MULTIVALENT FUNCTIONS AND \( Q_K \) SPACES

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We give a criterion for \( q \)-valent analytic functions in the unit disk to belong to \( Q_K \), a Möbius-invariant space of functions analytic in the unit disk in the plane for a nondecreasing function \( K : [0, \infty) \to [0, \infty) \), and we show by an example that our condition is sharp. As corollaries, classical results on univalent functions, the Bloch space, BMOA, and \( Q_p \) spaces are obtained.

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1. Introduction. For analytic univalent function \( f \) in the unit disk \( \Delta \), Pommerenke [8] proved that \( f \in H_{5102} \) if and only if \( f \in \text{BMOA} \), which easily implies a result of Baernstein II [4] about univalent Bloch functions: if \( g(z) \neq 0 \) is an analytic univalent function in \( \Delta \), then \( \log g \in \text{BMOA} \). We know that Pommerenke’s result mentioned above was generalized to \( Q_p \) spaces for all \( p, 0 < p < \infty \), by Aulaskari et al. (cf. [2, Theorem 6.1]). Their result can be stated as follows.

**Theorem 1.1.** Let \( f \) be an analytic function in \( \Delta \) such that

\[
\iint_{|w-w_0|<1} n(w,f)\,dA(w) \leq A < \infty,
\]

for all \( w_0 \in \mathbb{C} \), where \( n(w,f) \) denotes the number of roots of the equation \( f(z) = w \) in \( \Delta \) counted according to their multiplicity and \( dA(z) \) is the Euclidean area element on \( \Delta \). Then \( f \in \mathbb{B}(\mathbb{B}_0) \) if and only if \( f \in Q_p(Q_{p,0}) \) for all \( p \in (0, \infty) \).

Here, \( Q_p \) and its subspace \( Q_{p,0} \), \( 0 < p < \infty \), denote the spaces of analytic functions \( f \) in \( \Delta \) defined, respectively, as follows (cf. [1, 3]):

\[
Q_p = \left\{ f : f \text{ analytic in } \Delta, \sup_{a \in \Delta} \iint_{\Delta} |f''(z)|^2 (g(z,a))^p \,dA(z) < \infty \right\},
\]

\[
Q_{p,0} = \left\{ f \in Q_p : \lim_{|a| \to 1} \iint_{\Delta} |f''(z)|^2 (g(z,a))^p \,dA(z) = 0 \right\},
\]

where \( g(z,a) = \log 1/|\varphi_a(z)| \) is a Green’s function in \( \Delta \) with pole at \( a \in \Delta \), and \( \varphi_a(z) = (a-z)/(1-\bar{a}z) \) is a Möbius transformation of \( \Delta \).

We know that \( Q_1 = \text{BMOA} \), the space of all analytic functions of bounded mean oscillation (cf. [5]), and for each \( p \in (1, \infty) \), the space \( Q_p \) is the Bloch space \( \mathbb{B} \) (cf. [1]), which
is defined as follows:

\[ \mathcal{B} = \left\{ f : f \text{ analytic in } \Delta, \|f\|_{\mathcal{B}} = \sup_{z \in \Delta} (1 - |z|^2) |f'(z)| < \infty \right\}. \]  

Similar to the above we have \( Q_{1,0} = \text{VMOA} \), the space of all analytic functions of vanishing mean oscillation (cf. [5]), and \( Q_{p,0} = \mathcal{B}_0 \) for all \( p \in (1, \infty) \), where \( \mathcal{B}_0 \) denotes the little Bloch space defined by

\[ \mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{|z| \to 1} (1 - |z|^2) |f'(z)| = 0 \right\}. \]  

In the present paper, we consider a more general space \( Q_K \) (see below) and show that all the above-mentioned results are true for space \( Q_K \). Our contribution gives an extended version of Pommerenke’s theorem, which is also a slight improvement of all the above results, and the proof presented here is independently developed.

Let \( K : [0, \infty) \to [0, \infty) \) be a right-continuous and nondecreasing function. Recall that the space \( Q_K \) consists of analytic functions \( f \) in \( \Delta \) for which

\[ \|f\|_{Q_K}^2 = \sup_{a \in \Delta} \iint_{\Delta} |f'(z)|^2 K(g(z,a)) \, dA(z) < \infty; \]  

\( f \in Q_K \) belongs to the space \( Q_{K,0} \) if

\[ \iint_{\Delta} |f'(z)|^2 K(g(z,a)) \, dA(z) \to 0, \quad |a| \to 1. \]  

Modulo constants, \( Q_K \) is a Banach space under the norm defined in (1.5). It is clear that \( Q_K \) is Möbius-invariant and a subspace of the Bloch space \( \mathcal{B} \) (cf. [6]). For \( 0 < p < \infty \), \( K(t) = t^p \) gives the space \( Q_p \). Choosing \( K(t) = 1 \), we get the Dirichlet space \( \mathfrak{D} \).

