

THE CONVERGENCE ESTIMATES FOR GALERKIN-WAVELET SOLUTION OF PERIODIC PSEUDODIFFERENTIAL INITIAL VALUE PROBLEMS

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Using the discrete Fourier transform and Galerkin-Petrov scheme, we get some results on the solutions and the convergence estimates for periodic pseudodifferential initial value problems.

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1. Introduction. In recent years, wavelets have been developing intensively and have become a powerful tool to study mathematics and technology, for example, the theory of the singular integral, singular integro-differential equations, the areas such as sound analysis, image compression, and so on (see [9, 10] and references therein). In this paper, we use a scaling function and a multilevel approach to estimate the error of the problem

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} &= a \cdot Au(x,t), \quad x \in \mathcal{F}^n, \quad t > 0, \quad a \in \mathbb{R}, \\ u(x,0) &= [u_0](x), \quad x \in \mathcal{F}^n,\end{aligned}\tag{1.1}$$

where A is a pseudodifferential operator (see [1, 2, 3, 4, 6, 8, 9, 12]) with a symbol $\sigma \in C^\infty(\mathbb{R}^n)$, σ is positively homogeneous of degree $r > 0$ such that

$$|D^\alpha \sigma(\xi)| \leq C_\alpha (1 + |\xi|)^{r-|\alpha|}, \quad \text{for all multi-index } \alpha \in \mathbb{N}^n,\tag{1.2}$$

$\mathcal{F}^n = \mathbb{R}^n / \mathbb{Z}^n$, and $[u_0](x) = \sum_{k \in \mathbb{Z}^n} u_0(x+k)$ is a periodic operator.

We discuss only problem (1.1) with the following condition:

$$a\sigma(\xi) \leq 0, \quad \forall \xi \in \mathbb{Z}^n.\tag{1.3}$$

2. Preliminaries and notations. The continuous Fourier transform of the function $f \in L_2(\mathbb{R}^n)$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{R}^n\tag{2.1}$$

with the inverse Fourier formula

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \xi} \hat{f}(\xi) d\xi, \quad \xi \in \mathbb{R}^n \tag{2.2}$$

(see [4, 8, 11]).

The discrete Fourier transform of the function $f \in L_2(\mathcal{F}^n)$ is

$$\mathcal{F}(f)(\xi) = \tilde{f}(\xi) := \int_{[0,1]^n} e^{-2\pi i x \xi} f(x) dx, \quad \xi \in \mathbb{Z}^n, \tag{2.3}$$

and the inverse Fourier transform is

$$f(x) := \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) e^{2\pi i x \xi} \tag{2.4}$$

(see [6]).

Some simple properties of the discrete Fourier transform are

$$(f, g)_0 = \sum_{\xi \in \mathbb{Z}^n} \tilde{f}(\xi) \overline{\tilde{g}(\xi)}, \tag{2.5}$$

where $(\cdot, \cdot)_0$ is the $L_2(\mathcal{F}^n)$ -inner product,

$$\|f\|_0^2 = \sum_{\xi \in \mathbb{Z}^n} |\tilde{f}(\xi)|^2 = \|\tilde{f}\|_{l_2}^2, \tag{2.6}$$

where $\|\cdot\|_0$ is $L_2(\mathcal{F}^n)$ -norm and $\|\cdot\|_{l_2}$ is l_2 -norm.

Let $s \in \mathbb{R}$. Denote

$$H^s(\mathcal{F}^n) = \{u \in D'(\mathcal{F}^n) \mid \langle D \rangle^s u \in L_2(\mathcal{F}^n)\}, \tag{2.7}$$

where

$$\langle \xi \rangle = \begin{cases} 1 & \text{if } \xi = 0, \\ |\xi| & \text{if } \xi \neq 0, \end{cases} \tag{2.8}$$

then $H^s(\mathcal{F}^n)$ is the Sobolev space endowed with the norm

$$\|u\|_s^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\tilde{u}(\xi)|^2 \tag{2.9}$$

and the inner product

$$\langle u, v \rangle_s = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} \tilde{u}(\xi) \overline{\tilde{v}(\xi)}. \tag{2.10}$$

Here, we also define the discrete Sobolev space $H_d^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, of the functions $f \in H^s(\mathbb{R}^n)$ such that the following norm is finite:

$$\|f\|_{s,d}^2 = \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2s} |\hat{f}(\xi)|^2. \tag{2.11}$$

