Let $C$ be a closed convex subset of a uniformly smooth Banach space $E$, and $T : C \to E$ a nonexpansive nonself-mapping satisfying the weakly inwardness condition such that $F(T) \neq \emptyset$, and $f : C \to C$ a fixed contractive mapping. For $t \in (0, 1)$, the implicit iterative sequence $\{x_t\}$ is defined by $x_t = P(tf(x_t) + (1 - t)Tx_t)$, the explicit iterative sequence $\{x_n\}$ is given by $x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)$, where $\alpha_n \in (0, 1)$ and $P$ is a sunny nonexpansive retraction of $E$ onto $C$. We prove that $\{x_t\}$ strongly converges to a fixed point of $T$ as $t \to 0$, and $\{x_n\}$ strongly converges to a fixed point of $T$ as $\alpha_n$ satisfying appropriate conditions. The results presented extend and improve the corresponding results of Hong-Kun Xu (2004) and Yisheng Song and Rudong Chen (2006).

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1. Introduction

Let $C$ be a nonempty closed convex subset of a Banach space $E$, and let $T : C \to C$ be a nonexpansive mapping (i.e., $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$). We use Fix($T$) to denote the set of fixed points of $T$; that is, Fix($T$) = $\{x \in C : x = Tx\}$. Recall that a self-mapping $f : C \to C$ is a contraction on $C$ if there exists a constant $\beta \in (0, 1)$ such that

\[ \|f(x) - f(y)\| \leq \beta \|x - y\|, \quad x, y \in C. \]  

(1.1)

Xu (see [6]) defined the following two viscosity iterations for nonexpansive mappings:

\[ x_t = tf(x_t) + (1 - t)Tx_t, \quad x \in C, \]  

(1.2)

\[ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n)Tx_n, \]  

(1.3)

where $\alpha_n$ is a sequence in $(0, 1)$. Xu proved the strong convergence of $\{x_t\}$ defined by (1.2) as $t \to 0$ and $\{x_n\}$ defined by (1.3) in both Hilbert space and uniformly smooth Banach space.
Recently, Song and Chen [2] proved if \( C \) is a closed subset of a real reflexive Banach space \( E \) which admits a weakly sequentially continuous duality mapping from \( E \) to \( E^* \), and if \( T : C \rightarrow E \) is a nonexpansive nonself-mapping satisfying the weakly inward condition, \( F(T) \neq \emptyset \), \( f : C \rightarrow C \) is a fixed contractive mapping, and \( P \) is a sunny nonexpansive retraction of \( E \) onto \( C \), then the sequences \( \{x_n\} \) and \( \{x_n\} \) defined by

\[
x_t = P(tf(x_t) + (1 - t)Tx_t),
\]

\[
x_{n+1} = P(\alpha_n f(x_n) + (1 - \alpha_n)Tx_n)
\]

strongly converge to a fixed point of \( T \).

In this paper, we establish the strong convergence of both \( \{x_t\} \) defined by (1.4) and \( \{x_n\} \) defined by (1.5) for a nonexpansive nonself-mapping \( T \) in a uniformly smooth Banach space. Our results extend and improve the results in [2, 6].

2. Preliminaries

Let \( E \) be a real Banach space and let \( J \) denote the normalized duality mapping from \( E \) into \( 2^{E^*} \) given by

\[
J(x) = \{ f \in E^* : \langle x, f \rangle = \|x\| \|f\|, \|x\| = \|f\| \} \quad \forall x \in E,
\]

where \( E^* \) denotes the dual space of \( E \) and \( \langle \cdot, \cdot \rangle \) denotes the generalized duality pairing. In the sequence, we will denote the single-valued duality mapping by \( j \), and \( x_n \rightharpoonup x \) will denote strong convergence of the sequence \( \{x_n\} \) to \( x \). In Banach space \( E \), the following result is well known [1, 3] for all \( x, y \in E \), for all \( j(x + y) \in J(x + y) \), for all \( j(x) \in J(x) \),

\[
\|x\|^2 + 2\langle y, j(x) \rangle \leq \|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y) \rangle.
\]