By [6, Proposition 2.1] we know that if the integral

\[ \int_{0}^{1/e} K\left( \log \frac{1}{\rho} \right) \rho \, d\rho = \int_{1}^{\infty} K(t) e^{-2t} \, dt \]  

is divergent, then the space \( Q_K \) is trivial; that is, the space \( Q_K \) contains only constant functions. From now on, we assume that the function \( K : [0, \infty) \to [0, \infty) \) is right-continuous and nondecreasing and that the integral (1.7) is convergent. Without loss of generality, we can assume that \( K(1) > 0 \). For a general theory for \( Q_K \) spaces, see [6, 11].

2. Main results. A function \( f \) analytic in the unit disk is said to be \( q \)-valent if the equation \( f(z) = w \) has never more than \( q \) solutions. Let

\[ p(\rho) = \frac{1}{2\pi} \int_{0}^{2\pi} n(\rho e^{i\phi}, f) \, d\phi. \]  

If

\[ \int_{0}^{R} p(\rho) \, d(\rho^2) \leq qR^2, \quad R > 0, \]  

then \( f \) is \( q \)-valent. In particular, if \( q = 1 \), then \( f \) is univalent, and if \( q = 1 \), then \( f \) is subharmonic. The main result of this paper is the following theorem:

**Theorem.** Let \( f \) be analytic in \( \Delta \). Then

\[ \int_{0}^{1} p(\rho) \, d(\rho^2) \leq qR^2, \quad R > 0, \]  

implies that \( f \) is \( q \)-valent.
or

\[ p(R) \leq q, \quad R > 0, \tag{2.3} \]

where \( q \) is a positive number, we say that \( f \) is areally mean \( q \)-valent or circumferentially mean \( q \)-valent, respectively (cf. [7, pages 38 and 144]). It is clear that if \( f \) is circumferentially mean \( q \)-valent, then \( f \) is areally mean \( q \)-valent.

Note that if (1.1) holds, \( f \) will be areally mean \( q \)-valent in \( \Delta \) for some \( q > 0 \). We know that if \( f \) is univalent, then \( f \) must be areally and circumferentially mean 1-valent. Thus, it is natural to conjecture that Pommerenke’s result and Theorem 1.1 are also true for the areally and circumferentially mean \( q \)-valent functions.

We know that the space \( Q_K \) can be nontrivial if \( K \) is not too big at infinity (see condition (1.7)). For such functions \( K \), the properties of \( Q_K \) depend essentially on the behavior of \( K \) near the origin. From [6, Theorems 2.3 and 2.5], we know that \( Q_K = \mathcal{B} (Q_{K,0} = \mathcal{B}_0) \) if and only if

\[ \int_0^1 (1 - r^2)^{-2} K \left( \log \frac{1}{r} \right) r \, dr < \infty. \tag{2.4} \]

A natural idea is to look for an integral condition which is weaker than that given by (2.4) such that \( f \in \mathcal{B} (\mathcal{B}_0) \) if and only if \( f \in Q_K (Q_{K,0}) \) for some special \( f \). For the areally mean \( q \)-valent case, we present the main result in this paper as follows.

**Theorem 2.1.** Let \( f \) be an areally mean \( q \)-valent function in \( \Delta \). If

\[ \int_0^1 \left( \log \frac{1}{1 - r} \right)^2 (1 - r)^{-1} K \left( \log \frac{1}{r} \right) r \, dr < \infty, \tag{2.5} \]

then

(i) \( f \in \mathcal{B} \) if and only if \( f \in Q_K \);
(ii) \( f \in \mathcal{B}_0 \) if and only if \( f \in Q_{K,0} \).

Note that (2.4) implies (2.5) since \( (\log 1/(1 - r))^2 \leq 4e^{-2}/(1 - r) \) for \( 0 < r < 1 \), but the converse is not true. For example, \( K(t) = t \) gives that (2.5) holds but (2.4) fails. By [6, Theorems 2.3 and 2.5], (2.5) is also necessary for Theorem 2.1(i) and (ii) in case \( f \) is an areally mean \( q \)-valent function in \( \Delta \).

