

ON HORIZONTAL AND COMPLETE LIFTS FROM A MANIFOLD WITH $f_\lambda(7, 1)$ -STRUCTURE TO ITS COTANGENT BUNDLE

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The horizontal and complete lifts from a manifold M^n to its cotangent bundles $\overset{*}{T}(M^n)$ were studied by Yano and Ishihara, Yano and Patterson, Nivas and Gupta, Dambrowski, and many others. The purpose of this paper is to use certain methods by which $f_\lambda(7, 1)$ -structure in M^n can be extended to $\overset{*}{T}(M^n)$. In particular, we have studied horizontal and complete lifts of $f_\lambda(7, 1)$ -structure from a manifold to its cotangent bundle.

1. Introduction

Let M be a differentiable manifold of class c^∞ and of dimension n and let C_{TM} denote the cotangent bundle of M . Then C_{TM} is also a differentiable manifold of class c^∞ and dimension $2n$.

The following are notations and conventions that will be used in this paper.

(1) $\mathfrak{F}_s^r(M)$ denotes the set of tensor fields of class c^∞ and of type (r, s) on M . Similarly, $\mathfrak{F}_s^r(C_{TM})$ denotes the set of such tensor fields in C_{TM} .

(2) The map Π is the projection map of C_{TM} onto M .

(3) Vector fields in M are denoted by X, Y, Z, \dots and Lie differentiation by L_X . The Lie product of vector fields X and Y is denoted by $[X, Y]$.

(4) Suffixes $a, b, c, \dots, h, i, j, \dots$ take the values 1 to n and $\bar{i} = i + n$. Suffixes A, B, C, \dots take the values 1 to $2n$.

If A is a point in M , then $\Pi^{-1}(A)$ is fiber over A . Any point $p \in \Pi^{-1}(A)$ is denoted by the ordered pair (A, p_A) , where p is 1-form in M and p_A is the value of p at A . Let U be a coordinate neighborhood in M such that $A \in U$. Then U induces a coordinate neighborhood $\Pi^{-1}(U)$ in C_{TM} and $p \in \Pi^{-1}(U)$.

2. Complete lift of $f_\lambda(7, 1)$ - structure

Let $f(\neq 0)$ be a tensor field of type $(1, 1)$ and class c^∞ on M such that

$$f^7 + \lambda^2 f = 0, \tag{2.1}$$

where λ is any complex number not equal to zero. We call the manifold M satisfying (2.1) as $f_\lambda(7,1)$ -structure manifold. Let f_i^h be components of f at A in the coordinate neighborhood U of M . Then the complete lift f^c of f is also a tensor field of type $(1,1)$ in C_{TM} whose components \tilde{f}_B^A in $\Pi^{-1}(U)$ are given by [2]

$$\tilde{f}_i^h = f_i^h; \quad \tilde{f}_i^i = 0, \tag{2.2}$$

$$\tilde{f}_i^{\bar{h}} = P_a \left(\frac{\partial f_h^a}{\partial x^i} \frac{\partial f_i^a}{\partial x^h} \right); \quad \tilde{f}_i^{\bar{i}} = f_h^i, \tag{2.3}$$

where (x^1, x^2, \dots, x^n) are coordinates of A relative to U and p_A has a component (p_1, p_2, \dots, p_n) .

Thus we can write

$$f^C = (\tilde{f}_B^A) = \begin{pmatrix} f_i^h & 0 \\ p_a(\partial_i f_h^a - \partial_h f_i^a) & f_h^i \end{pmatrix}, \tag{2.4}$$

where $\partial_i = \partial/\partial x^i$.

If we put

$$\partial_i f_h^a - \partial_h f_i^a = 2\partial[i f_h^a], \tag{2.5}$$

then we can write (2.4) in the form

$$f^C = (f_B^A) = \begin{pmatrix} f_i^h & 0 \\ 2p_a \partial[i f_h^a] & f_h^i \end{pmatrix}. \tag{2.6}$$

Thus we have

$$(f^C)^2 = \begin{pmatrix} f_i^h & 0 \\ 2p_a \partial[i f_h^a] & f_h^i \end{pmatrix} \begin{pmatrix} f_j^i & 0 \\ 2p_t \partial[j f_t^i] & f_i^j \end{pmatrix}, \tag{2.7}$$

or

$$(f^C)^2 = \begin{pmatrix} f_i^h f_j^i & 0 \\ 2p_a f_j^i \partial[i f_h^a] + 2p_t f_h^i \partial[j f_t^i] & f_i^j f_h^i \end{pmatrix}. \tag{2.8}$$