Recall that the norm of \( E \) is said to be Gâteaux differentiable (and \( E \) is said to be smooth) if

\[
\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t}
\]

exists for each \( x, y \) in its unit sphere \( U = \{x \in E : \|x\| = 1\} \). It is said to be uniformly Gâteaux differentiable if, for each \( y \in U \), this limit is attained uniformly for \( x \in U \). Finally, the norm is said to be uniformly Fréchet differentiable (and \( E \) is said to be uniformly smooth) if the limit in (2.3) is attained uniformly for \( (x, y) \in U \times U \). A Banach space \( E \) is said to be smooth if and only if \( J \) is single valued. It is also well known that if \( E \) is uniformly smooth, \( J \) is uniformly norm-to-norm continuous. These concepts may be found in [3].

If \( C \) and \( D \) are nonempty subsets of a Banach space \( E \) such that \( C \) is nonempty closed convex and \( D \subset C \), then a mapping \( P : C \rightarrow D \) is called a retraction from \( C \) to \( D \) if \( P^2 = P \). It is easily known that a mapping \( P : C \rightarrow D \) is retraction, then \( Px = x \), for all \( x \in D \). A mapping \( P : C \rightarrow D \) is called sunny if

\[
P(Px + t(x - Px)) = Px \quad \forall x \in C,
\]
whenever \( Px + t(x - Px) \in C \) and \( t > 0 \). A subset \( D \) of \( C \) is said to be a sunny nonexpansive retract of \( C \) if there exists a sunny nonexpansive retraction of \( C \) onto \( D \). For more detail, see [1, 3–5].

The following lemma is well known [3].

**Lemma 2.1.** Let \( C \) be a nonempty convex subset of a smooth Banach space \( E \), \( D \in C \), \( J : E \rightarrow E^* \) the (normalized) duality mapping of \( E \), and \( P : C \rightarrow D \) a retraction. Then the following are equivalent:

(i) \( \langle x - Px, j(y - Px) \rangle \leq 0 \) for all \( x \in C \) and \( y \in D \);

(ii) \( P \) is both sunny and nonexpansive.

Let \( C \) be a nonempty convex subset of a Banach space \( E \), then for \( x \in C \), we define the inward set [4, 5]:

\[
I_C(x) = \{ y \in E : y = x + \lambda(z - x), z \in C \text{ and } \lambda \geq 0 \}. \tag{2.5}
\]

A mapping \( T : C \rightarrow E \) is said to be satisfying the inward condition if \( Tx \in I_C(x) \) for all \( x \in C \). \( T \) is also said to be satisfying the weakly inward condition if for each \( x \in C \), \( Tx \in IC(x) \) (\( IC(x) \) is the closure of \( I_C(x) \)). Clearly \( C \subset I_C(x) \) and it is not hard to show that \( I_C(x) \) is a convex set as \( C \) is. Using above these results and definitions, we can easily show the following lemma.

**Lemma 2.2 (‖2‖, Lemma 1.2).** Let \( C \) be a nonempty closed subset of a smooth Banach space \( E \), let \( T : C \rightarrow E \) be nonexpansive nonself-mapping satisfying the weakly inward condition, and let \( P \) be a sunny nonexpansive retraction of \( E \) onto \( C \). Then \( F(T) = F(PT) \).

**Lemma 2.3 (‖2‖, Lemma 2.1).** Let \( E \) be a Banach space and let \( C \) be a nonempty closed convex subset of \( E \). Suppose that \( T : C \rightarrow E \) is a nonexpansive mapping such that for each fixed contractive mapping \( f : C \rightarrow C \), and \( P \) is a sunny nonexpansive retraction of \( E \) onto \( C \). For each \( t \in (0, 1) \), \( \{x_t\} \) is defined by (1.4). Suppose \( u \in C \) is a fixed point of \( T \), then

(i) \( \langle x_t - f(x_t), j(x_t - u) \rangle \leq 0 \);

(ii) \( \{x_t\} \) is bounded.