In the light of the following example it is impossible to drop the assumption of areally mean \( q \)-valence of the functions \( f \) in Theorem 2.1. Indeed, choose \( K_1(t) = t^{2\alpha - 1} \) and

\[ f_1(z) = \sum_{j=1}^{\infty} 2^{-j(1-\alpha)} z^{2^j}, \quad \frac{1}{2} < \alpha < 1. \tag{2.6} \]

It is easy to see that \( f_1 \in \mathcal{B} \) and (2.5) holds for \( K_1 \). Since \( f_1 \) has a gap series representation, \( f_1 \) is not an areally mean \( q \)-valent in \( \Delta \). The following argument shows that \( f \notin Q_{K_1} \).
For \( r \in [\frac{3}{4}, 1) \), we find \( k \) so that \( 1/2 \leq 2^k (1 - r) < 1 \). Using the inequality \( \log r \geq 2(r - 1) \), we see that

\[
\int_0^{2\pi} |f'(r e^{i\theta})|^2 d\theta = 2\pi \sum_{j=1}^\infty 2^{j}\alpha r^{2j+1-2} \geq 2\pi (1 - r)^{-2\alpha} \sum_{j=1}^\infty (2^j (1 - r))^{2\alpha} \exp(-2^{j+2}(1 - r)) \geq 2^{-2\alpha + 1} \pi (1 - r)^{-2\alpha} \sum_{j=1}^\infty 2^{(j-k)(2\alpha)} \exp(-2^{j-k+2}) \geq 2^{-2\alpha + 1} \pi (1 - r)^{-2\alpha} \sum_{j=0}^\infty (2^{j\alpha} \exp(-2^{j+2})) = C(\alpha) (1 - r)^{-2\alpha}. \tag{2.7}
\]

Hence

\[
\sup_{a \in \Delta} \iint_{\Delta} |f'_1(z)|^2 K_1(g(z, a)) dA(z) \geq \int_{\Delta} |f'_1(z)|^2 K_1 \left( \log \frac{1}{|z|} \right) dA(z) = \int_0^1 K \left( \log \frac{1}{r} \right) r dr \int_0^{2\pi} |f'(r e^{i\theta})|^2 d\theta \geq C(\alpha) \int_{\frac{3}{4}}^1 (1 - r)^{-2\alpha} \left( \log \frac{1}{r} \right)^{2\alpha - 1} r dr.
\tag{2.8}
\]

Since the last integral is divergent, we conclude that \( f_1 \notin Q_K \).

**Theorem 2.2.** Let \( f \) be a circumferentially mean \( q \)-valent and nonvanishing function in \( \Delta \). If (2.5) holds, then \( \log f \in Q_K \).

It is clear that the integral in (2.5) is convergent for \( K(t) = t^p \), \( p > 0 \). Thus, we have the following result which extends Theorem 1.1.

**Corollary 2.3.** Let \( f \) be an areally mean \( q \)-valent function in \( \Delta \), \( 0 < p < \infty \). Then

(i) \( f \in \mathcal{B} \) if and only if \( f \in Q_p \);
(ii) \( f \in \mathcal{B}_0 \) if and only if \( f \in Q_{p,0} \).

3. Proofs. In the proofs of Theorems 2.1 and 2.2, we need two lemmas, the first one can be considered as a generalization of a result of Pommerenke (cf. [9, page 174]).

**Lemma 3.1.** Let \( f \) be areally mean \( q \)-valent in \( \Delta \). Then

\[
\int_0^{2\pi} |f'(r e^{i\theta})|^2 d\theta \leq \frac{4q\pi (M(\sqrt{r}, f))^2}{1 - r}, \quad \frac{1}{2} < r < 1, \tag{3.1}
\]

where \( M(r, f) = \sup_{|z|=r} |f(z)|, 0 < r < 1 \).
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**Proof.** If $1/2 < r < 1$, we obtain

$$\iint_{|z| < \sqrt{r}} |f'(z)|^2 dA(z) = \int_0^{\sqrt{r}} \rho \int_0^{2\pi} |f'(\rho e^{i\theta})|^2 d\theta d\rho$$

$$\geq \frac{1}{4} (1-r) \int_0^{2\pi} |f''(re^{i\theta})|^2 d\theta. \quad (3.2)$$

