REALCOMPACTIFICATION AND REPLETENESS OF WALLMAN SPACES

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ABSTRACT. The extension of bounded lattice continuous functions on an arbitrary set $X$ to the set of lattice regular zero-one measures on an algebra generated by a lattice (a Wallman-type space) is investigated.

Next the subset of lattice regular zero-one measures on an algebra generated by a lattice which integrates all lattice continuous functions on $X$ is introduced and various properties of it are presented.

Finally conditions are established using repleteness criteria whereby the space of lattice regular zero-one measures on an algebra generated by a lattice which are countably additive (a Wallman-type space) is realcompact.

KEY WORDS AND PHRASES. Realcompact, repleteness, Wallman spaces, normal lattice, lattice continuous functions.

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1. INTRODUCTION.

Let $X$ be an arbitrary set and $L$ a lattice of subsets of $X$. $A(L)$ denotes the algebra generated by $L$, and $M(L)$ those bounded finitely additive measures on $A(L)$, and $M^R(L)$ those elements of $M(L)$ which are $L$-regular while $M^R_s(L)$ denotes those elements of $M^R(L)$ which are countably additive. The zero-one valued members of the above are designated by $I(L)$, $I^R(L)$, and $I^R_s(L)$ respectively. For $A \in A(L)$, $u(A) = \{ u \in I^R(L) \mid u(A) = 1 \}$, $w(L) = \{ w \in I(L) \mid L \in L \}$. Then $I^R(L)$ with the topology of closed sets $\tau w(L)$ of arbitrary intersections of sets of $w(L)$ is a compact, $T_1$ topological space. It is one of the Wallman type spaces. Assuming $L$ is disjunctive then it is $T_2$ if and only if $L$ is normal.

We begin by considering briefly, because of their importance, certain fundamental properties of normal lattices. Then we proceed to a consideration of $I^R(L)$, and the extension of bounded lattices continuous functions on $X$ to $I^R(L)$. These results are generally known (see [8]) but we give somewhat shorter more direct proofs here.

We next consider the space $Q(L)$ of measures in $I^R(L)$ which integrate all lattice continuous functions on $X$, and show its relationship to $I^R_s(L)$, and under suitable conditions, its relationship to the $G_\delta$-closure of $X$ in $I^R(L)$.
Finally, we consider the Wallman type space $I^\sigma_R(L)$, and the lattice $w_\sigma(L)$, where for $A \in A(L)$, $w_\sigma(A) = \{ u \in I^\sigma_R(L) \mid u(A) = 1 \}$, and where $w_\sigma(L) = \{ w_\sigma(A) \mid L \in L \}$. It is well-known that if $L$ is disjunctive then $w_\sigma(L)$ is replete. We consider in this space the lattice of closed sets $rw_\sigma(L)$ and its associated lattice of zero sets, and investigate their repleteness - thus obtaining sufficient conditions for the space $I^\sigma_R(L)$ to be realcompact.

Our notations and terminology is consistent with [1, 3, 5, 6, 11]. However, the main definitions and notations used throughout the paper are presented for the reader’s convenience in section 2(a). We note also that a number of results on normal lattices in section 2(b) are related to work of [4, 9].

2.(a) BACKGROUND AND NOTATION.

Let $X$ be an abstract set, and $L$ the lattice of subsets of $X$. We assume that $\phi, X \in L$ for most of our results. First:

Lattice Terminology:

$A(L)$ is the algebra generated by $L$.

$\sigma(L)$ is the $\sigma$-algebra generated by $L$.

$\delta(L)$ is the lattice of all countable intersections of sets from $L$. $L$ is a delta lattice ($\delta$-lattice) if $\delta(L) = L$.

$\tau(L)$ is the lattice of arbitrary intersections of sets of $L$.

$L$ is complemented if $L \in L > L' \in L$ (prime denotes complement), that is, $L$ is an algebra.

$L$ is separating, if for any two elements $x \neq y$ of $X$, there exists an element $L \in L$ such that $x \in L$ and $y \notin L$.

$L$ is $T_2$ if, for any two elements $x \neq y$ of $X$, there exists $A, B \in L$ such that $X \in A'$ and $y \in B'$ and $A \cap B' = \phi$.

$L$ is disjunctive if for any $x \in X$ and $A \in L$ such that $x \notin A$, there exists a $B \in L$ such that $x \in B$ and $A \cap B = \phi$.

$L$ is regular if for any $x \in X$, and $A \in L$ such that $x \notin A$ there exist $B, C \in L$ such that $x \in B'$ and $A \subseteq C'$ and $B' \cap C' = \phi$.

$L$ is normal if for all $L_1, L_2 \in L$ such that $L_1 \cap L_2 = \phi$ there exists $L_1', L_2' \in L$ such that $L_1 \subseteq L_1'$, $L_2 \subseteq L_2'$, and $L_1' \cap L_2' = \phi$.

$L$ is compact if every covering of $X$ by elements of $L'$ has a finite subcovering.

$L$ is countably compact if every countable covering of $X$ by elements of $L'$ has a finite subcovering.

$L$ is Lindelöf if every covering of $X$ by elements of $L'$ has a countable subcovering.

$L$ is countably paracompact if whenever $A_n \subseteq \phi, A_n \in L$ there exists $B_n \in L$ such that $A_n \subseteq B_n$ and $B_n' \subseteq \phi$.

$L$ is complement generated if, for $L \in L$ there exists $L_n \in L$ such that $L = \bigcap_{n=1}^{\infty} L_n'$.

It is well known that if $L$ is complement generated then $L$ is countably paracompact.

Measure Terminology

We denote by $M(L)$ the finitely additive bounded measures on $A(L)$ (we may and do assume all elements of $M(L)$ are $\geq 0$).
$u \in M(\mathcal{L})$ is $\mathcal{L}$-regular if for any $A \in A(\mathcal{L})$, $u(A) = \sup\{u(L) \mid L \subset A, L \in \mathcal{L}\}$; (equivalently) $= \inf\{u(L) \mid A \subset L, L \in \mathcal{L}\}$. $u \in M(\mathcal{L})$ is $\sigma$-smooth on $\mathcal{L}$ if $L_n \in \mathcal{L}, n = 1, 2, \ldots$ and $L_n \upharpoonright \phi \to u(L_n)$. $u \in M(\mathcal{L})$ is $\sigma$-smooth on $A(\mathcal{L})$ if $A_n \in A(\mathcal{L}), n = 1, 2, \ldots$ and $A_n \upharpoonright \phi \to u(A_n)$. Note $u$ is $\sigma$-smooth on $A(\mathcal{L})$ if $u$ is countably additive.

We will use the following notations.

$M_{\mathcal{L}}(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $M(\mathcal{L})$.

$M_{\mathcal{L}}(\mathcal{L})$ the set of $\sigma$-smooth measures on $\mathcal{L}$ of $M(\mathcal{L})$.

$M^*(\mathcal{L})$ the set of $\sigma$-smooth measures on $A(\mathcal{L})$ of $M(\mathcal{L})$.

$M_{\mathcal{L}}^*(\mathcal{L})$ the set of $\mathcal{L}$-regular measures of $M^*(\mathcal{L})$.

Note that if $u \in M_{\mathcal{L}}(\mathcal{L})$ and $u \in M_{\mathcal{L}}(\mathcal{L})$ then $u \in M_{\mathcal{L}}^*(\mathcal{L})$.