**Definition 2.4.** \( \mu \) is called a Banach limit if \( \mu \) is a continuous linear functional on \( l^\infty \) satisfying

(i) \( \|\mu(e)\| = 1 = \mu(1), e = (1, 1, 1, ...) \);

(ii) \( \mu_n(a_n) = \mu_n(a_{n+1}) \), for all \( a_n \in (a_0, a_1, ...) \in l^\infty \);

(iii) \( \liminf_{n \to \infty} a_n \leq \mu(a_n) \leq \limsup_{n \to \infty} a_n \), for all \( a_n \in (a_0, a_1, ...) \in l^\infty \).

According to time and circumstances, we use \( \mu_n(a_n) \) instead of \( \mu(a_0, a_1, ...) \).

Further, we know the following result.

**Lemma 2.5 ([3], Lemma 4.5.4).** Let \( C \) be a nonempty closed convex subset of a Banach space \( E \) with a uniformly Gâteaux differentiable norm and let \( \{x_n\} \) be a bounded sequence in \( E \). Let \( \mu \) be a Banach limit and \( u \in C \). Then

\[
\mu_n\|x_n - u\|^2 = \min_{y \in C}\mu_n\|x_n - y\|^2 \tag{2.6}
\]
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if and only if

\[ \mu_n \langle x - u, J(x_n - u) \rangle \leq 0 \quad (2.7) \]

for all \( x \in C \).

3. Main results

Theorem 3.1. Let \( E \) be a uniformly smooth Banach, suppose that \( C \) is a nonempty closed convex subset of \( E \) and \( T : C \to E \) is a nonexpansive nonself-mapping satisfying the weakly inward condition and \( F(T) \neq \emptyset \). Let \( f : C \to C \) be a fixed contractive mapping, and let \( \{x_t\} \) be defined by (1.4), where \( P \) is a sunny nonexpansive retraction of \( E \) onto \( C \). Then as \( t \to 0 \) \( \{x_t\} \) converges strongly to some fixed point \( q \) of \( T \) that \( q \) is the unique solution in \( F(T) \) to the following variational inequality:

\[ \langle (I - f)q, j(q - u) \rangle \leq 0 \quad \forall u \in F(T). \quad (3.1) \]

Proof. For all \( u \in F(T) \) by Lemma 2.3(ii), \( \{x_t\} \) is bounded, therefore the sets \( \{Tx_t : t \in (0,1)\} \) and \( \{f(x_t) : t \in (0,1)\} \) are also bounded. From \( x_t = P(tf(x_t) + (1 - t)Tx_t) \), we have

\[
\|x_t - PTx_t\| = \|P(tf(x_t) + (1 - t)Tx_t) - PTx_t\|
\leq \|tf(x_t) + (1 - t)Tx_t - Tx_t\|
= t\|Tx_t - f(x_t)\| \to 0 \quad \text{as} \quad t \to 0. \quad (3.2)
\]

This implies that

\[ \lim_{t \to 0} \|x_t - PTx_t\| = 0. \quad (3.3) \]

Assume \( t_n \to 0 \), set \( x_n := x_{t_n} \), and define \( g : C \to \mathbb{R} \) by \( g(x) = \mu_n\|x_n - x\|^2 \), \( x \in C \), where \( \mu_n \) is a Banach limit on \( \ell^\infty \). Let

\[ K = \left\{ x \in C : g(x) = \min_{y \in C} \mu_n\|x_n - y\|^2 \right\}. \quad (3.4) \]

It is easily seen that \( K \) is a nonempty closed convex bounded subset of \( E \), since (note \( \|x_n - Tx_n\| \to 0 \))

\[ g(Tx) = \mu_n\|x_n - Tx\|^2 = \mu_n\|Tx_n - Tx\|^2 \leq \mu_n\|x_n - x\|^2 = g(x). \quad (3.5) \]