Since $f$ is areally mean $q$-valent, we deduce that

$$\int_0^{2\pi} |f'(re^{i\theta})|^2 d\theta \leq \frac{4}{1-r} \iint_{|z| < \sqrt{r}} |f'(z)|^2 dA(z) \leq \frac{4}{1-r} \iint_{|w| < M(\sqrt{r},f)} n(w,f) dA(w)$$

$$\leq \frac{4q\pi (M(\sqrt{r},f))^2}{1-r}, \quad (3.3)$$

which proves Lemma 3.1. \qed

**Lemma 3.2.** Let $K$ be defined as in Section 1. Then

(i) $Q_{K,0} \subset \mathcal{B}_0$;

(ii) an analytic function $f$ belongs to $\mathcal{B}_0$ if and only if there exists an $r \in (0,1)$ such that

$$\lim_{|a| \to 1} \iint_{\Delta(a,r)} |f'(z)|^2 K(g(z,a)) dA(z) = 0, \quad (3.4)$$

where $\Delta(a,r) = \{ z \in \Delta : |\varphi_a(z)| < r \}.$

**Proof.** See [6, Thereom 2.4]. \qed

Now we turn to give the proofs of our main theorems.

**Proof of Theorem 2.1.** We first prove (i). Since $Q_K \subset \mathcal{B}_0$, it suffices to prove that if a Bloch function $f$ is areally mean $q$-valent in $\Delta$, then $f \in Q_K$. We use the change of variable $w = \varphi_a(z)$ to deduce that

$$\iint_{\Delta(a,1/2)} |f'(z)|^2 K(g(z,a)) dA(z)$$

$$= \iint_{\Delta(a,1/2)} \left| (f(z) - f(a))' \right|^2 K\left( \log \frac{1}{|\varphi_a(z)|} \right) dA(z)$$

$$= \iint_{|w| < 1/2 \mid |w| < 1} \left| (f \circ \varphi_a(w) - f(a))' \right|^2 K\left( \log \frac{1}{|w|} \right) dA(w)$$

$$= \int_{1/2}^1 K\left( \log \frac{1}{r} \right) r \int_0^{2\pi} \left| (f \circ \varphi_a(re^{i\theta}) - f(a))' \right|^2 d\theta dr. \quad (3.5)$$
It is known that if \( g \in \mathcal{B}_0 \), then
\[
|g(z) - g(0)| \leq \frac{1}{2} \|g\|_{\mathcal{B}_0} \log \frac{1 + |z|}{1 - |z|}.
\] (3.6)

Choosing \( g = f \circ \varphi_a - f(a) \) and observing that \( \|g\|_{\mathcal{B}_0} = \|f\|_{\mathcal{B}_0} \), we obtain
\[
M(r, f \circ \varphi_a - f(a)) \leq \frac{1}{2} \|f\|_{\mathcal{B}_0} \log \frac{1 + r}{1 - r}.
\] (3.7)

It follows from (3.5) and Lemma 3.1 that
\[
\iint_{\Delta \setminus \Delta(a,1/2)} |f'(z)|^2 K(g(z,a))dA(z)
= \int_{1/2}^{1} K\left(\frac{\log \frac{1}{r}}{r}\right) r \int_{0}^{2\pi} |(f \circ \varphi_a(r e^{i\theta}) - f(a))'|^2 d\theta dr
\leq 4q\pi \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1 - r)^{-1} r dr
\leq q\pi C \|f\|_{\mathcal{B}_0}^2 \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) \left(\log \frac{1}{1 - r}\right)^2 (1 - r)^{-1} r dr.
\] (3.8)

On the other hand, we have
\[
\iint_{\Delta(a,1/2)} |f'(z)|^2 K(g(z,a))dA(z)
\leq \|f\|_{\mathcal{B}_0}^2 \int_{\Delta(a,1/2)} (1 - |z|^2)^{-1} K(g(z,a))dA(z)
= \|f\|_{\mathcal{B}_0}^2 \int_{\Delta(0,1/2)} (1 - |w|^2)^{-1} K\left(\log \frac{1}{|w|}\right)dA(w)
\leq 4\pi \|f\|_{\mathcal{B}_0}^2 \int_{1/2}^{1} K\left(\log \frac{1}{r}\right) r dr.
\] (3.9)

Combining the upper bounds given by (3.8), (3.9), and (2.5), we see that \( f \in Q_K \), which proves part (i) of Theorem 2.1.

