STRONG BOUNDEDNESS OF ANALYTIC FUNCTIONS IN TUBES

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ABSTRACT. Certain classes of analytic functions in tube domains $T^C = \mathbb{R}^n + iC$ in $n$-dimensional complex space, where $C$ is an open connected cone in $\mathbb{R}^n$, are studied. We show that the functions have a boundedness property in the strong topology of the space of tempered distributions $\mathcal{S}'$. We further give a direct proof that each analytic function attains the Fourier transform of its spectral function as distributional boundary value in the strong (and weak) topology of $\mathcal{S}'$.

KEY WORDS AND PHRASES. Analytic Function in Tubes, Strong Boundedness, Tempered Distributions, Distributional Boundary Value.


1. INTRODUCTION.

Vladimirov [1, p. 230] has defined the spectral function $V_t$ of a function $f(z)$ which is analytic in a tubular domain $T^B = \mathbb{R}^n + iB$ to be the distribution
$V_t \in \mathcal{S}'$, the space of distributions of L. Schwartz [2], which possesses the following properties:

\begin{align}
e^{-yt} V_t \in \mathcal{S}' & \text{ for all } y \in \mathbb{B}; \\
f(z) = \langle V_t, e^{izt} \rangle & \text{ for all } z \in \mathbb{T}.
\end{align}

Here $\mathcal{S}'$ is the space of tempered distributions of Schwartz [2] and $\langle V_t, e^{izt} \rangle$ is the Fourier-Laplace transform of the spectral function $V_t$.

In [3] Vladimirov defined certain classes of analytic functions in tubular cones $\mathbb{T}^C = \mathbb{R}^n + i\mathbb{C}$, where $\mathbb{C}$ is an open cone, and analyzed the spectral functions of these analytic functions corresponding to $\mathbb{C}$ being an open connected cone. The results of [3] have been incorporated into the book [1] of Vladimirov [1, section 26.4].

In this paper we add information to the main results of [3] and [1, section 26.4] which are [1, pp. 238-239, Theorems 1 and 2]. We show that the analytic functions considered by Vladimirov in these results have boundedness properties in the strong topology of the space of tempered distributions $\mathcal{S}'$. Further, we give a direct proof by elementary means that each analytic function attains the Fourier transform of its spectral function as distributional boundary value in the strong (and weak) topology of $\mathcal{S}'$, a fact which has been recognized by Vladimirov [1, p. 238] and which is obtained by him as a special case of a more general result.

2. NOTATION AND DEFINITIONS.

Our $n$-dimensional notation is that of Vladimirov [1, p. 1]. $x$, $y$, and $t$ will be points in $\mathbb{R}^n$ in this paper and $z \in \mathbb{C}^n$, $n$-dimensional complex space. Note the inner products $zt = z_1 t_1 + \ldots + z_n t_n$ and $yt = y_1 t_1 + \ldots + y_n t_n$ for $t$ and $y$ in $\mathbb{R}^n$ and $z \in \mathbb{C}^n$. Note also the differential operator $D^\alpha$ in [1, p. 1], and we shall write $D^\alpha_z$ or $D^\alpha_t$ to indicate that the differentiation is with respect to $z$ or $t$, respectively. Here $\alpha$ is an $n$-tuple of nonnegative integers. The
definitions of cone $C$ in $\mathbb{R}^n$, compact subcone of a cone, indicatrix $u_C(t)$ of a cone, and of the number $\rho_C$, which characterizes the nonconvexity of a cone $C$, can all be found in [1, section 25.1]. Note that $\rho_C \geq 1$ [1, p. 220] for any cone $C$. The cone $C^* = \{t \in \mathbb{R}^n : yt \geq 0, y \in C\}$ is the dual cone of $C$ and $C'$ will denote $C^* = \mathbb{R}^n \setminus C^*$. $0(C)$ will denote the convex envelope (hull) of the cone $C$, and we define the tubes $T^C$ and $T^0(C)$ by $T^C = \mathbb{R}^n + iC$ and $T^0(C) = \mathbb{R}^n + i0(C)$, respectively.

