let $M$ be a module. Then $M$ is lifting or (quasi-)discrete if and only if $M$ is $M$-lifting or $M$-(quasi-)discrete if and only if $M$ is $N$-lifting or $N$-(quasi-)discrete for any module $N$.

Proof. It is clear by Lemma 2.8.

Proposition 2.10. Let $M$ be an amply supplemented module. Then the condition $\mathcal{S}(N,M)-D_1$ is inherited by coclosed submodules of $M$.

Proof. Let $M$ satisfy $\mathcal{S}(N,M)-D_1$ and $H$ be a coclosed submodule of $M$. $H$ is amply supplemented by Lemma 1.4. For any $A \in \mathcal{S}(N,H)$, $A$ has an $s$-closure $\overline{A} \in \mathcal{S}(N,H)$ in $H$ by Proposition 2.4. Since $\overline{A} \in \mathcal{S}(N,H) \subseteq \mathcal{S}(N,M)$ and $M$ satisfies $\mathcal{S}(N,M)-D_1$, there is a direct summand $K$ of $M$ such that $K \leq \overline{A}$ and $\overline{A}/K \ll M/K$. By Lemma 1.2, $\overline{A}/K \ll H/K$. Now $\overline{A} = K$ since $\overline{A}$ is coclosed in $H$. Thus $H$ satisfies $\mathcal{S}(N,H)-D_1$.

Corollary 2.11. Let $M$ be an amply supplemented module. Then the condition $\mathcal{S}(N,M)-D_1$ is inherited by direct summands of $M$.

Proposition 2.12. Let $M$ be an amply supplemented module. Then $\mathcal{S}(N,M)-D_1$ ($i = 2,3$) is inherited by direct summands of $M$.

Proof. (1) Let $M$ satisfy $\mathcal{S}(N,M)-D_2$ and $H$ be a direct summand of $M$. We will show that $H$ satisfies $\mathcal{S}(N,H)-D_2$.

Let $A \in \mathcal{S}(N,H) \subseteq \mathcal{S}(N,M)$ and $H/A$ is isomorphic to a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M/A = (H \oplus H')/A \cong (H/A) \oplus H'$, and so $M/A$ is isomorphic to a direct summand of $M$. $A$ is a direct summand of $M$ since $M$ satisfies $\mathcal{S}(N,M)-D_2$, and hence $A$ is a direct summand of $H$.

(2) Let $A \in \mathcal{S}(N,H) \subseteq \mathcal{S}(N,M)$ and $A, L$ be direct summands of $H$ with $A + L = H$. We will show that $A \cap L$ is a direct summand of $H$. Since $H$ is a direct summand of $M$, there exists $H' \leq M$ such that $M = H \oplus H'$. Thus $M = (A + L) \oplus H' = A + (L \oplus H')$. Now $A \cap (L \oplus H')$ is a direct summand of $M$ since $M$ satisfies $\mathcal{S}(N,M)-D_3$. Note that $A \cap (L \oplus H') = A \cap L$, so $A \cap L$ is a direct summand of $H$.

Theorem 2.13. Let $M$ be an amply supplemented module and $A \in \mathcal{S}(N,M)$ a direct summand of $M$. If $M$ is $N$-(quasi-)discrete, then $A$ is (quasi-)discrete.
Proof. The proof follows from Lemma 2.2, Corollary 2.11, and Proposition 2.12.

Proposition 2.14. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence. Then $(N', M) \cup (N'', M) \subseteq (N, M)$. Therefore, if $M$ is $N$-lifting (resp., (quasi-)discrete), then $M$ is $N'$-lifting and $N''$-lifting (resp., (quasi-)discrete).

Proof. Without loss of generality we can assume that $N' \subseteq N$ and $N'' = N/N'$. By definition, $N' \subseteq N$ implies $(N', M) \subseteq (N, M)$. Next, let $A_2 \in (N'', M)$. Then there exist $X \subseteq N'' = N/N'$ and $f \in \text{Hom}(X, M)$ such that $A_2/f(X) \ll M/f(X)$. Write $X = Y/N'$, $Y \subseteq N$ and let $\delta : Y \to Y/N'$ be the canonical homomorphism. It is clear that $g = f \delta \in \text{Hom}(Y, M)$ and $g(Y) = f(X)$, hence $A_2/g(Y) \ll M/g(Y)$. Thus $A_2 \in (N, M)$. Therefore $(N', M) \cup (N'', M) \subseteq (N, M)$. The rest is obvious.

Dual to [8, Proposition 2.7], we have the following.