To prove (ii), we assume that \( f \) is an areally mean \( q \)-valent function in \( \Delta \) which is also in \( \mathcal{B}_0 \). By Lemma 3.2(i), it suffices to prove that \( f \in Q_{K,0} \). By Lemma 3.2(ii), there exists an \( r_0, 1/2 < r_0 < 1 \), such that
\[
\lim_{|a| \to 1} \iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))dA(z) = 0.
\] (3.10)

Now we show that
\[
\lim_{|a| \to 1} \iint_{\Delta(a,r_0)} |f'(z)|^2 K(g(z,a))dA(z) = 0.
\] (3.11)
By the proof of part (i) and assumption (2.5), we see that

\[
\int_{\Delta \setminus \Delta(a,r_0)} |f'(z)|^2 K(g(z,a)) dA(z) \\
= \int_{r_0}^1 K\left(\log \frac{1}{r}\right) r^2 d\theta dr \\
\leq 4q\pi \int_{r_0}^1 K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \\
\leq q\pi \|f\|_H^2 \int_{r_0}^1 K\left(\log \frac{1}{r}\right) \left(\log \frac{1+r}{1-r}\right)^2 (1-r)^{-1} r dr < \infty
\]

(3.12)

for all \(a \in \Delta\). Thus, for any given \(\varepsilon > 0\), there exists an \(r_1, r_0 < r_1 < 1\), such that

\[
\int_{r_1}^1 K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr < \varepsilon
\]

(3.13)

for all \(a \in \Delta\). Hence, what we need to prove is that

\[
\lim_{|a| \to 1} \int_{r_0}^{r_1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr = 0.
\]

(3.14)

In fact, we have

\[
\int_{r_0}^{r_1} K\left(\log \frac{1}{r}\right) (M(\sqrt{r}, f \circ \varphi_a - f(a)))^2 (1-r)^{-1} r dr \\
\leq C(r_0, r_1)K\left(\log \frac{1}{r_0}\right) (M(r_2, f \circ \varphi_a - f(a)))^2,
\]

(3.15)

where \(r_2 = \sqrt{r_1}\) and \(C(r_0, r_1)\) is a constant depending on \(r_0\) and \(r_1\). Define \(f_t(z) = f(tz)\) for \(0 < t < 1\) and then

\[
(M(r_2, f \circ \varphi_a - f(a)))^2 \\
\leq 2\left(\frac{1}{4} \|f - f_t\|_H^2 \left(\log \frac{1+r_2}{1-r_2}\right)^2 + (M(r_2, f_t \circ \varphi_a - f_t(a)))^2\right).
\]

(3.16)

Since \(f \in \mathcal{B}_0\), \(\|f - f_t\|_H \to 0\), \(t \to 1\). Also,

\[
\max_{|z| \leq r_2} |f_t \circ \varphi_a(z) - f_t(a)| \leq \frac{1-|a|^2}{(1-r_2)^2} \max_{|w| \leq t} |f'(w)|,
\]

(3.17)

which implies that

\[
\lim_{|a| \to 1} M(r_2, f_t \circ \varphi_a - f_t(a)) = 0.
\]

(3.18)
Thus we have (3.14). Hence
\begin{equation}
\lim_{|a|\to 1} \int_{\Delta} |f''(z)|^2 K(g(z,a)) dA(z) = 0,
\end{equation}
which shows that \( f \in Q_{K,0} \). The proof of Theorem 2.1 is complete.

**Proof of Theorem 2.2.** Assume that \( f \) is a nonvanishing circumferentially mean \( q \)-valent function in \( \Delta \). According to [7, Theorem 5.1], we have \( \log f \in \mathcal{R} \). From [7, Lemma 5.2] and the argument in the beginning of the proof of [7, Theorem 5.1], we see that we can define a single-valued branch of \( f(z)^{1/q} \) which is circumferentially mean 1-valent in \( \Delta \) and such that on each circle \( \{|w|=R\} \) there exists a point which is not assumed by \( f(z)^{1/q} \). It follows that
\begin{equation}
\int_{-\infty}^{\infty} n\left(\log \rho + i\phi, \frac{1}{q} \log f\right) d\phi = \int_{0}^{2\pi} n(\rho e^{i\phi}, f^{1/q}) d\phi \leq 2\pi,
\end{equation}
\begin{equation}
\int_{|w|<R} n(w, \log f) dA(w) \leq 4\pi Rq,
\end{equation}
which means that \( \log f \) is areally mean \( q_1 \)-valued in \( \Delta \) for some \( q_1 > 0 \). It follows from Theorem 2.1 that \( \log f \in Q_K \).