Let $C$ be a cone in $\mathbb{R}^n$. We make the convention throughout this paper that by $z \in T^C(\in T^0(C))$ and $y \in C(\in 0(C))$ we mean that $z \in T^C(\in T^0(C))$ and $y \in C(\in 0(C))$ for an arbitrary compact subcone $C' \subset C (C' \subset 0(C))$.

The space of functions of rapid decrease $\mathcal{S} = \mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}' = \mathcal{S}'(\mathbb{R}^n)$ are defined and discussed in Schwartz [2, Chapter 7]. The Fourier (inverse Fourier) transform of an $L^1(\mathbb{R}^n)$ function $\phi(t)$, denoted $\mathcal{F}[\phi(t); x]$ ($\mathcal{F}^{-1}[\phi(t); x]$), will be as defined in Vladimirov [1, p. 21]. The Fourier transform of a tempered distribution $V_t$, denoted $\mathcal{F}[V_t]$, is defined in Schwartz [2, p. 250, (VII 6; 6)]. All terminology and definitions concerning distributions in this paper, such as support of a distribution, will be that of Schwartz [2].

Let $C$ be an open connected cone. The analytic function $f(z)$, $z \in T^C$, obtains $U \in \mathcal{S}'$ as boundary value in the weak topology of $\mathcal{S}'$ if

$$\lim_{y \to 0} \langle f(x + iy), \phi(x) \rangle = \langle U, \phi \rangle$$

(2.1)

for each $\phi \in \mathcal{S}$. $U \in \mathcal{S}'$ is the boundary value of $f(z)$ in the strong topology of $\mathcal{S}'$ if the convergence (2.1) holds uniformly for $\phi$ varying over arbitrary bounded sets in $\mathcal{S}$. The set $\{U_y \in \mathcal{S}' : y \in C\}$, where $U_y \in \mathcal{S}'$ in some sense depends on $y \in C$, is said to be a bounded set in the strong topology of $\mathcal{S}'$ if for any bounded set $\phi$ in $\mathcal{S}$, $\{(U_y, \phi) : \phi \in \mathcal{S}, y \in C\}$ is a bounded set in the complex plane.
3. **THEOREMS OF VLADIMIROV.**

Let \( C \) be an open cone. A function \( f(z) \) belongs to the class \( H_p(a;C) \), where \( p \geq 1 \) and \( a \geq 0 \), if \( f(z) \) is analytic in the tubular cone \( T^C \) and, for an arbitrary compact subcone \( C' \) in \( C \), the inequality

\[
|f(z)| \leq M(C') \left(1 + |z| \right)^N \left(1 + |y|^{-K}\right) e^{a|y|^p}, \quad z = x+iy \in T^C',
\]

is satisfied where \( M(C') \) is a constant which depends at most on the compact subcone \( C' \subset C \) and \( N \) and \( K \) are nonnegative real numbers which do not depend on \( C' \subset C \). We define

\[
H_p(a+\varepsilon;C) = \bigcap_{a' > a} H_p(a';C), \quad H_0(C) = H_1(0;C).
\]

For the convenience of the reader we now state the theorems of Vladimirov with which we are concerned in this paper.

**Theorem 1.** [1, p. 238] Let \( f(z) \in H_p(a+\varepsilon;C) \), where \( C \) is an open connected cone, \( p \geq 1 \), and \( a > 0 \). The spectral function \( V_t \) of \( f(z) \) can be represented in the form of a finite sum of distributional derivatives of continuous functions \( g_\alpha(t) \) of power increase,

\[
V_t = \sum_{\alpha} D_\alpha^\alpha(g_\alpha(t))
\]

which, for all \( t \in C_*' \), where \( C_*' \) is an arbitrary compact subcone of \( C_* = \mathbb{R}^\mathbb{R} \setminus C_* \), and for all \( \varepsilon > 0 \), satisfy

\[
|g_\alpha(t)| \leq M_\varepsilon(C_*') \exp[-(a'-\varepsilon)(u_0(t))^p']
\]

where the numbers \( p \) and \( a \) are connected with \( p' \) and \( a' \) by the relations

\[
\frac{1}{p} + \frac{1}{p'} = 1, \quad (p'a')^p(pa)^{p'} = 1.
\]

Conversely, if \( V_t \) satisfies these conditions for certain numbers \( a' > 0, \ p' > 1 \) and the cone \( C_*' \), then all derivatives \( D_\beta^\beta(f(z)) \) of its Fourier-Laplace transform \( f(z) \) belong to the class \( H_p(a_0p + \varepsilon;0(C)) \).