Denote

$$\mathcal{L}_2 = \left\{ f \in L_2(\mathbb{R}^n) : \sum_{\xi \in \mathbb{Z}^n} |f(\cdot - \xi)| \in L_2([0, 1]^n) \right\}. \tag{2.12}$$

It is clear that any function $f \in L_2(\mathbb{R}^n)$, which has compact support, or any function, for which $\int_{k+[0,1]^n} |f(x)|^2 dx$ decays exponentially as $|k|$ tends to infinity, belongs to \mathcal{L}_2 . The periodic operator $[u]$ is totally defined if $u \in \mathcal{L}_2$. Here, we assume that $u_0 \in \mathcal{L}_2$.

REMARK 2.1. (1) It follows from (2.1) and (2.3) that if $u \in \mathcal{L}_2$, then $\mathcal{F}([u])(\xi) = \hat{u}(\xi), \xi \in \mathbb{Z}^n$.

(2) It is clear that if $t \leq s, s, t \in \mathbb{R}$, then $H^t(\mathcal{F}^n) \subset H^s(\mathcal{F}^n)$.

Using the variable separate method and the discrete Fourier transform, the solution of problem (1.1) can be represented as

$$u(x, t) = E(t)[u_0](x) = \sum_{\xi \in \mathbb{Z}^n} \exp(a\sigma(\xi)t) \mathcal{F}([u_0])(\xi) e^{2\pi i x \xi}, \tag{2.13}$$

where $E(t)$ is a differentiable function and $E(0) = 1$.

We recall that a multiresolution approximation (MRA) of $L_2(\mathbb{R}^n)$ is, as a definition, an increasing sequence $V_j, j \in \mathbb{Z}$, of closed linear subspaces of $L_2(\mathbb{R}^n)$ with the following properties:

$$\bigcap_{j \in \mathbb{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j \in \mathbb{Z}} V_j} = L_2(\mathbb{R}^n); \tag{2.14}$$

for all $f \in L_2(\mathbb{R}^n)$ and all $j \in \mathbb{Z}$,

$$f(x) \in V_j \iff f(2x) \in V_{j+1}; \tag{2.15}$$

for all $f \in L_2(\mathbb{R}^n)$ and $k \in \mathbb{Z}^n$,

$$f(x) \in V_0 \iff f(x - k) \in V_0. \tag{2.16}$$

There exists a function, called the scaling function (SF) $\phi(x) \in V_0$, such that the sequence

$$\{\phi(x - k), k \in \mathbb{Z}^n\} \tag{2.17}$$

is a Riesz basic of V_0 (see [5, 9]).

An SF ϕ is called μ -regular ($\mu \in \mathbb{N}$) if, for each $m \in \mathbb{N}$, there exists c_m such that the following condition holds:

$$|D^\alpha \phi(x)| \leq c_m (1 + |x|)^{-m}, \quad \forall \alpha, |\alpha| \leq \mu. \tag{2.18}$$

REMARK 2.2. (1) Denote $\phi_{jk}(x) = 2^{nj/2}\phi(2^jx - k)$, $k \in \mathbb{Z}^n$. It follows from (2.14), (2.15), (2.16), and (2.17) that $V_j = \overline{\text{span}}\{\phi_{jk}(x), k \in \mathbb{Z}^n\}$, $j \in \mathbb{Z}$.

(2) For each $\mu \in \mathbb{N}$, there exists an SF $\phi(x)$ with compact support, and $\phi(x)$ is μ -regular; so in what follows, we always assume that ϕ has compact support and is μ -regular (see [9]).

Using the periodic operator and an MRA of $L_2(\mathbb{R}^n)$, we can build an MRA of $L_2(\mathcal{F}^n)$ with the SF $[\phi]$ as follows.

Denote

$$\phi_k^j(x) = 2^{nj/2} \sum_{l \in \mathbb{Z}^n} \phi_{jk}(x+l) = 2^{nj/2} \sum_{l \in \mathbb{Z}^n} \phi(2^j(x+l) - k), \quad j \geq 0, \tag{2.19}$$

$$[V_j] = \overline{\text{span}}\{\phi_k^j(x), k \in \mathbb{Z}^{nj}\}, \quad j \geq 0, \tag{2.20}$$

where $\mathbb{Z}^{nj} = \mathbb{Z}^n / 2^j \mathbb{Z}^n$.