Also we denote by $I(\mathcal{L}), I_{\mathcal{L}}(\mathcal{L}), I_{\mathcal{L}}(\mathcal{L}), I^*(\mathcal{L}),$ and $I_{\mathcal{L}}^*(\mathcal{L})$ the subsets of $M(\mathcal{L}), M_{\mathcal{L}}(\mathcal{L}), M_{\mathcal{L}}(\mathcal{L}), M^*(\mathcal{L}),$ and $M_{\mathcal{L}}^*(\mathcal{L})$ consisting of zero-one valued measures.

Let $J(\mathcal{L})$ denote those $u \in I(\mathcal{L})$ such that whenever $L_n \in \mathcal{L}, n = 1, 2, \ldots$ and $\bigcap_{n=1}^{\infty} L_n \in \mathcal{L}$ then $u\left(\bigcap_{n=1}^{\infty} L_n\right) = \inf_{n=1}^{\infty} u(L_n)$.

Clearly, $I(\mathcal{L}) \subset J(\mathcal{L}) \subset I_{\mathcal{L}}(\mathcal{L})$.

For $u \in M(\mathcal{L})$ the support of $u, S(u) = \cap\{L \in \mathcal{L} \mid u(L) = u(X)\}$. $\mathcal{L}$ is replete if for any $u \in I_{\mathcal{L}}^*(\mathcal{L}), u \neq 0$, $S(u) \neq \phi$.

Let $C(\mathcal{L})$ be the set of all real-valued $\mathcal{L}$-continuous functions defined on $X$, where $f : X \to R$ is called $\mathcal{L}$-continuous if $f^{-1}(E) \in \mathcal{L}$ for any closed set $E \subset R$. If $X$ is a topological space, $C(X)$ denotes the continuous functions on $X$ or equivalently we can write $C(X) = C(F)$ where $F$ is the lattice of closed sets of $X$, $z(\mathcal{L})$ is the lattice of zero sets of functions in $C(\mathcal{L})$.

$C(\mathcal{L})$ set of all real valued bounded $\mathcal{L}$-continuous functions defined on $X$.

Next we define $w(\mathcal{L}) = \{u \in I_{\mathcal{L}}^*(\mathcal{L}) \mid u(\mathcal{L}) = 1\}$ for $A \in A(\mathcal{L})$, and $w(\mathcal{L}) = \{u(\mathcal{L}) \mid L \in \mathcal{L}\}$.

We have for $A, B \in A(\mathcal{L})$:

\begin{align*}
(1) & \quad w(A \cup B) = w(A) \cup w(B) \\
(2) & \quad w(A \cap B) = w(A) \cap w(B) \\
(3) & \quad w(A') = w(A') \\
(4) & \quad w(A(\mathcal{L})) = A(w(\mathcal{L})) \\
(5) & \quad A \subset B \Rightarrow w(A) \subset w(B)
\end{align*}

Note $w(\mathcal{L})$ is a lattice and if $\mathcal{L}$ is disjunctive then $w(A) = w(B)$ if and only if $A = B$.

The Wallman topology is obtained by taking $w(\mathcal{L})$ as a base for the closed sets of a topology on $I_{\mathcal{L}}(\mathcal{L})$. $< I_{\mathcal{L}}(\mathcal{L}), \tau w(\mathcal{L})>$ is the general Wallman space associated with $X$ and $\mathcal{L}$. Note we have $w(\mathcal{L}) = \mathcal{L}$ for $L \in \mathcal{L}$ if $\mathcal{L}$ is separating and disjunctive. We also define: $w_{\mathcal{L}}(A) = \{u \in I_{\mathcal{L}}^*(\mathcal{L}) \mid u(A) = 1\}$ where $A \in A(\mathcal{L})$, and note $w(\mathcal{L}) \cap I_{\mathcal{L}}^*(\mathcal{L}) = w_{\mathcal{L}}(\mathcal{L})$.

We now consider two lattices. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote lattices of subsets of $X$ where $\mathcal{L}_1 \subset \mathcal{L}_2$.

$\mathcal{L}_1$ semi-separates $\mathcal{L}_2$ if $A \in \mathcal{L}_1, B \in \mathcal{L}_2$ and $A \cap B = \phi$ implies there exists $C \in \mathcal{L}_1, B \subset C$ and $A \cap C = \phi$. $\mathcal{L}_1$ separates $\mathcal{L}_2$ if $A, B \in \mathcal{L}_2$ and $A \cap B = \phi$ implies there exists $C, D \in \mathcal{L}_1$ such that $A \subset C, B \subset D, \text{ and } C \cap D = \phi$. $\mathcal{L}_2$ is $\mathcal{L}_1$-countable paracompact if for every sequences $\{B_n\}$ of sets of $\mathcal{L}_2$, such that $B_n \upharpoonright \phi \text{ there exists } \{A_n \in \mathcal{L}_1\} \text{ such that } A_n \upharpoonright \phi \text{ and } B_n \subset A_n$.
Let $L_2$ be $L_1$-closed if given $B_n \uparrow \phi$, $B_n \in L_2$ there exists $\{A_n\}, A_n \in L_1$ such that $A_n \uparrow \phi$ and $B_n \subseteq A_n$.

Clearly if $L_1$ separates $L_2$ then $L_1$ semiseparates $L_2$.

If $\nu \in M(L_2)$ then by $\nu \upharpoonright A(L_1)$ we mean $\nu$ restricted to $A(L_1)$. We state the following well known results:

Let $L_1 \subseteq L_2$ be two lattices of subsets of $X$. If $L_1$ semiseparates $L_2$ then for $\nu \in M_R(L_2)$, $u = \nu \upharpoonright A(L_1) \in M_R(L_1)$.

Suppose $L_1 \subseteq L_2$ are two lattices of subsets of $X$. Then if $u \in M_R(L_1)$, $u$ extends to $\nu \in M_R(L_2)$.

We will frequently assume in the sequel that $L_1 \subseteq L_2$ and $L_2$ is $L_1$ countably paracompact or countably bounded, but we note that this is unnecessary in certain situations as the following facts listed below show:

1. If $L_2$ is $L_1$ countably bounded and if $L_1$ is countably paracompact (e.g., if $L_1$ is complement generated) then $L_2$ is $L_1$ countably paracompact.

2. If $L_2$ is countably paracompact and if $L_1$ separates $L_2$ then $L_2$ is $L_1$ countably paracompact.

3. Suppose $L_2$ is $L_1$ countably paracompact and $L_1$ semiseparates $L_2$ then $L_2$ is $L_1$ countably bounded.

4. If $L_2$ is countably paracompact and if $L_1$ separates $L_2$ then $L_2$ is $L_1$ countably bounded.

2.(b) NORMAL LATTICES AND MEASURES.

In this section we will consider a number of measure implications of normal lattices and other special lattices as well as converses for implications. We first note:

**THEOREM 2.1.** Let $L$ be a complemented generated lattice. The $u \in I_R(L')$ implies $u \in I_R(L)$.

**PROOF.** Since $L$ is complemented generated then $L$ is countably paracompact and therefore $I_R(L') \subseteq I_R(L)$. Therefore it suffices to show $u \in I_R(L)$, but this is easy for if $L \subseteq L$ then $L = \{ \bigcap_{n=1}^{\infty} L_n' \}_{n \in \mathbb{N}} \subseteq L$ all $n$, and we may assume that the $L_n' \uparrow \phi$. Now if $u(L) = 0$, and if all $u(L_n') = 1$ then $\bigcap_{n=1}^{\infty} L_n' \cap L = \phi$ and $u(L_n' \cap L) = 1$ all $n$ which is a contradiction since $u \in I_R(L')$. It follows that $u(L) = \inf\{u(L') \mid L \subseteq L', L \in \mathcal{L} \}$ and this implies $u \in I_R(L)$.