Notice that the \( C_* \) as printed in [1, p. 239, line 8] should be \( C_*' \) instead as we have written in Theorem 1.
THEOREM 2. [1, p. 239] Let \( f(z) \in H^1_{\lambda}(a + \xi;C) \) where \( C \) is an open connected cone and \( \lambda \geq 0 \). Then its spectral function \( V_t \in \mathcal{S}' \) and \( V_t \) has support in \( \{ t : u_\lambda C(t) \leq a \} \). Conversely, if \( V_t \in \mathcal{S}' \) and has support in \( \{ t : u_\lambda C(t) \leq a \} \) for some \( a \geq 0 \) and some open connected cone \( C \), then all the derivatives \( D^\beta_z f(z) \) of the Fourier-Laplace transform \( f(z) \) of \( V_t \) belong to the class \( H^1_{\lambda}(a \xi C';0(C)) \).

4. LEMMAS.

As noted in the introduction, we shall add information to Theorems 1 and 2. We shall show that the analytic functions in these theorems have a strong boundedness property in \( \mathcal{S}' \). In addition we give a direct proof that the analytic functions attain the Fourier transform of their spectral functions as distributional boundary values in the strong (and weak) topology of \( \mathcal{S}' \).

The following lemma is the basis of the boundary value result, and its proof in turn is useful in obtaining our strong boundedness properties. Throughout this section \( C \) is an open connected cone.

**LEMMA 1.** Let \( f(z) \in H^p_{\lambda}(a + \xi;C) \), \( p > 1 \) and \( \lambda > 0 \). The spectral function \( V_t \) of \( f(z) \) is in \( \mathcal{S}' \) as is \( e^{-yt} V_t \), \( y \in 0(C) \), and

\[
\lim_{y \to 0} \mathcal{F}[e^{-yt} V_t] = \mathcal{F}[V] \quad \text{in the strong (and weak) topology of } \mathcal{S}'.
\]

**PROOF.** Let \( C' \) be an arbitrary compact subcone of \( 0(C) \). By the sufficiency of Theorem 1, the spectral function \( V_t \) of \( f(z) \) has the representation (3.2). Since each \( g_{\alpha}(t) \) in (3.2) is continuous and of power increase over \( \mathbb{R}^n \), we immediately have \( V_t \in \mathcal{S}' \). The fact that \( e^{-yt} V_t \in \mathcal{S}' \), \( y \in C' \subseteq 0(C) \), follows by the proof of Theorem 1 given in [1, section 26.5].

Let \( \phi \) be an arbitrary element of \( \mathcal{S} \). Using the notion of distributional differentiation and the generalized Leibnitz rule, we have for \( y \in C' \subseteq 0(C) \) that
\[ \langle v, (e^{-yt} - 1)\phi(t) \rangle = \]
\[ = \sum_{\alpha} (-1)^{\alpha} \int_{\mathbb{R}^n} g_{\alpha}(t) \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} D_{t}^{\beta}(e^{-yt} - 1) D_{t}^{\gamma}(\phi(t)) \, dt \quad (4.2) \]
\[ = \sum_{\alpha} (-1)^{\alpha} \sum_{\beta + \gamma = \alpha} \frac{\alpha!}{\beta! \gamma!} I_{y}(\alpha, \beta, \gamma) \]

where \( \alpha, \beta, \) and \( \gamma \) are \( n \)-tuples of nonnegative integers and

\[ I_{y}(\alpha, \beta, \gamma) = \int_{\mathbb{R}^n} g_{\alpha}(t) (-1)^{\beta} y^{\beta} e^{-yt} - D_{t}^{\beta}(1) D_{t}^{\gamma}(\phi(t)) \, dt. \quad (4.3) \]

For the arbitrary \( C \subset 0(C) \) we apply [1, p. 223, Lemma 2] to obtain a number \( \delta = \delta(C') > 0 \) and an open cone \( (C^\star)' \), both depending on \( C' \), such that \( (C^\star)' \) contains the cone \( C^\star = \{t \in \mathbb{R}^n : yt \geq 0, y \in C, y \in C, t \in (C^\star)' \). (4.4)