Then, the sequence $[V_j]_{j \geq 0}$ satisfies

$$[V_0] \subset [V_1] \subset \dots, \quad \overline{\bigcup_{j \geq 0} [V_j]} = L_2(\mathcal{F}^n). \tag{2.21}$$

It is clear that $\dim[V_j] = 2^{nj}$, and if $(\phi_{jk}, \phi_{jl}) = \delta_{kl}$, $k, l \in \mathbb{Z}^n$, then $(\phi_k^j, \phi_l^j) = \delta_{kl}$, $k, l \in \mathbb{Z}^{nj}$ (see [6]).

For each $j \geq 0$, let $P_j : L_2(\mathcal{F}^n) \rightarrow [V_j]$ be the orthogonal projection from $L_2(\mathcal{F}^n)$ on $[V_j]$, which has the following property.

THEOREM 2.3 (see [6, page 600]). *Let $-\mu - 1 \leq s \leq \mu$, $-\mu \leq q \leq \mu + 1$, and $s \leq q$, then*

$$\|u - P_j u\|_s \leq c 2^{j(s-q)} \|u\|_q \tag{2.22}$$

for all $u \in H^q(\mathcal{F}^n)$, where c is independent of j and u .

Denoting $h = 2^{-j}$ and $V_h = [V_j]$, we can write (2.22) as

$$\|v - P_j v\|_s \leq c h^{q-s} \|v\|_q. \tag{2.23}$$

3. The Galerkin-wavelet solution. Fix a distribution with compact support $\eta \in H^{-s'}(\Gamma)$, where $s' \geq 0$ satisfying $AV_h \subset H^{s'}(\mathcal{F}^n)$ and where $\Gamma \subset \mathbb{R}^n$ is some fixed compact domain such as a hypercube. For $f \in H^{s'}(\mathcal{F}^n)$, define

$$\eta_k^j(f) = 2^{-nj/2} \eta(f(2^{-j}(\cdot + k))). \tag{3.1}$$

The space

$$X^j := \text{span}\{\eta_k^j, k \in \mathbb{Z}^{nj}\} \tag{3.2}$$

is contained in $(AV_h)'$, which is the dual of AV_h . The corresponding Galerkin-Petrov-wavelet scheme is then given by

$$\eta_k^j \left(\frac{\partial u_h}{\partial t} \right) = a \eta_k^j (Au_h), \quad k \in \mathbb{Z}^{nj}, \tag{3.3}$$

$$u_h(x, 0) = R_h[u_0](x), \tag{3.4}$$

where $R_h v$ is a linear approximation of v in V_h and $u_h : [0, \infty) \rightarrow V_h$ is a differentiable operator.

Set

$$u_h(x, t) = \sum_{k \in \mathbb{Z}^{nj}} c_k(t) \phi_k^j(x), \tag{3.5}$$

$$R_h[u_0](x) := [u_0]_h(x) := \sum_{k \in \mathbb{Z}^{nj}} c_k(0) \phi_k^j(x). \tag{3.6}$$

Then the scheme (3.3) and (3.4) provides an algebra equation system and the solution can be solved by Fourier series.

LEMMA 3.1. *The following formulas hold true:*

$$\begin{aligned} \mathcal{F}(\phi_k^j)(\xi) &= h^{n/2} \hat{\phi}(h\xi) e^{-2\pi i h k \xi}, \\ \mathcal{F}(A\phi_k^j)(\xi) &= h^{n/2} \sigma(\xi) \hat{\phi}(h\xi) e^{-2\pi i h k \xi}. \end{aligned} \tag{3.7}$$

PROOF. (a) It follows from (2.3) and (2.19) that

$$\begin{aligned} \overline{\mathcal{F}}(\phi_k^j)(\xi) &= h^{-n/2} \sum_{l \in \mathbb{Z}^n} \int_{[0,1]^n} e^{-2\pi i x \xi} \phi(2^j(x+l)-k) dx \\ &= h^{n/2} \sum_{l \in \mathbb{Z}^n} \int_{2^j(l+[0,1]^n)-k} e^{-2\pi i h x \xi} \phi(x) dx e^{-2\pi i h k \xi} \\ &= h^{n/2} \int_{\mathbb{R}^n} e^{-2\pi i h x \xi} \phi(x) dx e^{-2\pi i h k \xi} \\ &= h^{n/2} \hat{\phi}(h\xi) e^{-2\pi i h k \xi}. \end{aligned} \tag{3.8}$$