It follows that \( T(K) \subset K \), that is, \( K \) is invariant under \( T \). Since a uniformly smooth Banach space has the fixed point property for nonexpansive mappings, \( T \) has a fixed point, say \( q \), in \( K \). From Lemma 2.5 we get

\[ \mu_n \langle x - q, j(x_n - q) \rangle \leq 0, \quad x \in C. \quad (3.6) \]
For all $q \in F(T)$, we have $tf(x_t) + (1 - t)q = P[tf(x_t) + (1 - t)q]$, then

$$
||x_t - [tf(x_t) + (1 - t)q]|| = ||P[tf(x_t) + (1 - t)Tx_t] - P[tf(x_t) + (1 - t)q]|| \leq ||(1 - t)(Tx_t - q)|| \leq (1 - t)||x_t - q||.
$$

Hence from (2.2) and the above inequality we get

$$
||x_t - [tf(x_t) + (1 - t)q]||^2 = ||(1 - t)(x_t - q) + t(x_t - f(x_t))||^2 \geq (1 - t)^2||x_t - q||^2 + 2t(1 - t)\langle x_t - f(x_t), j(x_t - q) \rangle.
$$

Therefore

$$
\langle x_t - f(x_t), j(x_t - q) \rangle \leq 0.
$$

Then

$$
0 \geq \langle x_t - f(x_t), j(x_t - q) \rangle = ||x_t - q||^2 + \langle q - f(q), j(x_t - q) \rangle + \langle f(q) - f(x_t), j(x_t - q) \rangle \geq (1 - \beta)||x_t - q||^2 + \langle q - f(q), j(x_t - q) \rangle.
$$

We get

$$
||x_t - q||^2 \leq \frac{1}{1 - \beta}\langle f(q) - q, j(x_t - q) \rangle.
$$

Now applying Banach limit to the above inequality, we get

$$
\mu_n||x_t - q||^2 \leq \mu_n\left(\frac{1}{1 - \beta}\langle f(q) - q, j(x_t - q) \rangle\right).
$$

Let $x = f(q)$ in (3.6), and noting (3.12), we have

$$
\mu_n||x_t - q||^2 \leq 0,
$$

that is,

$$
\mu_n||x_n - q||^2 = 0
$$

and then exists a subsequence which is still denoted by $\{x_n\}$ such that

$$
x_n \rightarrow q, \quad n \rightarrow \infty.
$$

We have proved that for any sequence $\{x_{tn}\}$ in $\{x_t : t \in (0, 1)\}$, there exists a subsequence which is still denoted by $\{x_{tn}^\prime\}$ that converges to some point $q$ of $T$. To prove that
the entire net \( \{x_t\} \) converges to \( q \), suppose that there exists another sequence \( \{x_{sk}\} \subset \{x_t\} \) such that \( x_{sk} \to p \), as \( s_k \to 0 \), then we also have \( p \in F(T) \) (using \( \lim_{t \to 0} \|x_t - PTx_t\| = 0 \)).

Next we show \( p = q \) and \( q \) is the unique solution in \( F(T) \) to the following variational inequality:

\[
\langle (I - f)q, j(q - u) \rangle \quad \forall u \in F(T).
\]  

(3.16)

Since the sets \( \{x_t - u\} \) and \( \{x_t - f(x_t)\} \) are bounded and the uniform smoothness of \( E \) implies that the duality map \( J \) is norm-to-norm uniformly continuous on bounded sets of \( E \), for any \( u \in F(T) \), by \( x_{sk} \to p \) \( (s_k \to 0) \), we have

\[
\| (I - f)x_{sk} - (I - f)p \| \to 0, \quad s_k \to 0,
\]

\[
\left| \langle x_{sk} - f(x_{sk}), j(x_{sk} - u) \rangle - \langle (I - f)p, j(p - u) \rangle \right|
\]

\[
= \left| \langle x_{sk} - f(x_{sk}) - (I - f)p, j(x_{sk} - u) \rangle - \langle (I - f)p, j(x_{sk} - u) - j(p - u) \rangle \right|
\]

\[
\leq \| (I - f)x_{sk} - (I - f)p \| \| x_{sk} - u \|
\]

\[
+ \left| \langle (I - f)p, j(x_{sk} - u) - j(p - u) \rangle \right| \to 0 \quad \text{as} \quad s_k \to 0.
\]