Put \( C_{\star}' = \mathbb{R}^n \setminus (C^\star)' \). \( C_{\star}' \) is a compact subcone of \( C_{\star} = \mathbb{R}^n \setminus C^\star \), and we have \( C_{\star}' \cap (C^\star)' = \emptyset \) and \( C_{\star}' \cup (C^\star)' = \mathbb{R}^n \). We now write the integral \( I_{y}(\alpha, \beta, \gamma) \) in (4.3) as

\[ I_{y}(\alpha, \beta, \gamma) = I_{y}^{1}(\alpha, \beta, \gamma) + I_{y}^{2}(\alpha, \beta, \gamma) \quad (4.5) \]

where

\[ I_{y}^{1}(\alpha, \beta, \gamma) = \int_{(C^\star)'} g_{\alpha}(t) (-1)^{\beta} y^{\beta} e^{-yt} - D_{t}^{\beta}(1) D_{t}^{\gamma}(\phi(t)) \, dt \quad (4.6) \]
\[ I_{y}^{2}(\alpha, \beta, \gamma) = \int_{C_{\star}'} g_{\alpha}(t) (-1)^{\beta} y^{\beta} e^{-yt} - D_{t}^{\beta}(1) D_{t}^{\gamma}(\phi(t)) \, dt. \]

For any \( n \)-tuple \( \beta \) of nonnegative integers we have

\[ (-1)^{\beta} y^{\beta} e^{-yt} - D_{t}^{\beta}(1) = \begin{cases} e^{-yt} - 1, & \beta = (0, \ldots, 0), \\
(-1)^{\beta} y^{\beta} e^{-yt}, & \beta \neq (0, \ldots, 0), \end{cases} \quad (4.7) \]

for all \( y \in C \subset 0(C) \) and in fact for all \( y \in \mathbb{R}^n \); hence for any \( \alpha \) in the last sum in (4.2) and any subsequent \( \beta \) and \( \gamma, \beta + \gamma = \alpha, \) (4.7) yields
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\[
\lim_{y \to 0} \int_{y \in 0(C)} g_\alpha(t) \left((-1)\beta y^\beta e^{-yt} - d^\beta_t(1) d^\gamma_t(\phi(t))\right) = 0
\]

(4.8)

for all \( t \in \mathbb{R}^n \). (The limit (4.8) actually holds as \( y \to 0, \ y \in \mathbb{R}^n \), because (4.7) holds for all \( y \in \mathbb{R}^n \).)

Recall that we desire a convergence result in this lemma as \( y \to 0, \ y \in 0(C) \). Hence to obtain (4.1) it suffices to consider \( y \in 0(C) \) such that

\[ |y| \leq Q \text{ for } Q > 0 \text{ fixed.} \]

Now consider the integrand of the integral

\[ I^1_y(\alpha, \beta, \gamma) \] in (4.6) for \( t \in (C^*)' \). Since each \( g_\alpha(t) \) in (3.2) is of power increase over \( \mathbb{R}^n \), we have the existence of a polynomial \( P_\alpha(t) \) corresponding to each \( g_\alpha(t) \) such that

\[ |g_\alpha(t)| \leq P_\alpha(|t|) , \ t \in \mathbb{R}^n . \] (4.9)

Using (4.9) and (4.4) we get

\[
\left|g_\alpha(t) \left((-1)\beta y^\beta e^{-yt} - d^\beta_t(1) d^\gamma_t(\phi(t))\right)\right| \\
\leq P_\alpha(|t|) (1 + |\beta| |\exp(-\delta|y| |t|)|) |d^\gamma_t(\phi(t))| \\
\leq P_\alpha(|t|) (1 + |\beta|) |d^\gamma_t(\phi(t))| \\
\] (4.10)

for \( t \in (C^*)' \) and \( y \in C \subseteq 0(C) \) such that \( |y| \leq Q \). Since \( \phi \in \mathcal{B} \), the right side of the last inequality in (4.10) is an \( L^1 \) function over \( \mathbb{R}^n \) which is independent of \( y \in C \subseteq 0(C) \) such that \( |y| \leq Q \). Using this fact, (4.8), and the Lebesgue dominated convergence theorem we obtain

\[
\lim_{y \to 0} \int_{y \in 0(C)} I^1_y(\alpha, \beta, \gamma) = 0
\]

(4.11)

for any \( \alpha \) in (4.2) and any subsequent \( \beta \) and \( \gamma, \ \beta + \gamma = \alpha \).