**REMARK.** It is equally easy to show if $L$ is complement generated and $u \in M_R(L')$ then $u \in M_R(L)$.

**THEOREM 2.2.** Let $u \in J(L)$ and let $L$ be a $\delta$-lattice then $u \left( \bigcup_{i=1}^{\infty} L_i' \right) \leq \sum_{i=1}^{\infty} u(L_i')$ where all $L_i \in L$.

**PROOF.** Suppose $u \left( \bigcup_{i=1}^{\infty} L_i' \right) = 1$ and $\sum_{i=1}^{\infty} u(L_i') = 0$. Now $\sum_{i=1}^{\infty} u(L_i') = 0$ implies $u(L_i') = 0$ all $i$ and $\bigcap_{i=1}^{\infty} L_i = \left( \bigcup_{i=1}^{\infty} L_i' \right)'$ therefore $u \left( \bigcap_{i=1}^{\infty} L_i \right) = 0$ where obviously $\bigcap_{i=1}^{\infty} L_i \in L$. Also $u \left( \bigcap_{i=1}^{\infty} L_i \right) = \inf\{u(L_i) \mid L \subseteq L_i \}$ since $u \in J(L)$. So $u \left( \bigcap_{i=1}^{\infty} L_i \right) = 0$ implies there exists an $i_0$ such that $L_{i_0} \subseteq L_{i_0}' \cap u(L_{i_0}') = 0$. Therefore $u(L_{i_0}') = 1$ which is a contradiction, therefore theorem is proved.

**THEOREM 2.3.** If $L$ is normal and complement generated then $u \in J(L) = \nu \in I_R(L)$.

**PROOF.** Let $u \in J(L)$; we know that $u \leq \nu$ on $L$ where $\nu \in I_R(L)$. This gives $\nu \leq u$ on $L'$. Suppose $u \neq \nu$. Then there exists $L \subseteq L$ such that $u(L) = 0$, $u(L) = 1$. However, $L = \bigcup_{n=1}^{\infty} L_n'$ since $L$ is complement generated so $L \subseteq L_n'$. Therefore $u(L) = 1 \Rightarrow u(L_n') = 1$ for all $n$ which implies $u(L_n') = 1$ for all $n$ as $\nu \leq u$ on $L'$. Now $L = \bigcap_{n=1}^{\infty} L_n' = \bigcap_{n=1}^{\infty} L_n'$ since $L$ is normal there exists $A_n', B_n' \in L'$ such that $L \subseteq A_n'$, $L_n \subseteq B_n'$, and $A_n' \cap B_n' = \phi$. Therefore $L \subseteq A_n' \subseteq B_n' \subseteq L_n$ from this
which gives \( \nu(A'_n) = 1 \) and \( \nu(B'_n) = 1 \) by monotonicity of \( \nu \). Therefore \( \nu(B_n) = 1 \) as \( \nu \leq \nu = \nu_L \). Also \( L \subseteq A'_n \subseteq B'_n \subseteq L_n = L \subseteq \bigcap_{n=1}^\infty A'_n \subseteq B'_n \subseteq \bigcap_{n=1}^\infty L_n = L \) which implies that \( L = \bigcap_{n=1}^\infty A'_n = \bigcap_{n=1}^\infty B'_n = \bigcap_{n=1}^\infty L_n' \), so \( \nu(L) = 0 \) \( \Rightarrow \nu(L_n) = 0 \) which \( \Rightarrow \nu(B_n) = 0 \) by \( \nu \in J(L) \). This is a contradiction as \( \nu(B_n) = 1 \). Therefore \( \nu = \nu \in I_R(L) \rightarrow \nu \in I_R(L) \). Now \( \nu \notin J(L) \rightarrow \nu \notin I_R(L) \), therefore \( \nu \in J_R^*(L) \).

**THEOREM 2.4.** Let \( L \) be a normal lattice, \( \nu \in I_R(L) \), \( \nu \leq \rho(L') \) where \( \rho \in I_R(L') \). Then for \( L \subseteq L', \nu(L') = \sup(\rho(L) \vert L \subseteq L', L \subseteq L_1 \).  

**PROOF.** Suppose \( \nu(L') = 1 \), where \( L \subseteq L' \) then since \( \nu \in I_R(L) \) there exists \( \tilde{L} \subseteq L' \subseteq \tilde{L} \subseteq L \), \( \nu(L) = 1 \). Since \( L \subseteq \tilde{L} \subseteq L' \subseteq L' \subseteq \tilde{L} \subseteq L \), therefore by normality there exists \( A, B \subseteq L \) such that \( L = A \cap B \subseteq L' \subseteq A \cap B \). Therefore \( L \subseteq B' \subseteq A \subseteq L' \), also \( \nu(L) = 1 \Rightarrow \nu(B) = 1 \) by monotonicity of \( \nu \). Therefore \( \rho(B) = 1 \) as \( \nu \leq \rho(L') \). \( \rho(A) = 1 \) follows by monotonicity of \( \rho \), proving the theorem.

**REMARK.** This theorem is equivalent to the following: Let \( L \) be normal and let \( \nu \leq \nu(L) \) where \( \nu \in I_R(L) \) and \( \nu \in I_R(L) \). Then \( \nu(L') = \sup(\nu(L) \vert L \subseteq L', L \subseteq L_1 \). Next we show that actually the property in Theorem 2.4 or equivalently the one in the remark characterizes normal lattices, i.e.,

**THEOREM 2.5.** Suppose \( \nu \in I_R(L) \) and \( \rho \leq \nu(L) \) where \( \rho \in I_R(L) \) and \( \nu(L') = 1 \), \( L \subseteq L' \) implies \( L' \subseteq A \subseteq L \) such that \( \rho(A) = 1 \). Then \( L \) is normal.

**PROOF.** Let \( \rho \leq \nu(L) \), \( \rho \leq \nu(L) \) where \( \nu, \nu \in I_R(L) \) and \( \rho \in I_R(L) \). Assume \( \nu \neq \nu \), this implies \( \nu(L_1) = 0 \), \( \nu(L_2) = 1 \), \( \nu(L_2) = 0 \) where \( L_1, L_2 \subseteq L \) and \( L_1 \cap L_2 = \phi \). Now \( \nu(L_1) = 0 \) implies \( \nu(L_1') = 1 \) which implies there exists \( L_1' \subseteq A \subseteq L \) such that \( \rho(A) = 1 \) and \( \nu(L_2) = 0 \) implies \( \nu(L_2') = 1 \) which implies there exists \( L_2' \subseteq B \subseteq L \) such that \( \rho(B) = 1 \). Since \( A \subseteq L_1' \), \( B \subseteq L_2' \) then \( L_1 \subseteq A \) and \( L_2 \subseteq B \). So \( \rho(B) = 1 \) implies \( \rho(B) = 0 \) which implies \( \rho(B) = 0 \) as \( \nu \leq \rho(L') \). However, by monotonicity \( \nu(L_2) \leq \nu(B) \) and \( \nu(L_2) = 1 \) which implies \( \nu(B) = 1 \) which contradicts \( \rho(B) = 0 \). Therefore \( \nu = \nu \) which means \( L \) is normal.