If we put

$$2p_a f_j^i \partial[i f_h^a] + 2p_t f_h^i \partial[j f_t^i] = L_{hj}, \tag{2.9}$$

then (2.8) takes the form

$$(f^C)^2 = \begin{pmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{pmatrix}. \tag{2.10}$$

Thus we have

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i & 0 \\ L_{hj} & f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_k^j f_l^k & 0 \\ L_{jl} & f_k^l f_j^k \end{pmatrix}, \tag{2.11}$$

or

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ f_k^j f_l^k L_{hj} + f_i^j f_h^i L_{jl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.12}$$

Putting again

$$f_k^j f_l^k L_{hj} + f_i^j f_h^i L_{jl} = P_{hl}, \tag{2.13}$$

then we can put (2.12) in the form

$$(f^C)^4 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ P_{hl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.14}$$

Thus,

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k & 0 \\ P_{hl} & f_k^l f_j^k f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_m^l f_n^m & 0 \\ L_{ln} & f_m^n f_l^m \end{pmatrix}, \tag{2.15}$$

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ P_{hl} f_m^l f_n^m + L_{ln} f_k^l f_j^k f_i^j f_h^i & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.16}$$

Putting again

$$P_{hl} f_m^l f_n^m + L_{ln} f_k^l f_j^k f_i^j f_h^i = Q_{hn}, \tag{2.17}$$

then (2.16) takes the form

$$(f^C)^6 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ Q_{hn} & f_m^n f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.18}$$

Thus,

$$(f^C)^7 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m & 0 \\ Q_{hn} & f_m f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix} \begin{pmatrix} f_p^n & 0 \\ 2p_r \partial[p f_n^r] & f_n^p \end{pmatrix}, \tag{2.19}$$

$$(f^C)^7 = \begin{pmatrix} f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m f_p^n & 0 \\ Q_{hn} f_p^n + 2p_r \partial[p f_n^r] & f_m f_l^m f_k^l f_j^k f_i^j f_h^i f_n^p f_m f_l^m f_k^l f_j^k f_i^j f_h^i \end{pmatrix}. \tag{2.20}$$

In view of (2.1), we have

$$f_i^h f_j^i f_k^j f_l^k f_m^l f_n^m f_p^n = -\lambda^2 f_p^h, \tag{2.21}$$

and also putting

$$Q_{hn} f_p^n + 2p_r \partial[p f_n^r] f_m f_l^m f_k^l f_j^k f_i^j f_h^i = -\lambda^2 p_s \partial[p f_h^s], \tag{2.22}$$

then (2.20) can be given by

$$(f^C)^7 = \begin{pmatrix} -\lambda^2 f_p^n & 0 \\ -\lambda^2 p_s \partial[p f_h^s] & -\lambda^2 f_h^p \end{pmatrix}. \tag{2.23}$$

In view of (2.6) and (2.23), it follows that

$$(f^C)^7 + \lambda^2 (f^C) = 0. \tag{2.24}$$

Hence the complete lift f^C of f admits an $f_\lambda(7,1)$ -structure in the cotangent bundle C_{TM} .

Thus we have the following theorem.

THEOREM 2.1. *In order that the complete lift of f^C of a $(1,1)$ tensor field f admitting $f_\lambda(7,1)$ -structure in M may have the similar structure in the cotangent bundle C_{TM} , it is necessary and sufficient that*

$$Q_{hn} f_p^n + 2p_r \partial[p f_n^r] f_m f_l^m f_k^l f_j^k f_i^j f_h^i = -\lambda^2 p_s \partial[p f_h^s]. \tag{2.25}$$

3. Horizontal lift of $f_\lambda(7,1)$ -structure

Let f, g be two tensor fields of type $(1,1)$ on the manifold M . If f^H denotes the horizontal lift of f , we have

$$f^H g^H + g^H f^H = (fg + gf)^H. \tag{3.1}$$

Taking f and g identical, we get

$$(f^H)^2 = (f^2)^H. \tag{3.2}$$

Multiplying both sides by f^H and making use of the same (3.2), we get

$$(f^H)^3 = (f^3)^H \tag{3.3}$$

and so on. Thus it follows that

$$(f^H)^4 = (f^4)^H, \quad (f^H)^5 = (f^5)^H, \tag{3.4}$$

and so on. Thus,

$$(f^H)^7 = (f^7)^H. \tag{3.5}$$

Since f gives on M the $f_\lambda(7, 1)$ -structure, we have

$$f^7 + \lambda^2 f = 0. \tag{3.6}$$

Taking horizontal lift, we obtain

$$(f^7)^H + \lambda^2 (f^H) = 0. \tag{3.7}$$

In view of (3.5) and (3.7), we can write

$$(f^H)^7 + \lambda^2 (f^H) = 0. \tag{3.8}$$

Thus the horizontal lift f^H of f also admits a $f_\lambda(7, 1)$ -structure. Hence we have the following theorem.