We now consider the integrand of the integral \( I^2_y(\alpha, \beta, \gamma) \) in (4.6) for \( t \in C^*_\gamma \). For such \( t \) each \( g_\alpha(t) \) in (3.2) satisfies (3.3). Using (3.3), the relations (3.4), the facts

\[
-yt \leq |y| u_0(C)(t) , \ u_0(C)(t) \leq \rho_C u_C(t) , \ t \in C^*_\gamma , \ y \in 0(C)
\]

(4.12)
contained in [1, section 25.1], and analysis as in [1, p. 244], we have for
t \in C_0 \subset C_\infty and y \in C \subset O(C) such that |y| \leq Q that

$$|g_\alpha(t)((-l)|\beta| y^\beta e^{-yt} - D_t^\beta(1) D_t^\gamma(\phi(t))| \leq$$

$$\leq M(E, C) \exp[-(a'-\varepsilon)(u_C(t))^p'] (1 + |y| \beta \exp[|y| \rho_C u_C(t)]) D_t^\gamma(\phi(t)) | (4.13)$$

$$\leq M(E, C) (1 + |y| \beta \exp[-(a'-\varepsilon)(u_C(t))^p' + |y| \rho_C u_C(t)]) D_t^\gamma(\phi(t)) | (4.13)$$

$$\leq M(E, C) (1 + Q |\beta| |\exp[\frac{1}{p} (\frac{1}{p'} (a' - 2\varepsilon) \rho_C \rho_C') P \rho_C^P] D_t^\gamma(\phi(t)) |$$

for all t \in C_0 \subset C_\infty and y \in C \subset O(C) such that |y| \leq Q. Since \phi \in \mathcal{G}
the right side of (4.14) is an L^1 function over \mathbb{R}^n and is independent of
y \in C \subset O(C) such that |y| \leq Q. Thus by (4.14), (4.8), and the Lebesgue
dominated convergence theorem we have

$$\lim_{y \to 0} 1_{(\alpha, \beta, \gamma)}(y) = 0$$

(4.15)

for each relevant \alpha, \beta, and \gamma. Combining (4.5), (4.11), and (4.15) we get

$$\lim_{y \to 0} 1_{(\alpha, \beta, \gamma)}(y) = 0$$

(4.16)

for each \alpha in (4.2) and each \beta and \gamma, \beta + \gamma = \alpha. Since \phi is an
arbitrary element of \mathcal{G}, we combine (4.2) and (4.16) to yield
\[
\lim_{y \to 0} e^{-yt} V_t = V_t \quad (4.17)
\]
in the weak topology of \( \mathcal{S}' \). But \( \mathcal{S} \) is a Montel space ([1, p. 21] and [4, p. 510]). Hence by Edwards [4, p. 510, Corollary 8.4.9] the convergence (4.17) is in the strong topology of \( \mathcal{S}' \) also. Since the Fourier transform on \( \mathcal{S}' \) [2, Chapter 7] is a strongly continuous mapping of \( \mathcal{S}' \) onto \( \mathcal{S}' \), the desired convergence (4.1) now follows in the strong (and weak) topology of \( \mathcal{S}' \). The proof is complete.

The next lemma is the basis of our strong boundedness results concerning the analytic functions \( H_p(a + \xi; C) \), \( p > 1 \) and \( a > 0 \).

**Lemma 2.** Let \( p > 1 \) and \( a > 0 \). Let \( C \) be an open connected cone. Let \( V_t \) be any generalized function of the form (3.2) where the \( g_\alpha(t) \) satisfy the conditions stated in Theorem 1. Then \( V_t \in \mathcal{S}' \), \( (e^{-yt} V_t) \in \mathcal{S}' \) for all \( y \in \mathcal{O}(C) \), and \( \{ e^{-yt} V_t \} \in \mathcal{S}' : y \in \mathcal{O}(C) \), \( |y| < Q \} \) is a strongly bounded set in \( \mathcal{S}' \) for \( Q > 0 \) being arbitrary but fixed.