Theorem 2.15. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence and $M$ an amply supplemented module. Then $M$ is $N$-lifting if and only if it is both $N'$-lifting and $N''$-lifting.

Proof. Let $M$ be $N$-lifting. Then it is both $N'$-lifting and $N''$-lifting by Proposition 2.14. Conversely suppose that $M$ is both $N'$-lifting and $N''$-lifting. For any submodule $A \subseteq (N, M)$, $A$ has a $s$-closure $\overline{A} \subseteq (N, M)$ by Proposition 2.4. Since $\overline{A} \in (N, M)$, there exist $X \subseteq N$ and $f \in \text{Hom}(X, M)$ such that $\overline{A}/f(X) \ll M/f(X)$. Since $\overline{A}$ is coclosed in $M$, $f(X) = \overline{A}$. Write $Y = X \cap N' \subseteq N'$ and $f|_Y : Y \to M$ is a homomorphism, then $f(Y) \subseteq f(X) = \overline{A}$. Let $f(Y)$ be an $s$-closure of $f(Y)$ in $\overline{A}$ (for $\overline{A}$ is amply supplemented). Thus we conclude that $f(Y)/f(Y) \ll M/f(Y)$ and $f(Y) \subseteq (N', M)$. Since $M$ is $N'$-lifting, there exists a direct summand $K$ of $M$ such that $f(Y)/K \ll M/K$. It is easy to see $f(Y)$ is coclosed in $M$, hence $f(Y) = K$ is a direct summand of $M$. Write $M = f(Y) \oplus K'$, $K' \subseteq M$ and $\overline{A} = \overline{A} \cap M = f(Y) \oplus (\overline{A} \cap K')$. Define $h : W = (X + N')/N' \to M$ by $h(x + N') = \pi f(x)$, where $\pi : \overline{A} \to \overline{A} \cap K'$ denotes the canonical projection. It is clear that $h(W) = \overline{A} \cap K'$, thus $(\overline{A} \cap K')/h(W) \ll M/h(W)$, and hence $(\overline{A} \cap K') \in (N'', M)$. Since $M$ is $N''$-lifting, there exists a direct summand $K''$ of $M$ such that $(\overline{A} \cap K')/K'' \ll M/K''$. Since $\overline{A} \cap K'$ is coclosed in $M$, $\overline{A} \cap K' = K''$. Now $\overline{A} \cap K'$ is a direct summand of $K'$. Thus $\overline{A}$ is a direct summand of $M$. It follows that $M$ is $N$-lifting.

Corollary 2.16. Let $M$ be an amply supplemented module. If $M$ is $N_i$-lifting for $i = 1, 2, \ldots, n$ and $N = \bigoplus^n_i N_i$, then $M$ is $N$-lifting.

Corollary 2.17. Let $M$ be an amply supplemented module. Then $M$ is lifting if and only if $M$ is $N$-lifting and $M/N$-lifting for every submodule $N$ of $M$ if and only if $M$ is $N$-lifting and $M/N$-lifting for some submodule $N$ of $M$.

Recall that a module $M$ is said to be distributive if $N \cap (K + L) = (N \cap K) + (N \cap L)$ for all submodules $N, K, L$ of $M$. A module $M$ has SSP (see [4]) if the sum of any pair of direct summands of $M$ is a direct summand of $M$.

Corollary 2.18. Let $0 \to N' \to N \to N'' \to 0$ be an exact sequence and let $M$ be a distributive and amply supplemented module with SSP. If $M$ is both $N'$-quasidiscrete and $N''$-quasidiscrete, then $M$ is $N$-quasidiscrete.
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Proof. We only need to show that \( M \) satisfies \((N,M)\)-\(D_3\) when \( M \) satisfies \((N',M)\)-\(D_3\) and \((N'',M)\)-\(D_3\) by Theorem 2.15. Let \( A \in (N,M) \) and \( A,H \) be direct summands of \( M \) with \( A + H = M \). We know that \( A = A_1 \oplus A_2 \), where \( A_1 \in (N',M) \), \( A_2 \in (N'',M) \) from the proof of Theorem 2.15. Since \( M \) is a distributive module with SSP, \( A_1 \cap H \) and \( A_2 \cap H \) are direct summands of \( M \). This implies that \( A \cap H \) is a direct summand of \( M \). Thus \( M \) satisfies \((N,M)\)-\(D_3\). \( \square \)

3. SSRS-modules

In [2], a module is called a CESS-module if every complement with essential socle is a direct summand. As a dual of CESS-modules, the concept of SSRS-modules is given in this section. It is proven that: (1) let \( M \) be an amply supplemented SSRS-module such that \( \mathrm{Rad}(M) \) is finitely generated, then \( M = K \oplus K' \), where \( K \) is a radical module and \( K' \) is a lifting module; (2) let \( M \) be a finitely generated amply supplemented module, then \( M \) is an SSRS-module if and only if \( M/K \) is a lifting module for every coclosed submodule \( K \) of \( M \).