THEOREM 3.1. *Let f be a tensor field of type $(1, 1)$ admitting $f_\lambda(7, 1)$ -structure in M . Then the horizontal lift f^H of f also admits the similar structure in the cotangent bundle C_{TM} .*

4. Nijenhuis tensor of complete lift of f^7

The Nijenhuis tensor of a $(1, 1)$ tensor field f on M is given by

$$N_{f,f}(X, Y) = [fX, fY] - f[fX, Y] - f[X, fY] + f^2[X, Y]. \tag{4.1}$$

Also for the complete lift of f^7 , we have

$$\begin{aligned} N(f^7)^C, (f^7)^C(X^C, Y^C) &= [(f^7)^C X^C, (f^7)^C Y^C] - (f^7)^C [(f^7)^C X^C, Y^C] \\ &\quad - (f^7)^C [X^C, (f^7)^C Y^C] + (f^7)^C (f^7)^C [X^C, Y^C]. \end{aligned} \tag{4.2}$$

In view of (2.1), the above (4.2) takes the form

$$\begin{aligned} N(f^7)^C, (f^7)^C(X^C, Y^C) &= [(-\lambda^2 f)^C X^C, (-\lambda^2 f)^C Y^C] - (-\lambda^2 f)^C [(-\lambda^2 f)^C X^C, Y^C] \\ &\quad - (-\lambda^2 f)^C [X^C, (-\lambda^2 f)^C Y^C] + (-\lambda^2 f)^C (-\lambda^2 f)^C [X^C, Y^C], \end{aligned} \tag{4.3}$$

or

$$N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^4 \left\{ \begin{aligned} & [(f)^C X^C, (f)^C Y^C] - (f)^C [(f)^C X^C, Y^C] \\ & - (f)^C [X^C, (f)^C Y^C] + (f)^C (f)^C [X^C, Y^C] \end{aligned} \right\}. \tag{4.4}$$

We also know that [3]

$$(f)^C X^C = (fX)^C + \nu(\mathcal{L}_X f), \tag{4.5}$$

where νf has components

$$\nu f = \begin{pmatrix} O^a \\ P_a f_i \end{pmatrix}. \tag{4.6}$$

In view of (4.5), (4.4) takes the form

$$\begin{aligned} & N(f^7)^C, (f^7)^C X^C, Y^C \\ & = \lambda^4 \left\{ \begin{aligned} & [(fX)^C, (fY)^C] + [\nu(\mathcal{L}_X f), (fY)^C] + [(fX)^C, \nu(\mathcal{L}_Y f)] \\ & + [\nu(\mathcal{L}_X f), \nu(\mathcal{L}_Y f)] - (f)^C [(fX)^C, Y^C] - (f)^C [\nu(\mathcal{L}_X f), Y^C] \\ & - (f)^C [X^C, (fY)^C] - (f)^C [X^C, \nu(\mathcal{L}_Y f)^C] + (f)^C (f)^C [X^C, Y^C] \end{aligned} \right\}. \end{aligned} \tag{4.7}$$

We now suppose that

$$\mathcal{L}_X f = \mathcal{L}_Y f = 0. \tag{4.8}$$

Then from (4.7), we have

$$N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^4 \left\{ \begin{aligned} & [(fX)^C, (fY)^C] - (f)^C [(fX)^C, Y^C] \\ & - (f)^C [X^C, (fY)^C] + (f)^C (f)^C [X^C, Y^C] \end{aligned} \right\}. \tag{4.9}$$

Further, if f acts as an identity operator on M [2], that is,

$$fX = X \quad \forall X \in \mathfrak{V}_0^1(M), \tag{4.10}$$

then we have from (4.9)

$$N(f^7)^C, (f^7)^C(X^C, Y^C) = \lambda^8 \{ [X^C, Y^C] - [X^C, Y^C] - [X^C, Y^C] + [X^C, Y^C] \} = 0. \tag{4.11}$$

Hence we have the following theorem.

THEOREM 4.1. *The Nijenhuis tensor of the complete lift of f^7 vanishes if the Lie derivatives of the tensor field f with respect to X and Y are both zero and f acts as an identity operator on M .*

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