**Proof.** Let \( C \) be an arbitrary compact subcone of \( \mathcal{O}(C) \). The facts that \( V_t \in \mathcal{S}' \) and \( (e^{-yt} V_t) \in \mathcal{S}' \) for all \( y \in \mathcal{C}' \subseteq \mathcal{O}(C) \) follow as at the beginning of the proof of Lemma 1. The locally convex topology of \( \mathcal{S} \) is defined by the norms
\[
\| \phi \|_k = \sup_{t \in \mathbb{R}^n} |t|^k |D^\alpha(\phi(t))|, \quad k = 1, 2, 3, \ldots . \quad (4.18)
\]
Let \( \phi \) be an arbitrary bounded set in \( \mathcal{S} \). For the arbitrary \( \mathcal{C}' \subseteq \mathcal{O}(C) \) we apply [1, p. 223, Lemma 2] as in the proof of Lemma 1 and obtain a number \( \delta = \delta(C') > 0 \) and an open cone \( (C^*)' \), both depending on \( \mathcal{C}' \), such that \( (C^*)' \) contains the cone \( C^* \) and (4.4) holds. We then put \( C^*_* = \mathbb{R}^n \setminus (C^*)' \), and \( C^*_* \) is a compact subcone of \( C_* = \mathbb{R}^n \setminus C^* \) as in the proof of Lemma 1. Using the form of \( V_t \) in (3.2) and the generalized Leibnitz rule we obtain for any \( \phi \in \Phi \) and \( y \in \mathcal{C}' \subseteq \mathcal{O}(C) \) that
\[
\langle e^{-yt} v_t, \phi(t) \rangle = \sum_{\alpha} (-1)^{|\alpha|} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (-1)^{|\beta|} y^\beta (I^1_y(\alpha, \gamma) + I^2_y(\alpha, \gamma)) \quad (4.19)
\]

where

\[
I^1_y(\alpha, \gamma) = \int_{(C^*)^*} g_\alpha(t) e^{-yt} D_t^\gamma(\phi(t)) \, dt
\]

\[
I^2_y(\alpha, \gamma) = \int_{(C^*)^*} g_\alpha(t) e^{-yt} D_t^\gamma(\phi(t)) \, dt.
\]

Using (4.4), (4.18), and the fact that each \( g_\alpha(t) \) satisfies (4.9) for some polynomial \( P_\alpha(t) \), we have

\[
|I^1_y(\alpha, \gamma)| \leq \int_{(C^*)^*} P_\alpha(|t|) \exp[-\delta|y||t|] |D_t^\gamma(\phi(t))| \, dt
\]

\[
\leq \int_{(C^*)^*} P_\alpha(|t|) (1 + |t|)^{n+1} |D_t^\gamma(\phi(t))| (1 + |t|)^{-n-1} \, dt \quad (4.21)
\]

\[
\leq R_\alpha \frac{||\phi||_{k_\alpha}}{\mathbb{R}^n} \int (1 + |t|)^{-n-1} \, dt
\]

where \( R_\alpha \) is a constant and \( k_\alpha \) is a positive integer with both depending on \( \alpha \); and (4.21) holds for each \( \alpha \) and \( \gamma \), \( \alpha = \beta + \gamma \), in (4.19). Also recall that each \( g_\alpha(t) \) satisfies (3.3). Using (3.3), (4.12), and analysis as in (4.21), (4.13), and (4.14) we have for \( y \in C \subseteq O(C) \) that

\[
|I^2_y(\alpha, \gamma)| \leq M_1'(C^*) \int_{(C^*)^*} \exp[-(a'-\xi)(u_c(t))' \, P_1' \, \exp[-(y|\rho_c u_c(t)) |D_t^\gamma(\phi(t))| \, dt
\]

\[
\leq M_2'(C^*) \frac{||\phi||_{k_\alpha}}{C^*} \int_{(C^*)^*} \exp[-(a'-\xi)(u_c(t))' + |y| |\rho_c u_c(t)| (1 + |t|)^{-n-1} \, dt \quad (4.22)
\]

\[
\leq M_3'(C^*) \frac{||\phi||_{k_\alpha}}{C^*} \int_{(C^*)^*} \exp[\frac{1}{p'}(\frac{1}{a'-2\xi})^{-p'/p'} \rho_c |y|^p] \int_{\mathbb{R}^n} (1 + |t|)^{-n-1} \, dt
\]

where \( M_3'(C^*) \) is a constant and \( k_\alpha \) is a positive integer depending on \( \alpha \).