Definition 3.1. A module is called an SSRS-module if every supplement with small radical is a direct summand.

Lifting modules are SSRS-modules, but the converse is not true. For example, \( _2\mathbb{Z} \) is an SSRS-module which is not a lifting module.

Proposition 3.2. Let \( M \) be an SSRS-module. Then any direct summand of \( M \) is an SSRS-module.

Proof. Let \( K \) be a direct summand of \( M \) and \( N \) a supplement submodule of \( K \) such that \( \mathrm{Rad}(N) \ll N \). Let \( N \) be a supplement of \( L \) in \( K \), that is, \( N + L = K \) and \( N \cap L \ll N \). Since \( K \) is a direct summand of \( M \), there exists \( K' \leq M \) such that \( M = K \oplus K' \). Note that \( M = N + (L \oplus K') \) and \( N \cap (L \oplus K') = N \cap L \ll N \). Therefore \( N \) is a supplement of \( L \oplus K' \) in \( M \). Thus \( N \) is a direct summand of \( M \) since \( M \) is an SSRS-module. So \( N \) is a direct summand of \( K \). The proof is complete. \( \square \)

Proposition 3.3. Let \( M \) be a weakly supplemented SSRS-module and \( K \) a coclosed submodule of \( M \). Then \( K \) is an SSRS-module.

Proof. It follows from the assumption and [4, Lemma 2.6(3)]. \( \square \)

Proposition 3.4. Let \( M \) be an amply supplemented module. Then \( M \) is an SSRS-module if and only if for every submodule \( N \) with small radical, there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N/K \ll M/K \).

Proof. “\( \Rightarrow \)” Let \( N \) be a supplement submodule with small radical. By assumption, there exists a direct summand \( K \) of \( M \) such that \( K \leq N \) and \( N/K \ll M/K \). Since \( N \) is coclosed in \( M \), \( N = K \). Thus \( N \) is a direct summand of \( M \).

“\( \Leftarrow \)” Let \( N \leq M \) with \( \mathrm{Rad}(N) \ll N \). There exists an \( s \)-closure \( \overline{N} \) of \( N \) since \( M \) is amply supplemented. Since \( \mathrm{Rad}(N) \ll M \) (for \( \mathrm{Rad}(N) \ll N \) and \( \mathrm{Rad}(\overline{N}) \leq \mathrm{Rad}(N) \),
Rad(\(N\)) \ll \overline{N} and \(\overline{N}\) is a supplement submodule by Lemma 1.2. Therefore \(\overline{N}\) is a direct summand of \(M\) by assumption. This completes the proof. \(\square\)

**Corollary 3.5.** Let \(M\) be an amply supplemented SSRS-module. Then every simple submodule of \(M\) is either a direct summand or a small submodule of \(M\).

**Proposition 3.6.** Let \(M\) be an amply supplemented module. Then \(M\) is an SSRS-module if and only if for every submodule \(N\) of \(M\), every \(s\)-closure of \(N\) with small radical is a lifting module and a direct summand of \(M\).

**Proof.** It is straightforward. \(\square\)

**Proposition 3.7.** Let \(M\) be an amply supplemented SSRS-module. Then \(M = K \oplus K'\), where \(K\) is semisimple and \(K'\) has small socle.

**Proof.** For \(\text{Soc}(M)\), there exists a direct summand \(K\) of \(M\) such that \(\text{Soc}(M)/K \ll M/K\) by Proposition 3.4. It is easy to see that \(K\) is semisimple. Since \(K\) is a direct summand of \(M\), there exists \(K' \leq M\) such that \(M = K \oplus K'\). Note that \(\text{Soc}(M) = \text{Soc}(K) \oplus \text{Soc}(K')\). So \(\text{Soc}(M)/K = (K \oplus \text{Soc}(K'))/K \ll M/K = (K \oplus K')/K\). Thus \(\text{Soc}(K') \ll K'\). \(\square\)

Recall that a module \(M\) is called a radical module if \(\text{Rad}(M) = M\). Dual to [2, Theorem 2.6], we have the following.