4. Further discussion. In [10] we studied the conditions for analytic univalent Bloch function \( f \) to belong to \( Q_K \) spaces. The log-order of the function \( K(r) \) is defined as
\begin{equation}
\rho = \lim_{r \to \infty} \frac{\log^+ \log^+ K(r)}{\log r},
\end{equation}
where \( \log^+ x = \max\{\log x, 0\} \), and if \( 0 < \rho < \infty \), the log-type of the function \( K(r) \) is defined as
\begin{equation}
\sigma = \lim_{r \to \infty} \frac{\log^+ K(r)}{r^\rho}.
\end{equation}

**Theorem 4.1.** Let \( f \) be an analytic univalent function in \( \Delta \) and let \( K: [0, \infty) \to [0, \infty) \) satisfy that \( K(t) = O((t \log 1/t)^p) \) as \( t \to 0 \) for some \( p > 0 \). If the log-order \( \rho \) and the log-type \( \sigma \) of \( K \) satisfy one of the conditions
\begin{enumerate}
  \item \( 0 \leq \rho < 1 \),
  \item \( \rho = 1 \) and \( \sigma < 2 \),
\end{enumerate}
then \( f \in \mathcal{B} \) if and only if \( f \in Q_K \).

We note that Theorem 4.1 can be viewed as a consequence of Theorem 2.1. In fact, conditions (i) and (ii) of Theorem 4.1 show that the space \( Q_K \) is not trivial. That is, the integral (1.7) is convergent in this case. Suppose that \( K(t) = O((t \log 1/t)^p) \), \( t \to 0 \). There exist an \( r_0 \in (1/2, 1) \) and a constant \( C > 0 \) such that both \( \log 1/r \leq 2(1-r) \) and
\begin{equation}
K\left(\log \frac{1}{r}\right) \leq C\left(\log \frac{1}{r} \log \left(\log \frac{1}{r}\right)^{-1}\right)^p
\end{equation}
hold for \( r_0 < r < 1 \). Thus

\[
\int_0^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left( \log \frac{1}{r} \right) r \, dr
\]

\[
= \int_0^{r_0} + \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} K \left( \log \frac{1}{r} \right) r \, dr
\]

\[
\leq \left( \log \frac{1}{1-r_0} \right)^2 (1-r_0)^{-1} \int_0^{r_0} K \left( \log \frac{1}{r} \right) r \, dr
\]

\[
+ C \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^2 (1-r)^{-1} \left( \log \frac{1}{r} \left( \log \frac{1}{r} \right)^{-1} \right)^p r \, dr
\]

\[
\leq C_1 + C_2 \int_{r_0}^1 \left( \log \frac{1}{1-r} \right)^{2+p} (1-r)^{p-1} r \, dr
\]

\[
\leq C_1 + C_2 \int_{r_0}^\infty e^{-p s^{2+p}} \, ds
\]

\[
\leq C_1 + C_2 p^{-3-p} \Gamma(3+p) < \infty.
\]

For a general analytic function \( f \), we have the following theorem.

**Theorem 4.2.** Suppose that (2.5) holds. If

\[
\sup_{a \in A} \iint_{|z| < r} |(f \circ \varphi_a(z))'|^2 \, dA(z) = O \left( \left( \log \frac{1}{1-r} \right)^2 \right),
\]

then

(i) \( f \in \mathcal{B} \) if and only if \( f \in Q_K \);

(ii) \( f \in \mathcal{B}_0 \) if and only if \( f \in Q_{K,0} \).

**Proof.** We know that

\[
\int_0^{2\pi} \left| (f \circ \varphi_a(\text{re}^{i\theta}))' \right|^2 \, d\theta \leq \frac{4}{1-r} \iint_{|z| < \sqrt{r}} \left| (f \circ \varphi_a(z))' \right|^2 \, dA(z)
\]

\[
\leq \frac{1}{1-r} O \left( \left( \log \frac{1}{1-\sqrt{r}} \right)^2 \right)
\]

\[
\leq \frac{C}{1-r} \left( \log \frac{1}{1-r} \right)^2.
\]

The proof can be completed by an argument similar to that used in the proof of Theorem 2.1.

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References


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