Because of (3.3), we can assume that \( \xi > 0 \) in (4.22) is fixed such that \( (a' - 2\xi) > 0 \). Since (4.22) holds for each \( \alpha \) and \( \gamma \), \( \beta + \gamma = \alpha \), in (4.19)
and since $\Phi$ is a bounded set in $\mathcal{S}$, it follows from the combination of 
(4.19), (4.20), (4.21), and (4.22) that

$$\{e^{-yt} V_t, \phi(t) : \phi \in \Phi, y \in \mathcal{O}(C), |y| \leq Q\}$$

is a bounded set in the complex plane for $Q > 0$ arbitrary but fixed. Since $\Phi$ was assumed to be an arbitrary bounded set in $\mathcal{S}$, this proves that $\{e^{-yt} V_t : y \in \mathcal{O}(C), |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}$; hence $\{\mathcal{F}[e^{-yt} V_t] : y \in \mathcal{O}(C), |y| \leq Q\}$ is a strongly bounded set in $\mathcal{S}'$ since the Fourier transform in $\mathcal{S}'$ [2, Chapter 7] is a strongly continuous mapping from $\mathcal{S}'$ onto $\mathcal{S}'$. The proof is complete.

5. ADDITIONS TO THEOREMS 1 AND 2.

Let us now consider Theorem 1. Let $C$ be an open connected cone. Let

$$f(z) \in H_p(a + \varepsilon; C), p > 1 \quad \text{and} \quad a > 0.$$ 

By the sufficiency of Theorem 1 we have that the spectral function $V_t$ of $f(z)$ has the form (3.2) and

$$f(z) = \langle V_t, e^{izt} \rangle, \quad z \in C.$$  \hspace{1cm} (5.1)

(Recall (1.2).) Further note that $V_t \in \mathcal{S}'$ and $(e^{-yt} V_t) \in \mathcal{S}'$ for all $y \in \mathcal{O}(C)$ as obtained in the proofs of Lemmas 1 and 2. For any fixed $y \in C$, $f(x + iy) \in \mathcal{S}'$ as a function of $x \in \mathbb{R}^n$ because of the growth (3.1) defining the $H_p(a + \varepsilon; C)$ spaces. Let $\psi \in \mathcal{S}$ and let $\phi \in \mathcal{S}$ be that unique element of $\mathcal{S}$ such that $\phi(t) = \mathcal{F}[\psi(x); t]$ [2, Chapter 7]. Using (5.1), (3.2), distributional differentiation, a change of order of integration, and differentiation under the integral sign we get

$$\langle f(z), \psi(x) \rangle = \sum_{\alpha} (-1)^{|\alpha|} i^{|\alpha|} \int_{\mathbb{R}^n} z^\alpha \psi(x) \int_{\mathbb{R}^n} g_\alpha(t) e^{izt} \, dt \, dx$$

$$= \sum_{\alpha} (-1)^{|\alpha|} \int_{\mathbb{R}^n} g_\alpha(t) (D^\alpha \int_{\mathbb{R}^n} \psi(x) e^{izt} \, dx) \, dt.$$  \hspace{1cm} (5.2)