(b) We have

$$\mathcal{F}(Au)(\xi) = \sigma(\xi) \tilde{u}(\xi); \tag{3.9}$$

consequently,

$$\overline{\mathcal{F}}(A\phi_k^j)(\xi) = \sigma(\xi) \overline{\mathcal{F}}(\phi_k^j)(\xi) = h^{n/2} \sigma(\xi) \hat{\phi}(h\xi) e^{-2\pi i h k \xi}. \tag{3.10}$$

The proof of the lemma is complete. □

COROLLARY 3.2. *The following formulas hold true:*

$$\begin{aligned} \eta_k^j(\phi_l^j) &= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{-2\pi i h(l-k)\xi}, \\ \eta_k^j(A\phi_l^j) &= h^n \sum_{\xi \in \mathbb{Z}^n} \sigma(\xi) \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{-2\pi i h(l-k)\xi}. \end{aligned} \tag{3.11}$$

PROOF. (a) Using (2.4), Lemma 3.1, and (3.1), we have

$$\begin{aligned} \eta_k^j(\phi_l^j) &= \eta_k^j \left(\sum_{\xi \in \mathbb{Z}^n} \mathcal{F}(\phi_l^j)(\xi) e^{2\pi i x \xi} \right) \\ &= \eta_k^j \left(\sum_{\xi \in \mathbb{Z}^n} h^{n/2} \hat{\phi}(h\xi) e^{-2\pi i h l \xi} e^{2\pi i x \xi} \right) \\ &= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) e^{-2\pi i h l \xi} \eta(e^{2\pi i h(x+k)\xi}) \\ &= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{-2\pi i h(l-k)\xi}. \end{aligned} \tag{3.12}$$

(b) Similarly, we can get the second assertion. □

The following lemma is extracted from [6].

LEMMA 3.3. *The following formula holds valid:*

$$\sum_{m \in \mathbb{Z}^{nj}} e^{-2\pi i h m(k-\xi)} = \begin{cases} 2^{nj} & \text{if } \xi = k + 2^j \theta, \theta \in \mathbb{Z}^n, \\ 0 & \text{otherwise.} \end{cases} \tag{3.13}$$

Set

$$\alpha(k) = \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{2\pi i h k \xi}, \tag{3.14}$$

$$\delta(k) = \sum_{\xi \in \mathbb{Z}^n} \sigma(h\xi) \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{2\pi i h k \xi}, \quad k \in \mathbb{Z}^{nj}. \tag{3.15}$$

The series

$$\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \alpha(k) e^{-2\pi i h k \zeta}, \tag{3.16}$$

$$\tilde{\delta}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \delta(k) e^{-2\pi i h k \zeta}, \tag{3.17}$$

$$\tilde{c}(\zeta, t) = h^n \sum_{k \in \mathbb{Z}^{nj}} c_k(t) e^{-2\pi i h k \zeta}, \quad \zeta \in \mathbb{Z}^n \tag{3.18}$$

are called discrete Fourier series.

It follows from (3.3), (3.5), the positively homogeneous condition, and Corollary 3.2 that

$$\sum_{k \in \mathbb{Z}^{nj}} c'_k(t) \alpha(l-k) = ah^{-r} \sum_{k \in \mathbb{Z}^{nj}} c_k(t) \delta(l-k), \quad l \in \mathbb{Z}^{nj}. \tag{3.19}$$

Thus

$$\tilde{c}'_t(\zeta, t) \tilde{\alpha}(\zeta) = ah^{-r} \tilde{c}(\zeta, t) \tilde{\delta}(\zeta), \tag{3.20}$$

$$\tilde{c}(\zeta, t) = \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\zeta)}{\tilde{\alpha}(\zeta)}\right) \tilde{c}(\zeta, 0). \tag{3.21}$$

For each $\tau = 0, 1$, set

$$g_{\phi, \tau}(\zeta) = \sum_{k \in \mathbb{Z}^n} \sigma(h\zeta + k)^\tau \hat{\phi}(h\zeta + k) \overline{\hat{\eta}(h\zeta + k)}. \tag{3.22}$$