(3.17)

Therefore, noting Lemma 2.3(i), for any \( u \in F(T) \), we get

\[
\langle (I - f)p, j(p - u) \rangle = \lim_{s_k \to 0} \langle x_{sk} - f(x_{sk}), j(x_{sk} - u) \rangle \leq 0.
\]  

(3.18)

Similarly, we also can show

\[
\langle (I - f)q, j(q - u) \rangle = \langle x_{tn} - f(x_{tn}), j(x_{tn} - u) \rangle \leq 0.
\]  

(3.19)

Interchange \( q \) and \( u \) to obtain

\[
\langle (I - f)p, j(p - q) \rangle \leq 0.
\]  

(3.20)

Interchange \( p \) and \( u \) to obtain

\[
\langle (I - f)q, j(q - p) \rangle \leq 0.
\]  

(3.21)

This implies that

\[
\langle (p - q) - (f(p) - f(q)), j(p - q) \rangle \leq 0,
\]  

(3.22)

that is,

\[
\| p - q \|^2 \leq \beta \| p - q \|^2.
\]  

(3.23)

This is a contradiction, so we must have \( q = p \).

The proof is complete. \( \square \)
From Theorem 3.1 we can get the following corollary directly.

**Corollary 3.2.** Let $E$ be a uniformly smooth space, suppose $C$ is a nonempty closed convex subset of $E$, $T : C \rightarrow E$ is a nonexpansive mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a fixed contractive mapping, and $\{x_n\}$ is defined by

$$x_t = tf(x_t) + (1 - t)PTx_t,$$

where $P$ is a sunny nonexpansive retraction of $E$ onto $C$, then $x_t$ converges strongly to some fixed point $q$ of $T$ as $t \rightarrow 0$ and $q$ is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)q, j(q - u) \rangle \quad \forall u \in F(T).$$

**Lemma 3.3 ([6], Lemma 2.1).** Let $\{\alpha_n\}$ be a sequence of nonnegative real numbers satisfying the property

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n \quad \forall n \geq 0,$$

where $\{\gamma_n\} \in (0, 1)$ and $\delta_n$ is a sequence in $\mathbb{R}$ such that:

(i) $\lim_{n \rightarrow \infty} \gamma_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n = \infty$;

(ii) either $\sum_{n=0}^{\infty} \delta_n < +\infty$ or $\limsup_{n \rightarrow \infty} (\delta_n/\gamma_n) \leq 0$.

Then $\lim_{n \rightarrow \infty} \alpha_n = 0$.

**Theorem 3.4.** Let $E$ be a uniformly smooth Banach space, suppose that $C$ is a nonempty closed convex subset of $E$, $T : C \rightarrow E$ is a nonexpansive nonself-mapping satisfying the weakly inward condition, and $F(T) \neq \emptyset$. Let $f : C \rightarrow C$ be a fixed contractive mapping, and $\{x_n\}$ is defined by (1.5), where $P$ is a sunny nonexpansive retraction of $E$ onto $C$, and $\alpha_n \in (0, 1)$ satisfies the following conditions:

(i) $\alpha_n \rightarrow 0$, as $n \rightarrow \infty$;

(ii) $\sum_{n=0}^{\infty} \alpha_n = \infty$;

(iii) either $\sum_{n=0}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty$ or $\lim_{n \rightarrow \infty} (\alpha_{n+1}/\alpha_n) = 1$.