But if $\phi(t) = \mathcal{F}[\psi(x); t]$ then

$$e^{-yt} \phi(t) = \int_{\mathbb{R}^n} \psi(x) e^{izt} \, dx.$$  \hspace{1cm} (5.3)
Putting (5.3) into (5.2) and using the Fourier transform on $S'$ [2, Chapter 7] we have

$$\langle f(z), \psi(x) \rangle = \sum_{\alpha} (-1)^{\alpha} \int_{\mathbb{R}^n} g_{\alpha}(t) \left( D_{\alpha}^{\mu}(e^{-yt} \phi(t)) \right) dt$$

$$= \left( e^{-yt} V_t, \phi(t) \right) = \left( \mathcal{F}[e^{-yt} V_t], \psi(x) \right)$$

for all $y = \text{Im}(z) \in C$ which proves that

$$f(z) = \mathcal{F}[e^{-yt} V_t], z = x + iy \in \mathbb{T}^C,$$

with this equality holding in $S'$. Thus by combining (5.5) and Lemma 2 we can also conclude in the sufficiency of Theorem 1 that

$$\{ f(z) : y = \text{Im}(z) \in C, |y| \leq Q \}$$

is a strongly bounded set in $S'$ for $Q > 0$ being arbitrary but fixed. Further, by combining (5.5) and Lemma 1 we have obtained a direct proof of the fact that

$$\lim_{y \to 0} f(x + iy) = \mathcal{F}[V]$$

in the strong (and weak) topology of $S'$.  

In the converse of Theorem 1 Vladimirov proves that if $V_t$ has the form (3.2) then all derivatives $D^\alpha f(z)$ of the Fourier-Laplace transform

$$f(z) = \langle V_t, e^{ixt} \rangle$$

of $V_t$ belong to the class $H_p(a \rho C + \varepsilon;O(\mathbb{C}))$, $C$ being an open connected cone. By the analysis in (5.2), (5.3), and (5.4) we conclude that (5.5) holds in this converse also for $z = x + iy \in T^0(\mathbb{C})$. Then combining this fact with Lemmas 1 and 2 we add the conclusions to the converse of Theorem 1 that

$$\{ f(x) : y = \text{Im}(z) \in O(\mathbb{C}), |y| \leq Q \}$$

is a strongly bounded set in $S'$, where $Q > 0$ is arbitrary but fixed, and (5.6), with $C$ replaced by $O(\mathbb{C})$, holds in the strong (and weak) topology of $S'$.

We now consider Theorem 2. For the element $f(z) \in H_1(a + \varepsilon;C)$ ($\in H_1(a \rho C;O(\mathbb{C}))$ in the converse), $a \geq 0$, and its corresponding spectral function $V_t \in S'$ in both the sufficiency and necessity of this theorem, we can
prove lemmas like Lemmas 1 and 2. Then using techniques as in our preceding additions to Theorem 1 we have the conclusions in both the sufficiency and necessity of Theorem 2 that

$$f(z) = \mathcal{F}[e^{-yt} v_t] , \quad z = x + iy \in T^C (\mathcal{E} T^0(C) \text{ in the converse}),$$

with this equality holding in $\mathcal{G}'$; \{f(z) : y = \text{Im}(z) \in C (\mathcal{E} 0(C) \text{ in the converse}) , |y| \leq Q\} is a strongly bounded set in $\mathcal{G}'$ for $Q > 0$ being arbitrary but fixed; and (5.6) holds in the strong (and weak) topology of $\mathcal{G}'$ with $0(C)$ replacing $C$ in the converse. The now evident details are left to the interested reader.

Let us also note the generalization of Theorems 1 and 2 given by Vladimirov in [1, section 26.7] concerning functions $f(z) \in H_p(a + \varepsilon; C)$ which are analytic in tubular cones $T^C$ where $C$ is an open cone that is the union of a finite number of open connected component cones $C_k, k = 1, 2, \ldots, r$. By our analysis in this paper one can also conclude our strong boundedness property in $\mathcal{G}'$ for the analytic function $f(z) \in H_p(a + \varepsilon; C)$ in [1, p. 247, Theorem] in each of the connected components $T^C_k, k = 1, 2, \ldots, r$, of $T^C$ and for the analytic extension function $f(z)$ in the conclusion of this result of Vladimirov for $z \in t^0(C)$.

The Theorems 1 and 2 of Vladimirov have recently motivated this author to define more general spaces of analytic functions in tubes than the $H_p(a; C)$ and $H_p(a + \varepsilon; C)$ spaces. The associated spectral functions are distributions of exponential growth, a class of distributions which contains the tempered distributions $\mathcal{G}'$. Our analysis will appear in [5].

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REFERENCES


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