LEMMA 3.4. *If the series (3.22) converges absolutely, then*

$$\tilde{\alpha}(\zeta) = g_{\phi, 0}(\zeta), \quad \tilde{\delta}(\zeta) = g_{\phi, 1}(\zeta). \tag{3.23}$$

PROOF. (a) From (3.14) and (3.16), it follows that

$$\tilde{\alpha}(\zeta) = h^n \sum_{k \in \mathbb{Z}^{nj}} \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} e^{-2\pi i h k(\zeta - \xi)}. \tag{3.24}$$

By the hypothesis of the lemma, we can interchange the summation in the above double sum; then by using the variable change and Lemma 3.3, it is easy to see that

$$\begin{aligned} \tilde{\alpha}(\zeta) &= h^n \sum_{\xi \in \mathbb{Z}^n} \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} \sum_{k \in \mathbb{Z}^{nj}} e^{-2\pi i h k(\zeta - \xi)} \\ &= \sum_{\theta \in \mathbb{Z}^n} \hat{\phi}(h\zeta + \theta) \overline{\hat{\eta}(h\zeta + \theta)} = g_{\phi, 0}(\zeta). \end{aligned} \tag{3.25}$$

(b) Similarly, the second assertion of the lemma will be checked.

From (3.5), (3.6), and (3.21), it follows that

$$\tilde{u}_h(\xi, t) = \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \mathcal{F}([u_0]_h)(\xi). \tag{3.26}$$

Let $F_h(t)$ be the operator defined by

$$\mathcal{F}(F_h(t)v(\cdot))(\xi) = \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \tilde{v}(\xi), \tag{3.27}$$

then the approximation $u_h(x)$ can be represented by

$$u_h(x) = F_h(t)R_h[u_0](x). \tag{3.28}$$

□

4. The error estimate of approximation solutions. Now to estimate the error, we need some restrictions on the σ , ϕ , and η used above. The triplet (σ, ϕ, η) is called *admissible* if the following properties hold:

- (i) there exists $p \in \mathbb{N}$, $p \geq r$, such that the series

$$\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \overline{\hat{\eta}(h\xi + k)} \tag{4.1}$$

converges absolutely and

$$\sum_{k \in \mathbb{Z}^n} \sigma(h\xi + k) \hat{\phi}(h\xi + k) \overline{\hat{\eta}(h\xi + k)} = \sigma(h\xi) \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} + o(|h\xi|^p) \tag{4.2}$$

as $|h\xi| \rightarrow 0$,

- (ii) $\hat{\phi}(\xi) \overline{\hat{\eta}(\xi)} \geq 0$, for all $\xi \in \mathbb{R}^n$, $\hat{\phi}(0) \overline{\hat{\eta}(0)} \neq 0$,

- (iii) the series

$$\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \overline{\hat{\eta}(h\xi + k)} \tag{4.3}$$

converges and

$$\sum_{k \in \mathbb{Z}^n} \hat{\phi}(h\xi + k) \overline{\hat{\eta}(h\xi + k)} = \hat{\phi}(h\xi) \overline{\hat{\eta}(h\xi)} + o(|h\xi|^p) \tag{4.4}$$

as $|h\xi| \rightarrow 0$.

REMARK 4.1. (1) If $\eta = \phi$ and σ is a pseudodifferential operator with symbol $\sigma(\xi) = |\xi|^r$, $0 < r \leq \mu$, then the triplet (σ, ϕ, ϕ) is automatically admissible at least for $p = \mu$, where $\mu \in \mathbb{N}$ is used in (2.18) (see [7] for detail).

(2) If $\eta = \phi$ and σ is a pseudodifferential operator with symbol $\sigma(\xi) = \langle \xi \rangle^2$, then the triplet $(\langle \xi \rangle^2, \phi, \phi)$ is admissible for $p = \mu$ (see [6]).