Then $x_n$ converges strongly to a fixed point $q$ of $T$ such that $q$ is the unique solution in $F(T)$ to the following variational inequality:

$$\langle (I - f)q, j(q - u) \rangle \leq 0 \quad \forall u \in F(T).$$

**Proof.** First we show $\{x_n\}$ is bounded. Take $u \in F(T)$, it follows that

$$||x_{n+1} - u|| \leq ||P((1 - \alpha_n)Tx_n + \alpha_n f(x_n)) - Pu||$$

$$\leq ||(1 - \alpha_n)Tx_n + \alpha_n f(x_n) - u||$$

$$\leq (1 - \alpha_n)||Tx_n - u|| + \alpha_n(||f(x_n) - f(u)|| + ||f(u) - u||)$$

$$\leq (1 - \alpha_n)||x_n - u|| + \alpha_n(\beta||x_n - u|| + ||f(u) - u||)$$

$$= (1 - (1 - \beta)\alpha_n)||x_n - u|| + \alpha_n||f(u) - u||$$

$$\leq \max\left\{||x_n - u||, \frac{1}{1 - \beta}||f(u) - u||\right\}.$$
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By induction,

$$||x_n - u|| \leq \max \left\{ ||x_0 - u||, \frac{1}{1 - \beta} ||f(u) - u|| \right\}, \quad n \geq 0, \quad (3.29)$$

and \(\{x_n\}\) is bounded, so are \(\{Tx_n\}\) and \(\{f(x_n)\}\). We claim that

$$x_{n+1} - x_n \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \quad (3.30)$$

Indeed we have (for some appropriate constant \(M > 0\))

$$||x_{n+1} - x_n|| = ||P(\alpha_n f(x_n) + (1 - \alpha_n) Tx_n) - P(\alpha_{n-1} f(x_{n-1}) + (1 - \alpha_{n-1}) Tx_{n-1})||$$

$$\leq ||\alpha_n f(x_n) + (1 - \alpha_n) Tx_n - \alpha_{n-1} f(x_{n-1}) - (1 - \alpha_{n-1}) Tx_{n-1}||$$

$$\leq ||(1 - \alpha_n) (Tx_n - Tx_{n-1}) + (\alpha_n - \alpha_{n-1}) (f(x_{n-1}) - Tx_{n-1})||$$

$$+ \alpha_n ||f(x_n) - f(x_{n-1})||$$

$$\leq (1 - \alpha_n)||x_n - x_{n-1}||[3 pt] + M|\alpha_n - \alpha_{n-1}| + \beta \alpha_n||x_n - x_{n-1}||$$

$$= (1 - (1 - \beta)\alpha_n)||x_n - x_{n-1}||[3 pt] + M|\alpha_n - \alpha_{n-1}|. \quad (3.31)$$

By Lemma 3.3 we have \(||x_{n+1} - x_n|| \rightarrow 0, \text{ as } n \rightarrow \infty.\) We now show that

$$||x_n - PTx_n|| \longrightarrow 0. \quad (3.32)$$

In fact,

$$||x_{n+1} - PTx_n|| = ||P(\alpha_n f(x_n) + (1 - \alpha_n) Tx_n) - PTx_n||$$

$$\leq \alpha_n ||f(x_n) - Tx_n||. \quad (3.33)$$

This follows from (3.30) that

$$||x_n - PTx_n|| \leq ||x_n - x_{n+1}|| + ||x_{n+1} - PTx_n||$$

$$\leq ||x_n - x_{n+1}|| + \alpha_n ||f(x_n) - Tx_n|| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \quad (3.34)$$

Let \(q = \lim_{t \rightarrow 0} x_t\), where \(\{x_t\}\) is defined in Corollary 3.2, we get that \(q\) is the unique solution in \(F(T)\) to the following variational inequality:

$$\langle (I - f)q, j(q - u) \rangle \leq 0 \quad \forall u \in F(T). \quad (3.35)$$