**Theorem 3.8.** Let \(M\) be an amply supplemented SSRS-module such that \(\text{Rad}(M)\) is finitely generated. Then \(M = K \oplus K'\), where \(K\) is a radical module and \(K'\) is a lifting module.

**Proof.** \(\text{Rad}(\text{Rad}(M)) \ll \text{Rad}(M)\) since \(\text{Rad}(M)\) is finitely generated. There exists a direct summand \(K\) of \(M\) such that \(\text{Rad}(M)/K \ll M/K\) by Proposition 3.4. Since \(K\) is a direct summand of \(M\), there exists \(K' \leq M\) such that \(M = K \oplus K'\). Note that \(\text{Rad}(M) = \text{Rad}(K) \oplus \text{Rad}(K')\). Therefore \(K = K \cap \text{Rad}(M) = \text{Rad}(K)\) and \(\text{Rad}(M)/K = (\text{Rad}(K) \oplus \text{Rad}(K'))/K \ll M/K = (K \oplus K')/K\). Thus \(\text{Rad}(K') \ll K'\).

Next, we show that \(K'\) is a lifting module. \(K'\) is amply supplemented since it is a direct summand of \(M\). So we only prove that every supplement submodule of \(K'\) is a direct summand of \(K'\). Let \(N\) be a supplement submodule of \(K'\). By Lemma 1.2 and \(\text{Rad}(K') \ll K'\), we know that \(\text{Rad}(N) \ll N\). \(N\) is a direct summand of \(K'\) since \(K'\) is an SSRS-module by Proposition 3.2. The proof is complete. \(\square\)

**Corollary 3.9.** Let \(M\) be an amply supplemented module with small radical. Then \(M\) is an SSRS-module if and only if \(M\) is a lifting module.

**Theorem 3.10.** Let \(M\) be a finitely generated amply supplemented module. Then the following statements are equivalent.

1. \(M\) is an SSRS-module.
2. \(M\) is a lifting module.
3. \(M/K\) is a lifting module for every coclosed submodule \(K\) of \(M\).

**Proof.** (1) \(\Leftrightarrow\) (2) follows from Corollary 3.9.

(3) \(\Rightarrow\) (1) is clear.

(1) \(\Rightarrow\) (3) we only prove that any supplement submodule of \(M/K\) is a direct summand. Let \(A/K\) be a supplement submodule of \(M/K\). \(A\) is coclosed in \(M\) since \(A/K\) is coclosed in
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$M/K$ and $K$ is coclosed in $M$. $\text{Rad}(A) \ll A$ since $M$ is finitely generated and $A$ is coclosed in $M$. $A$ is a direct summand of $M$ by assumption. Thus $A/K$ is a direct summand of $M/K$.

**Lemma 3.11.** Let $M$ be a module. Then the following statements are equivalent.

1. For every cyclic submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$.
2. For every finitely generated submodule $N$ of $M$, there exists a direct summand $K$ of $M$ such that $K \leq N$ and $N/K \ll M/K$.

**Proof.** See [12, 41.13].

**Corollary 3.12.** Let $M$ be a Noetherian module. Then the following statements are equivalent.

1. $M$ is $R$-lifting.
2. $M$ is $F$-lifting, for any free module $F$.
3. $M$ is lifting.
4. $M$ is an amply supplemented SSRS-module.

**Proof.** It is easy to see that $\mathcal{G}(R,M)$ and $\mathcal{G}(F,M)$ are closed under cyclic submodules. The rest follows immediately from Theorem 3.10 and Lemma 3.11.

**Corollary 3.13.** Let $R$ be a left perfect (semiperfect) ring. Then every SSRS-module (finitely generated SSRS-module) is a lifting module.

**Proof.** It follows from the fact that every module over a left perfect ring has small radical, [11, Theorems 1.6 and 1.7] and Corollary 3.9.

A module $M$ is uniserial (see [6]) if its submodules are linearly ordered by inclusion and it is serial if it is a direct sum of uniserial submodules. A ring $R$ is right (left) serial if the right (left) $R$-module $R_R R$ is serial and it is serial if it is both right and left serial.

**Corollary 3.14.** The following statements are equivalent for a ring $R$ with radical $J$.

1. $R$ is an artinian serial ring and $J^2 = 0$.
2. $R$ is a left semiperfect ring and every finitely generated module is an SSRS-module.
3. $R$ is a left perfect ring and every module is an SSRS-module.

**Proof.** It holds by [6, Theorem 3.15], [10, Theorem 1 and Proposition 2.13], and Corollary 3.13.

**References**


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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