**THEOREM 2.6.** Let \( L \) be a normal lattice, \( \nu \in I_R(L) \), \( \nu \leq \nu(L) \) where \( \nu \in I_R(L) \). Then \( \nu \in I_R(L') \).

**PROOF.** Suppose \( \nu \in I_R(L) \) we know \( \rho \leq \nu \leq \nu(L) \) where \( \nu, \nu \in I_R(L) \), \( \rho \in I_R(L) \). Assume \( \nu \neq \nu \), this implies \( \nu(L_1) = 0 \), \( \nu(L_2) = 1 \), \( \nu(L_2) = 0 \) where \( L_1, L_2 \subseteq L \) and \( L_1 \cap L_2 = \phi \). Now \( \nu(L_1) = 0 \) implies \( \nu(L_1') = 1 \) which implies there exists \( L_1' \subseteq A \subseteq L \) such that \( \rho(A) = 1 \) and \( \nu(L_2) = 0 \) implies \( \nu(L_2') = 1 \) which implies there exists \( L_2' \subseteq B \subseteq L \) such that \( \rho(B) = 1 \). Since \( A \subseteq L_1' \), \( B \subseteq L_2' \) then \( L_1 \subseteq A \) and \( L_2 \subseteq B \). So \( \rho(B) = 1 \) implies \( \rho(B) = 0 \) which implies \( \rho(B) = 0 \) as \( \nu \leq \rho(L') \). However, by monotonicity \( \nu(L_2) \leq \nu(B) \) and \( \nu(L_2) = 1 \) which implies \( \nu(B) = 1 \) which contradicts \( \rho(B) = 0 \). Therefore \( \nu = \nu \) which means \( L \) is normal.

**COROLLARY 2.7.** If \( L \) is normal and countably paracompact then the \( \nu \) (from Theorem 2.6) belongs to \( I_R(L) \).

**PROOF.** Since \( L \) is countably paracompact then \( I_R(L') \subseteq I_R(L) \) by Theorem 2.2. Then \( \nu \in I_R(L) \) and since \( \nu \in I_R(L) \) it follows that \( \nu \in I_R(L') \).

Next we consider a pair of lattices \( L_1, L_2 \) of \( X \) such that \( L_1 \subseteq L_2 \) and \( L_1 \cap L_2 = \phi \), then we have:

**THEOREM 2.8.** If \( L_1 \) separates \( L_2 \) then \( L_1 \) is normal if and only if \( L_2 \) is normal.

**PROOF.** The proof is not difficult. We just show \( L_2 \) normal implies \( L_1 \) normal. Now let \( L_2 \) be normal and \( \nu \in I(L_1) \), \( \nu \leq v_1(L_1) \), \( \nu \leq v_2(L_1) \) where \( v_1, v_2 \in I_R(L_1) \). Now we can extend \( u \in I(L_1) \) to \( L_2 \) and extend \( v_1 \) to \( L_2 \), \( v_2 \) to \( L_2 \). Now we have \( \lambda \leq \tau_1(L_1) \), \( \lambda \leq \tau_2(L_2) \) which is not difficult to see since \( L_1 \) separates \( L_2 \). Now \( L_2 \) is normal, therefore \( \tau_1 = \tau_2 \) and \( \nu_1 = \tau_1 \mid A(L_1) = \tau_2 \mid A(L_1) = \nu_2 \). Therefore \( L_1 \) is normal.

**3. THE WALLMAN SPACE \( I_R(L) \).**

We give here a brief discussion of the general Wallman space (see also [11]). Consider the set \( I_R(L) \) and the lattice of subsets \( u(L) \). It is well-known that \( u(L) \) is compact and it is not difficult to show:
THEOREM 3.1. The following are equivalent:
(a) \( w(L) \) is normal;
(b) \( w(L) \) is regular;
(c) \( w(L) \) is \( T_2 \).

Now since \( w(L) \) is compact, \( \tau w(L) \) the topology of closed sets, is compact and \( w(L) \) separates \( \tau w(L) \), and by Theorem 2.8 \( w(L) \) is normal if and only if \( \tau w(L) \) is normal. \(< I_R(L), \tau w(L) > \) is a compact topological space and it is always \( T_1 \). Assuming \( L \) is disjunctive, it is \( T_2 \) if and only if \( L \) is normal.

Next, let \( L \) be a \( \delta \)-normal lattice of subsets of \( X \), then the Alexandroff representation theorem (see [1]) yields for the conjugate space of \( C_b(L) \), namely \( C_b(L)^\prime = M_R(L) \) where to any \( \Phi \in C_b(L)^\prime \) there corresponds a unique \( u \in M_R(L) \) such that \( \Phi(f) = \int f du \) for all \( f \in C_b(L) \).

A net \( \{u_\alpha\} \) in \( M_R(L) \) converges to \( u \) in \( M_R(L) \) in the weak * topology if and only if
\[
\int f du_\alpha \to \int f du \quad \text{for all } f \in C_b(L).
\]
We shall denote weak * convergence by \( u^* \).

THEOREM 3.2. Now let \( L \) be \( \delta \)-normal and consider convergence in \( M_R^+(L) \). The following are equivalent:
(1) \( u^*_\alpha \to u \)
(2) \( u_\alpha(X) \to u(X) \) and \( \liminf_{\alpha} u_\alpha(A) \leq u(A) \) for all \( A \in L \)
(3) \( u_\alpha(X) \to u(X) \) and \( \limsup_{\alpha} u_\alpha(A) \geq u(A) \) for all \( A' \in L' \). For the proof in this particular setting see [7].

THEOREM 3.3. Let \( u_\alpha \in I_R(L) \) be \( u \in M_R(L) \) then \( u \in I_R(L) \). Thus \( I_R(L) \) is \( u^* \)-closed in \( M_R(L) \).

PROOF. Suppose \( u_\alpha \in I_R(L) \) \( u^* \to u \in M_R(L) \). Therefore \( u_\alpha(X) \to u(X) \) by Theorem 3.2. Now \( u_\alpha(X) = 1 \) since \( u_\alpha \in I_R(L) \), therefore \( u(X) = 1 \), which means for \( A \in A(L) \): \( 0 \leq u(A) \leq 1 \). Suppose \( A \in A(L) \) and \( 0 < u(A') < 1 \). Since \( u \in M_R(L) \) there exists \( L \in L \subseteq A \) such that \( 0 < u(L) \leq u(A) \) and there exists \( A \subseteq L' \subseteq L' \) such that \( u(A) \leq u(L') < 1 \). Therefore \( 0 < u(L) \leq u(L') < 1 \). Now \( L \subseteq L' \) therefore \( L \cap L = \emptyset \) which implies \( L \subseteq A' \subseteq B' \subseteq L' \) such that \( L \subseteq A' \subseteq B' \subseteq L' = \emptyset \) by \( L \) normal.

Therefore \( L \subseteq A' \subseteq B' \subseteq L' \) which implies \( 0 < u(L) \leq u(A') \leq u(B) \leq u(L') < 1 \) so that \( u(A') \leq u(B) < 1 \). Now \( u \in C_b(L) \) then \( \liminf_{\alpha} u_\alpha(B) \leq u(B) \) for \( B \in L \). Now \( u(B) < 1 \), therefore \( \liminf_{\alpha} u_\alpha(B) \leq 1 \) which means \( u_\alpha(B) = 0 \) since \( u_\alpha \in I_R(L) \). Also \( \limsup_{\alpha} u_\alpha(A') \geq u(A') \) for \( A' \subseteq L' \) but \( u(A') < 1 \), therefore \( \limsup_{\alpha} u_\alpha(A') \geq 1 \) for \( 0 < u(A') < 1 \). Therefore \( u_\alpha(A) = 1 \) since \( u_\alpha \in I_R(L) \). However for \( A' \subseteq B \) we have \( u_\alpha(A') = 1 \), \( u_\alpha(B) = 0 \) which is impossible. Therefore \( u(A') = 0 \) or \( 1 \), which implies \( u \in I_R(L) \).