We next show that

$$\limsup_{n \rightarrow \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0. \quad (3.36)$$
Form Corollary 3.2, let \( x_t = tf(x_t) + (1 - t)PTx_t \), indeed we can write
\[
x_t - x_n = t(f(x_t) - x_n) + (1 - t)(PTx_t - x_n). \tag{3.37}
\]

Noting (3.32), putting
\[
a_n(t) = \|x_n - PTx_n\|\left(\|x_n - PTx_n\| + 2\|x_n - x_t\|\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty, \tag{3.38}
\]
and using (2.2), we obtain
\[
\|x_t - x_n\|^2 \\
\leq (1 - t)^2\|PTx_t - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n) \rangle \\
\leq (1 - t)^2\|PTx_t - PTx_n + PTx_n - x_n\|^2 + 2t\langle f(x_t) - x_n, j(x_t - x_n) \rangle \\
+ 2t\|x_t - x_n\|^2 \leq (1 - t)^2\|x_t - x_n\|^2 + (1 - t)^2\|x_n - PTx_n\|^2 \\
+ 2(1 - t)^2\|PTx_n - x_n\|\|x_t - x_n\| + 2t\langle f(x_t) - x_n, j(x_t - x_n) \rangle + 2t\|x_t - x_n\|^2 \\
\leq (1 + t^2)\|x_t - x_n\|^2 + a_n(t) + 2t\langle f(x_t) - x_n, j(x_t - x_n) \rangle. \tag{3.39}
\]

The last inequality implies
\[
\langle f(x_t) - x_n, j(x_n - x_t) \rangle \leq \frac{t}{2}\|x_t - x_n\|^2 + \frac{1}{2t}a_n(t). \tag{3.40}
\]

From \( a_n(t) \rightarrow 0 \) as \( n \rightarrow \infty \) we get
\[
\limsup_{n \to \infty} \langle f(x_t) - x_n, j(x_n - x_t) \rangle \leq M \cdot \frac{t}{2}, \tag{3.41}
\]
where \( M > 0 \) is a constant such that \( M \geq \|x_t - x_n\|^2 \) for all \( n \geq 0 \) and \( t \in (0, 1) \). By letting \( t \to 0 \) in (3.41) we have
\[
\lim_{t \to 0} \limsup_{n \to \infty} \langle f(x_t) - x_n, j(x_n - x_t) \rangle \leq 0. \tag{3.42}
\]

On the one hand, for all \( \varepsilon > 0, \exists \delta_1 \) such that \( t \in (0, \delta_1) \),
\[
\limsup_{n \to \infty} \langle f(x_t) - x_n, j(x_n - x_t) \rangle \leq \frac{\varepsilon}{2}, \tag{**}
\]
On the other hand, \( \{x_t\} \) strongly converges to \( q \), as \( t \to 0 \), the set \( \{x_t - x_n\} \) is bounded, and the duality map \( J \) is norm-to-norm uniformly continuous on bounded sets of uniformly smooth space \( E \); from \( x_t \to q \ (t \to 0) \), we get

\[
\| f(q) - q - (f(x_t) - x_t) \| \to 0, \quad t \to 0,
\]

\[
\| \langle f(q) - q, j(x_n - q) \rangle - \langle f(x_t) - x_t, j(x_n - x_t) \rangle \| \leq \| f(q) - q \| | j(x_n - q) - j(x_n - x_t) | + \| f(q) - q - (f(x_t) - x_t) \| | x_n - x_t | \to 0, \quad t \to 0.
\]

Hence for the above \( \varepsilon > 0 \), \( \exists \delta_2 \), such that for all \( t \in (0, \delta_2) \), for all \( n \), we have

\[
\| \langle f(q) - q, j(x_n - q) \rangle - \langle f(x_t) - x_t, j(x_n - x_t) \rangle \| \leq \frac{\varepsilon}{2}.
\]

Therefore, we have

\[
\langle f(q) - q, j(x_n - q) \rangle \leq \langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2}.
\]