Write

$$u - u_h = \{u - F_h(t)[u_0]\} + F_h(t)\{[u_0] - R_h[u_0]\}. \tag{4.5}$$

We have

$$\begin{aligned} \mathcal{F}(F_h(t)[u_0](\cdot))(\xi) &= \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \mathcal{F}([u_0])(\xi) \\ &= \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \hat{u}_0(\xi), \quad \xi \in \mathbb{Z}^n, \end{aligned} \tag{4.6}$$

thus

$$\begin{aligned} \mathcal{F}(u - F_h(t)[u_0])(\xi) &= \left\{ \exp(at\sigma(\xi)) - \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \right\} \hat{u}_0(\xi), \quad \xi \in \mathbb{Z}^n. \end{aligned} \tag{4.7}$$

If the triplet (σ, ϕ, η) is admissible, then it follows from (3.22) and Lemma 3.4 that

$$\frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)} = \sigma(h\xi) + O(|h\xi|^p) \quad \text{as } |h\xi| \rightarrow 0. \tag{4.8}$$

THEOREM 4.2. *Suppose that $r + s' \leq s \leq p$, $0 \leq m \leq s$, and it is assumed that the triplet (σ, ϕ, η) is admissible. Then, for $u_0 \in \mathcal{L}_2 \cap H_d^{m+s}(\mathbb{R}^n)$, $0 \leq t \leq T$, with h small enough, we get*

$$\|u - F_h(t)[u_0]\|_m \leq ch^{s-r} \|u_0\|_{s+m,d}, \tag{4.9}$$

where c is independent of u , h , and u_0 .

PROOF. It follows from (4.8) that

$$\left| at\sigma(\xi) - \frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)} \right| \leq ch^{p-r} |\xi|^p \quad \text{as } |h\xi| \leq 1. \tag{4.10}$$

The equality

$$e^{ta} - e^{tb} = t(a-b) \int_0^1 e^{sta+(1-s)tb} ds, \tag{4.11}$$

(4.10), and (1.3) imply that, for $r \leq s \leq p$ and $0 \leq t \leq T$,

$$\left| \exp(at\sigma(\xi)) - \exp\left(\frac{at}{h^r} \frac{\tilde{\delta}(\xi)}{\tilde{\alpha}(\xi)}\right) \right| \leq ch^{s-r} |\xi|^s \quad \text{as } |h\xi| \leq 1. \tag{4.12}$$

Hence, from (4.7) and (4.12), we obtain

$$|\mathcal{F}(u(\cdot, t) - F_h(t)[u_0])(\xi)| \leq ch^{s-r} |\xi|^s |\hat{u}_0(\xi)| \quad \text{as } |h\xi| \leq 1. \tag{4.13}$$

By (1.3) and the admissibility of the triplet (σ, ϕ, η) , inequality (4.13) is also valid for all $\xi \in \mathbb{Z}^n$. Hence, for each $0 \leq m \leq s$, $r + s' \leq s \leq p$, and $0 \leq t \leq T$, we get

$$\begin{aligned} \|u - F_h(t)[u_0]\|_m^2 &= \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2m} |\mathcal{F}\{u(\cdot, t) - F_h(t)[u_0]\}(\xi)|^2 \\ &\leq ch^{2(s-r)} \sum_{\xi \in \mathbb{Z}^n} \langle \xi \rangle^{2(m+s)} |\hat{u}_0(\xi)|^2 \\ &\leq ch^{2(s-r)} \|u_0\|_{m+s,d}^2. \end{aligned} \tag{4.14}$$

The theorem is thus proved. □

From the admissibility of the triplet (σ, ϕ, η) and (1.3), it follows that $F_h(t) : H^m(\mathbb{R}^n) \rightarrow H^m(\mathbb{R}^n)$, $0 \leq m \leq s$, is a continuous linear operator. Consequently,

$$\|F_h(t)([u_0] - R_h[u_0])\|_m \leq c\|[u_0] - R_h[u_0]\|_m. \quad (4.15)$$

Therefore, if we assume that

$$\|(I - R_h)[u_0]\|_m \leq ch^s\|[u_0]\|_{m+s}, \quad (4.16)$$

then

$$\|F_h(t)([u_0] - R_h[u_0])\|_m \leq ch^s\|[u_0]\|_{m+s}. \quad (4.17)$$

REMARK 4.3. It follows from (2.23) that the assumption (4.17) is satisfied, when $R_h = P_j$ for $0 \leq m$, $m + s \leq \mu + 1$.

Thus from (4.5), (4.9), and (4.17), we obtain the following theorem.

THEOREM 4.4. *If all the hypotheses of Theorem 4.2 and assumption (4.17) are satisfied, then*

$$\|u - u_h\|_m \leq ch^{s-r}\|u_0\|_{m+s,d} + ch^s\|[u_0]\|_{m+s}, \quad (4.18)$$

where c is independent of u_0 , h .

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