THEOREM 3.4. \( \{u_\alpha\} = M_R(L) \)

PROOF. The proof of this is not difficult and can be modelled after the well-known special case of \( L \) being the lattice of zero sets in a Tychonoff space.

THEOREM 3.5. The \( u^* \)-topology of \( M_R(L) \) when restricted to \( I_R(L) \) gives the Wallman topology \( \tau w(L) \) for closed sets.

PROOF. Let \( u^*_\alpha \to u \) we will show \( u^*_\alpha \to u \) where \( w \) is convergence in Wallman. Consider \( u_0 \in w(L') \), therefore \( u_0(L') = 1 \). Using Theorem 3.2 we have \( \limsup_{\alpha} u_\alpha(L') \geq u_0(L') \), therefore \( \limsup_{\alpha} u_\alpha(L') = 1 \).

But \( 1 = \liminf_{\alpha} u_\alpha(L') \leq \limsup_{\alpha} u_\alpha(L') = 1 \), therefore \( \limsup_{\alpha} u_\alpha(L') = 1 \). So there exists \( \alpha_0 \) such that for all \( \alpha \geq \alpha_0 \)
\( u_\alpha(L') = 1 \), therefore \( u_\alpha \in w(L') \) for all \( \alpha \geq \alpha_0 \). This gives \( u^*_\alpha \to u \) which proves the theorem.

We assume now that \( L \) is \( \delta \)-normal, separating and disjunctive. Let \( f \in C_b(L) \) we define \( \tilde{f} \) on \( I_R(L) \) by \( \tilde{f}(u) = \int f du \) where \( u \in I_R(L) \).

THEOREM 3.6. \( \tilde{f} \in C(I_R(L)) \).
Proof. Let \( u_\alpha \uparrow u_0 \). We must show that \( \tilde{f}(u_\alpha) \to \tilde{f}(u_0) \) which means \( u_\alpha \uparrow u_0 \). For \( u_0 \in w(L) \) we have \( u_\alpha \in w(L) \) for all \( \alpha \geq \alpha_0 \) as \( u_\alpha \uparrow u_0 \). Therefore, \( u_\alpha(L') = 1, \alpha \geq \alpha_0 \), which implies \( \lim \alpha u_\alpha(L') = 1 \).

Therefore \( \lim \alpha u_\alpha(L') = \lim \alpha u_\alpha(L') = \lim \alpha u_\alpha(L') = 1 \) and \( u_0(L') = 1 \) as \( u_0 \in w(L) \). So \( \lim \alpha u_\alpha(L') \geq u_0(L') \) and therefore by Theorem 3.2 we have \( u_\alpha \uparrow u_0 \) which proves \( \tilde{f} \in C(I_R(L')) \).

Theorem 3.7. The correspondence \( f \to \tilde{f} \) is a bijection between \( C_b(L) \) and \( C(I_R(L)) \); the continuous functions on the Wallman space \( I_R(L) \).

Proof. Let \( A = \{ f \in C_b(L) \} \). Then \( A \subset C(\tau w(L)) = C(I_R(L)) \). Since \( u_\alpha \uparrow u \Rightarrow \tilde{f}(u) \in C(I_R(L)) \).

Now it is easy to show the following:

1. \( \tilde{f} + \tilde{g} = \tilde{f} + \tilde{g} \)
2. \( \tilde{a f} = a \tilde{f} \) for \( a \in R \)
3. \( \tilde{f} g = \tilde{f g} \)
4. \( || \tilde{f} || = || f || \), therefore \( A \) is a closed subalgebra of \( C(\tau w(L)) \)
5. \( A \) separates points. We can prove this by showing for \( u, v \in I_R(L) \), \( u \neq v \) there exists \( \tilde{f} \in A \) such that \( \tilde{f}(u) = 1 \) and \( \tilde{f}(v) = 0 \). This is done by using the normality of \( L \).
6. \( 1 \in A \). Therefore given \( u \in I_R(L) \) there exists \( \tilde{f} \in A \) such that \( \tilde{f}(u) \neq 0 \).

So by the Stone-Weierstrass theorem \( A = \bar{A} = C(\tau w(L)) \) which proves the theorem.

4. THE SPACE \( Q(L) \).

In this section, we consider the important measures of \( I_R(L) \) which integrate all \( f \in C(L) \) and consider their relationship to \( I_R^*(L) \). Let \( L \) be \( \delta \)-normal lattice. We define \( Q(L) = \{ v \in I_R(L) \} \).

Theorem 4.1. \( I_R^*(L) \subset Q(L) \).

Proof. Let \( v \in I_R^*(L) \) and \( L_n = \{ |f| \geq n \} \). One can see \( L_n \downarrow \phi \) which implies \( v(L_n) \to 0 \) since \( v \in I_R^*(L) \). Therefore \( v(L_N) = 0 \) for \( N \) big. Now

\[
\int_X |f| \, dv = \int_{L_N} |f| \, dv + \int_{L_N} |f| \, dv \\
\leq NV L_N' \\
\leq N
\]

Therefore \( \int_X |f| \, dv \leq N \) which proves \( v \in Q(L) \).

Theorem 4.2. \( I_R(L) \cap I_\phi(L') \subset Q(L) \).

Proof. Let \( \{ |f| > n \} = A_n^* \). Clearly \( A_n^* \downarrow \phi \) and \( A_n \in L \) for all \( n \). Now let \( v \in I_R(L) \cap I_\phi(L') \), therefore \( v(A_n^*) = 0 \), \( n \geq N \). Now \( \int_X |f| \, dv = \int_{A_N^*} |f| \, dv + \int_{A_N^*} |f| \, dv \). Therefore \( \int_X |f| \, dv \leq NV(A_N) \), so

\[
\int_X |f| \, dv \leq NV(A_N) \leq N.
\]

Therefore \( \int_X |f| \, dv \leq N \) which provides the theorem.

Theorem 4.3. \( I_R^*(L) \subset I_R(L) \cap I_\phi(L') \subset Q(L) \).

Proof. By Theorems 4.1 and 4.2 and the trivial observation that \( I_R^*(L) \subset I_R(L) \cap I_\phi(L') \), the result is proved.

Following Varadarajan who considered the lattice of zero sets in a Tychonoff space, we introduce

Definition. The Sequence \( \{ B_n \} \) in \( L \) is called regular if \( B_n \downarrow \phi \) and there exists \( A_n \in L \) such that \( B_n \subset A_n \subset B_{n+1} \) for all \( n \).

Theorem 4.4. Let \( \{ B_n \} \) be a regular sequence. Then there exists \( \{ f_n \}, f_n \in C_b(L), 0 \leq f_n \leq 1 \) such that \( f_n \downarrow \phi, f_n(B_n) = 0, f_n(B_n') = 1 \) for \( n = 1, 2, ... \).
THEOREM 4.5. Let $X$ be an abstract set and $L$ a $\delta$-normal lattice of subsets which is also countably paracompact. Let $\{A_n\}$ in $L$. $A_n \not\perp \phi$. Then there exists a regular sequence $\{C_n\}$ such that $C_n \subset A_n$ for all $n$.