Noting (***) and taking \( \delta = \min \{\delta_1, \delta_2\} \), for all \( t \in (0, \delta) \), we have

\[
\limsup_{n \to \infty} \langle f(q) - q, j(x_n - q) \rangle \leq \limsup_{n \to \infty} \left( \langle f(x_t) - x_t, j(x_n - x_t) \rangle + \frac{\varepsilon}{2} \right) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

Since \( \varepsilon \) is arbitrary, we get

\[
\limsup_{n \to \infty} \langle f(q) - q, j(x_n - q) \rangle \leq 0.
\]

Finally we show \( x_n \to q \). Indeed

\[
x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n) q) = (x_{n+1} - q) - \alpha_n (f(x_n) - q).
\]
By (2.2) we have
\[
||x_{n+1} - q||^2 = ||x_{n+1} - (\alpha_n f(x_n) + (1 - \alpha_n) q) + \alpha_n (f(x_n) - q)||^2 \\
\leq ||x_{n+1} - P(\alpha_n f(x_n) + (1 - \alpha_n) q)||^2 + 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q) \rangle \\
\leq ||P[\alpha_n f(x_n) + (1 - \alpha_n) Tx_n] - P(\alpha_n f(x_n) + (1 - \alpha_n) q)||^2 \\
+ 2\alpha_n \langle f(x_n) - q, j(x_{n+1} - q) \rangle \\
\leq (1 - \alpha_n)^2 ||Tx_n - q||^2 + 2\alpha_n \langle f(x_n) - f(q), j(x_{n+1} - q) \rangle \\
+ 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle \\
\leq (1 - \alpha_n)^2 ||x_n - q||^2 + 2\alpha_n ||f(q) - f(x_n)|| ||x_{n+1} - q|| \\
+ 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle \\
\leq (1 - \alpha_n)^2 ||x_n - q||^2 + \alpha_n (||f(q) - f(x_n)||^2 + ||x_{n+1} - q||^2) \\
+ 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle. \\
\tag{3.49}
\]

Therefore, we have
\[
(1 - \alpha_n) ||x_{n+1} - q||^2 \\
\leq (1 - \alpha_n)^2 ||x_n - q||^2 + \alpha_n \beta^2 ||x_n - q||^2 + 2\alpha_n \langle f(q) - q, j(x_{n+1} - q) \rangle. \\
\tag{3.50}
\]

That is,
\[
||x_{n+1} - q||^2 \leq (1 - \alpha_n) ||x_n - q||^2 + \frac{\alpha_n^2}{1 - \alpha_n} ||x_n - q||^2 \\
+ \frac{2\alpha_n}{1 - \alpha_n} \langle f(q) - q, j(x_{n+1} - q) \rangle \\
\leq (1 - y_n) ||x_n - q||^2 + \lambda y_n \alpha_n + \frac{2}{1 - \beta^2} y_n \langle f(q) - q, j(x_{n+1} - q) \rangle, \\
\tag{3.51}
\]

where \( y_n = ((1 - \beta^2)/(1 - \alpha_n)) \alpha_n \) and \( \lambda \) is a constant such that \( \lambda > (1/(1 - \beta^2)) ||x_n - q||^2 \).

Hence,
\[
||x_{n+1} - q||^2 \leq (1 - y_n) ||x_n - q||^2 + y_n \left( \lambda \alpha_n + \frac{2}{1 - \beta^2} \langle f(q) - q, j(x_{n+1} - q) \rangle \right). \tag{3.52}
\]

It is easily seen that \( y_n \to 0, \sum_{n=1}^{\infty} y_n = \infty \), and (noting (3.36))
\[
\limsup_{n \to \infty} \left( \lambda \alpha_n + \frac{2}{1 - \beta^2} \langle f(q) - q, j(x_{n+1} - q) \rangle \right) \leq 0. \tag{3.53}
\]

Applying Lemma 3.3 onto (3.52), we have \( x_n \to q \).

The proof is complete.
12 Strong convergence of approximation fixed points

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References


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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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