PROOF. Since $A_n \not\perp \phi$ and since $L$ is countably paracompact then there exists $\{B_n\}$ in $L$ with $A_n \subset B_n \not\perp \phi$. Now we show by induction that for any $n$ we have $\{C_K\}, \{D_K\}$ in $L$ with $A_K \subset C_K \subset D_K \subset (B_K \cap C_{K-1})$ where $K = 1, \ldots, n$. (1) For $n = 1$, take $C_0 = \phi$, and $A_1 \subset C_1 \subset D_1 \subset B_1$ follows by normality. (2) Assume expression is true for $n$. Now $A_{n+1} \subset B_{n+1}$ and $A_{n+1} \subset C_{n+1}$, therefore $A_{n+1} \subset B_{n+1} \cap C_{n+1}$. Using normality, there exists $C_{n+1}, D_{n+1} \in L$ such that $A_{n+1} \subset C_{n+1} \subset D_{n+1} \subset (B_{n+1} \cap C_{n+1})$ which finishes the induction argument. Since $C_n \subset A_n$ we must show $\{C_n\}$ is regular. Now $C_n \subset B_n \not\perp \phi$ implies $C_n \not\perp \phi$ and $C_n \subset C_{n+1} \subset C_{n+1}$. Therefore $\{C_n\}$ is regular as $D_{n+1} \in L$. Finally using the previous two results it is not difficult to show using an argument similar to Varadarajan that the following holds:

THEOREM 4.6. Let $L$ be $\delta$-normal and countably paracompact, then $Q(L) \subset IR^\sigma(L)$.

So using Theorems 4.1 and 4.6 we have:

THEOREM 4.7. Let $L$ be $\delta$-normal and countably paracompact, then $Q(L) = IR^\sigma(L)$.

We also have:

THEOREM 4.8. If $Q(L) = IR^\sigma(L) \cap I_{\phi}(L)$ and if $I_{\phi}(L) \subset I_{\phi}(L)$ then $Q(L) = IR^\sigma(L)$.

PROOF. $Q(L) = IR^\sigma(L) \cap I_{\phi}(L) \subset IR^\sigma(L) \cap I_{\phi}(L)$, but we know if $v \in M_{\phi}(L)$ and $v \in M_{\phi}(L)$ then $v \in M_{\phi}(L)$. Therefore $Q(L) \subset IR^\sigma(L) \cap I_{\phi}(L) = IR^\sigma(L)$, so $Q(L) \subset IR^\sigma(L)$. However, from Theorem 4.1 we have $IR^\sigma(L) \subset Q(L)$, therefore $Q(L) = IR^\sigma(L)$.

Note: $I_{\phi}(L) \subset I_{\phi}(L)$ if $L$ is countably paracompact, also if $L$ is regular and Lindelöf.

Now we consider two lattices $L_1$ and $L_2$ such that $L_1 \subset L_2$. Then $C(L_1) \subset C(L_2)$.

THEOREM 4.9. Let $L_1$, $L_2$ be lattices of subsets such that $L_1$ semi-separates $L_2$. If $u \in Q(L_2)$ and if $u(v) = v | A(L_1)$, then $u \in Q(L_1)$.

PROOF. Since $L_1$ semi-separates $L_2, u \in IR^\sigma(L_1)$. Also, since $C(L_1) \subset C(L_2)$ and since $v$ integrates all $f \in C(L_2), u$ integrates all $g \in C(L_1)$. Hence $u \in Q(L_1)$.

THEOREM 4.10. Let $L_1, L_2$ be lattice of subsets such that $L_1$ separates $L_2$. Let $u \in Q(L_2)$ and $u(v) = v | A(L_1)$. If $Q(L_1) = IR^\sigma(L_1)$ then $v \in I_{\phi}(L_2)$.

PROOF. By the previous theorem $u \in Q(L_1) = IR^\sigma(L_1)$ by hypothesis, and since $L_1$ separates $L_2$ it is easy to see $v$, the extension of $u$, is in $I_{\phi}(L_2)$.

THEOREM 4.11. Let $L_1, L_2$ be lattice of subsets such that $L_1$ separates $L_2$. If $Q(L_1) = IR^\sigma(L_1)$ then $Q(L_2) = IR^\sigma(L_2) \cap I_{\phi}(L_2)$.

PROOF. $v \in Q(L_2)$ implies $v \in IR^\sigma(L_2)$, but $v \in I_{\phi}(L_2)$ from Theorem 4.10, therefore $Q(L_2) \subset IR^\sigma(L_2) \cap I_{\phi}(L_2)$. However we know if $v \in IR^\sigma(L_2) \cap I_{\phi}(L_2)$ then $v \in Q(L_2)$ from Theorem 4.2 which proves the result.

We have the following application: For $L$ $\delta$-normal, $z(L) \subset L$ where $z(L)$ consists of all sets of $L$ of the form $L = \bigcap_{n=1}^{\infty} L_n$, $L_n \in L$ for all $n$, (see [1]). Now $z(L)$ separates $L$ and $z(L)$ is normal and countably paracompact. Therefore by Theorem 4.7 we have $I_{\phi}(z(L)) = Q(z(L))$. Now using Theorem 4.11 we have $Q(L) = IR^\sigma(L) \cap I_{\phi}(L)$. Also if $I_{\phi}(L') \subset I_{\phi}(L)$ then $Q(L) = IR^\sigma(L)$ by Theorem 4.8.

REMARK. We recall that if $X$ is Tychonoff space and if $L = z$, the lattice of zero sets then $(IR^\sigma(z), \tau_{w(z)})$ is the realcompactification $\nu(X)$ of $X$.

Now we consider other criterion for $Q(L) = IR^\sigma(L) \cap I_{\phi}(L)$. If $X$ is a topological space and if $A \subset X$ we denote by $\overline{A}$ the $G_\delta$-closure of $A$. Now if $X$ is an abstract set and $L$ as usual is a
separating disjunctive $\delta$-normal lattice of subsets then we can view $X$ embedded in $Q(X)$; we have $X \subset Q(X) \subset I_R(X)$. In fact, using Theorem 4.3 we have $X \subset I^\delta_R(X) \subset I_R(X) \cap I_m(X) \subset Q(X) \subset I_R(X)$.

**THEOREM 4.12.** $X^\delta \subset Q(X)$ where $X^\delta$ is the $G_\delta$-closure of $X$ in the Wallman space $I_R(X)$.

**PROOF.** Suppose $u \in X^\delta$. If $u \notin Q(X)$ then there exists $f \in C(Q(X))$, $f \geq 0$ such that $\int f du = \infty$. Let $A_n = \{ f > n \} \in L$. Then $A_n \uparrow \phi$ and $w(A_n) = 1$. Therefore $u \in \bigcap_{n=1}^\infty w(A_n) \subset I_R(X) - X$ which contradicts the fact $u \notin X^\delta$. Therefore $u \in Q(X)$, so $X^\delta \subset Q(X)$.

**THEOREM 4.13.** If $Q(X) \subset R\subseteq Q(X)$, then $G_{\delta}$-closure of $X$ in $I_R(X)$, then $u \in I_m(X)$ where $u \in Q(X)$.

**PROOF.** Suppose $u \in Q(X)$ which implies $u \in I_R(X)$. If $u \notin I_m(X)$ then there exists $L_n \uparrow \phi$, $L_n \subseteq L$, $w(L_n) = 1$. Therefore $u \in \bigcap_{n=1}^\infty w(L_n) \subset I_R(X) - X$. Therefore $u \notin X^\delta$, so $Q(X) \subset X^\delta$ implies $u \in I_m(X)$.

**THEOREM 4.14.** $Q(X) = X^\delta$ if and only if $u \in I_m(X)$ for all $u \in Q(X)$.

**PROOF.** If $Q(X) = X^\delta$ and if $u \in Q(X)$ then $u \in I_m(X)$ by the previous theorem. While if $Q(X) \subset X^\delta$ then we must have $Q(X) \subset X^\delta$ for if not then there exists $G \in G_\delta$ such that $u \in G \subset I_R(X) - X$ where $u \in I_R(X)$. Therefore $u \in \bigcap_{n=1}^\infty O_n \subset I_R(X)$ where $O_n$ is an open set, which implies $u \in O_n$ for all $n$. Now $w(L_n)$ is an open set for $L_n \subseteq L$, therefore $u \in w(L_n) \subset Q(X)$ which yields $u \in \bigcap_{n=1}^\infty w(L_n) \subset \bigcap_{n=1}^\infty O_n$. Therefore there exists $u \in Q(X)$ such that $u \in \bigcap_{n=1}^\infty w(L_n)$ where the $w(L_n) \uparrow \phi$ and where $\bigcap_{n=1}^\infty w(I_n) \subseteq I_R(X)$ and $\bigcap_{n=1}^\infty w(I_n) \subset I_R(X) - X$, but then $w(L_n) = 1$ for all $n$ and $L_n \uparrow \phi$ which is a contradiction. Thus $Q(X) \subset X^\delta$ and then by Theorem 4.12, $Q(X) = X^\delta$.

Using the previous theorem and Theorem 4.2 we have:

**COROLLARY 4.15.** If $L$ is $\delta$-normal separating and disjunctive then $Q(L) = X^\delta$ if and only if $Q(L) = I_m(L)$.

**REMARK.** We note that $Q(L) = I_m(L)$ if and only if $C(L) = C(L)$; this situation arises in particular if $C(L)$ consists only of constant functions. (see below)

5. **THE WALLMAN SPACE $I_R^\delta(L)$.**

First we note $I_R^\delta(L)$ may be empty. Let $X = \{0,1,2,...\}$ where $L$ consists of $\phi$ and all sets of the form $\{n,n+1,...\}$ for all $n$, and $v_1,v_2 \in I_R(L)$. If $v_1 \neq v_2$ then there exists $L_1,L_2 \subseteq L$ such that $v_1(L_1) = 1$, $v_2(L_1) = 0$, $v_1(L_2) = 0$, $v_2(L_2) = 1$ and $L_1 \cap L_2 = \phi$. However, this is impossible here as $L_1 \cap L_2 \neq \phi$ unless $L_1$ or $L_2 = \phi$. Therefore $I_R(L) = \{\phi\}$. Now clearly if $L_n = \{n,n+1,...\}$, $L_n \subseteq L$ and $L_n \uparrow \phi$. However, $w(L_n) = 1$ for all $n$, therefore $I_R^\delta(L) = \phi$. We also have in this example: $C(L) = C(L) = \{\text{constant functions}\}$; $L$ is not disjunctive, $L$ is not countably paracompact; $L$ is not regular; $L$ is a $\delta$-lattice.

Now we state a familiar result:

**THEOREM 5.1.** Let $L$ be disjunctive then $< I_R^\delta(L),W_{\phi}(L) >$ is replete.

Next we give facts about $C(L)$: we denote by $M_R(L)$ the set $\{u \in M_R^\delta(L) : \int f |d| du < \infty \text{ for all } f \in C(L)\}$. Note $I_R^\delta(L) \subset M_R(L)$ and we denote by, similar to Varadarajan, $W_I(L)$ the topology on $M_R(L)$. A net $\{u_n\} \subset M_R(L)$ converges to $u \in M_R^\delta(L)$ with respect to $W_I$ if and only if $\int f du_n \rightarrow \int f du$ for all $f \in C(L)$. The topology $W_I$ restricted to $I_R^\delta(L)$ is the Wallman topology. Now using this it is easy to show that $f(u) = \int f du$, $u \in I_R^\delta(L)$ is continuous with respect to the Wallman topology $\tau_{\phi}(L)$ on $I_R^\delta(L)$, i.e., $f(u) \in C(I_R^\delta(L)) = C(\tau_{\phi}(L))$. Let $L$ be separating, disjunctive and $\delta$-normal throughout and $f \in C(L)$.

**THEOREM 5.2.** Let $f \in C(L)$ then $f^{-1}[0,\infty) = \tau(\widehat{g})$ where $g = (f - a) \wedge 0 \in C(L)$ and similarly $f^{-1}(-\infty,a] = \tau(\widehat{h})$ where $h \in C(L)$.

**PROOF.** Omitted.

**THEOREM 5.3.** Let $z(L)$ be the zero lattice of $C(L)$ then $w_{\phi}(z(L)) = z(w_{\phi}(L))$. 
PROOF. Let $Z \in \tau(L) \subset L$. Therefore by a theorem of Alexandroff $Z = \bigcap_{n=1}^{\infty} L_n$, $L_n \in L$ all $n$. Thus $w_\sigma(Z) = \bigcap_{n=1}^{\infty} w_\sigma(L_n)$. But $w_\sigma(L_n)$ is $\delta$-normal therefore by Alexandroff theorem we get $w_\sigma(Z) \in \tau(w_\sigma(L))$. Conversely if $w_\sigma(L) \in \tau(w_\sigma(L))$, where $L \in L$ then $w_\sigma(L) = \bigcap_{n=1}^{\infty} w_\sigma(L_n) = w_\sigma(\bigcap_{n=1}^{\infty} L_n)$ and since $L$ is disjunctive, $L = \bigcap_{n=1}^{\infty} L_n \in \tau(L)$ again by Alexandroff's result and the proof is completed.

We have seen that if $f \in C(L)$ then $\tilde{f} \in C(\tau w_\sigma(L))$, i.e., it is continuous with respect to Wallman topology on $I_R^\sigma(L)$. However we can do better.

THEOREM 5.4. If $f \in C(L)$ then $\tilde{f} \in C(w_\sigma(L))$ (where $\tilde{f}(u) = \int f \, du$ for all $u \in I_R^\sigma(L)$).

PROOF. We must show $\tilde{f}^{-1}(E) \in w_\sigma(L)$ for any closed set $E \subset R$. It will suffice to show this for $E = [a, b] \subset R$. Now $[a, b] = (-\infty, b] \cap [a, \infty)$ so $\tilde{f}^{-1}[a, b] = \tilde{f}^{-1}(-\infty, b] \cap [a, \infty) = \tilde{f}^{-1}(-\infty, b) \cap \tilde{f}^{-1}[a, \infty] = \tilde{f}^{-1}(Z(h)) \cap Z(g)$ using Theorem 5.2. Next we note if $g \in C(L)$ then $Z(g) = Z(\tilde{g})$ where the closure is taken in the Wallman space $I_R^\sigma(L)$ with topology of closed sets $\tau w_\sigma(L)$. Therefore $\tilde{f}^{-1}[a, b] = \tilde{f}^{-1}(Z(g)) \cap Z(h)$ and $Z(g), Z(h) \in \tau(L) \subset L$ so $\tilde{f}^{-1}[a, b] = \tilde{f}^{-1}(Z(g)) \cap Z(h)$ (using $A \cap B = A \cap B$ for $A, B \in L$). In addition $\tilde{f}^{-1}[a, b] = Z(g^2 + h^2)$ and $Z(g^2 + h^2) = \tilde{Z} = w_\sigma(Z)$. Therefore $\tilde{f}^{-1}[a, b] = \tilde{Z} = w_\sigma(Z)$ which implies $\tilde{f}^{-1}[a, b] \in w_\sigma(L)$). However using Theorem 5.3 we get $\tilde{f}^{-1}[a, b] \in \tau w_\sigma(L)$. However $z(w_\sigma(L)) \subset w_\sigma(L)$ therefore $\tilde{f}^{-1}[a, b] \in w_\sigma(L)$ which implies $\tilde{f} \in C(w_\sigma(L))$.

Now we intend to prove the converse. Suppose that $h \in \tau w_\sigma(L)$ then clearly $h \mid X \in C(L)$ and let $h \mid X = f \in C(L)$ then $h = \tilde{f}$ since both are continuous with respect to the Wallman topology and they agree on $X$ which is dense in $I_R^\sigma(L)$.

Using the above results we have the following:

THEOREM 5.5. The correspondence $f \mapsto \tilde{f}$ is a bijection between $C(L)$ and $C(w_\sigma(L))$: the $w_\sigma(L)$-continuous functions on the Wallman space $I_R^\sigma(L)$.

Next let $u \in I_R(L)$, then we define $M^u = \{ f \in C(L) \mid u \in Z(f) \mid I_R(L) \}$. The following facts we list for completeness (proofs can be found for this setting in [8]):
1) If $u_1, u_2 \in I_R(L)$ and if $u_1 \neq u_2$ then $M^{u_1} \neq M^{u_2}$.
2) $M^u$ is a maximal ideal in $C(L)$.
3) (Generalized Gelfand-Kolmogoroff) If $M$ is a maximal ideal in $C(L)$ then there exists $u \in I_R(L)$ such that $M = M^u$.

Thus there exists a one to one correspondence between elements of $I_R(L)$ and maximal ideals of $C(L)$.

Now we return to the Wallman space $<I_R^\sigma(L), \tau w_\sigma(L)>$ and give conditions when this topological space is realcompact. We know that for $L$ disjunctive $w_\sigma(L)$ is replete; the question we are now concerned with is: when is the lattice $\tau(w_\sigma(L))$ replete? or i.e., when is the Wallman space realcompact?

THEOREM 5.6. Let $L$ be $\delta$-normal, separating, disjunctive, and countably paracompact then $Q(L) = I_R^\sigma(L)$ and if $I_R^\sigma(L)$ with the Wallman topology is a c.b. space then it is realcompact.

PROOF. $Q(L) = I_R^\sigma(L)$ by Theorem 4.7. Now $<I_R^\sigma(L), \tau w_\sigma(L)>$ is replete from Theorem 5.1. Now $w_\sigma(L) \subset \tau w_\sigma(L)$ (of course) and consequently $z(\tau w_\sigma(L)) \subset \tau w_\sigma(L)$. Now $L$ $\delta$-normal implies $w_\sigma(L)$ $\delta$-normal and $L$ countably paracompact implies $w_\sigma(L)$ is countably paracompact. Then by Theorem 5.3 of [2] we have $\tau w_\sigma(L)$ is replete. Now by hypothesis $\tau w_\sigma(L)$ is $\tau(w_\sigma(L))$ countably bounded (c.b.). Thus $z(\tau w_\sigma(L))$ is replete by Theorem 3.4 of [2]. Hence $<I_R^\sigma(L), \tau w_\sigma(L)>$ is realcompact.

Note. If $\tau w_\sigma(L)$ is $\tau(w_\sigma(L))$ countably paracompact the same conclusion can be drawn.
We continue to assume that \( L \) is separating, disjunctive and \( \delta \)-normal. Let \( h \in C(I_R^*(L)) \) or, i.e., \( h \in C(rw_\sigma(L)) \) in lattice notation, then \( f = h \vert X \in C(\tau L) \), clearly. If \( f \in C(L) \) then by our earlier work in this section we would have \( h = \overline{f} \in C(\omega(L)) \). This situation arises if \( X \) is a Tychonoff topological space and \( L = z \) lattice of zero sets of continuous functions on \( X \) for in this case if \( h \in C(I_R^*(z)) \) then \( h \vert X \in C(\tau z) = C(F) = C(z) \) where \( F \) is the lattice of closed sets of \( X \). Thus, in this case, \( \omega(z) = z(rw_\sigma(z)) \) and since \( \omega(z) \) is replete, we have that \( I_R^*(z) \) is realcompact with respect to the Wallman space.

**Theorem 5.7.** Let \( L \) be separating, disjunctive and \( \delta \)-normal. If \( C(rw_\sigma(L)) = C(w_\sigma(L)) \) then \( z(w_\sigma(L)) = z(rw_\sigma(L)) \) and if \( w_\sigma(L) \) is \( z(w_\sigma(L)) \) c.b. or countably paracompact then \( I_R^*(L) \) with the Wallman topology is realcompact.

**Proof.** Since \( w_\sigma(L) \subseteq rw_\sigma(L) \) then \( z(w_\sigma(L)) \subseteq z(rw_\sigma(L)) \). Now let \( Z(f) \in z(rw_\sigma(L)) \) where \( f \in C(rw_\sigma(L)) \), but \( C(rw_\sigma(L)) = C(w_\sigma(L)) \). This implies \( Z(f) \in z(w_\sigma(L)) \). Therefore \( z(rw_\sigma(L)) \subseteq z(w_\sigma(L)) \). Now if \( w_\sigma(L) \) is \( z(w_\sigma(L)) \) countably bounded or countably paracompact then since \( w_\sigma(L) \) is replete we have using the same argument as in the proof of Theorem 5.6 that \( z(w_\sigma(L)) \) is replete, therefore \( z(rw_\sigma(L)) \) is replete.

Finally we extend Theorem 5.7 but first note \( z(w_\sigma(L)) \subseteq w(L) \subseteq rw_\sigma(L) \) and \( z(w_\sigma(L)) \subseteq z(w(L)) \subseteq z(rw_\sigma(L)) \).

**Theorem 5.8.** Let \( L \) be separating, disjunctive and \( \delta \)-normal. If \( L \) is \( z(L) \) countably bounded (c.b.) or \( L \) is \( z(L) \) countably paracompact and assume \( z(rw_\sigma(L)) \subseteq \tau z(L) \), then \( z(rw_\sigma(L)) \) is replete, i.e., \( I_R^*(L) \) with the Wallman topology is realcompact.

**Proof.** \( z(w_\sigma(L)) \) is complement generated since \( z(L) \) is complement generated. (Use Theorem 5.3 and \( z(w_\sigma(L)) \subseteq z(rw_\sigma(L)) \subseteq \tau z(w_\sigma(L)) \), therefore by Theorem 3.1 part (1) of [2] we have \( z(rw_\sigma(L)) \) is replete, as \( z(w_\sigma(L)) \) is replete from the fact \( L \) is \( z(L) \) countably bounded or \( L \) is \( z(L) \) countably paracompact.

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**References**


This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

We intend to publish in this special issue papers reporting research on time-dependent billiards. The topic includes both conservative and dissipative dynamics. Papers discussing dynamical properties, statistical and mathematical results, stability investigation of the phase space structure, the phenomenon of Fermi acceleration, conditions for having suppression of Fermi acceleration, and computational and numerical methods for exploring these structures and applications are welcome.

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