ON SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS OF HIGHER ORDER

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ABSTRACT. The classes $T_k(\rho)$, $0 < \rho < 1$, $k > 2$, of analytic functions, using the class $V_k(\rho)$ of functions of bounded boundary rotation, are defined and it is shown that the functions in these classes are close-to-convex of higher order. Covering theorem, arc-length result and some radii problems are solved. We also discuss some properties of the class $V_k(\rho)$ including distortion and coefficient results.

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1. THE CLASS $P_k(\rho)$

Let $P_k(\rho)$ be the class of functions $p(z)$ analytic in the unit disc $E = \{z:|z|<1\}$ satisfying the properties $p(0) = 1$ and

$$\int_0^{2\pi} \frac{\text{Re} p(z) - \rho}{1 - \rho} \, d\theta < k\pi, \quad (1.1)$$

where $z = re^{i\theta}$, $k > 2$ and $0 < \rho < 1$. This class has been introduced in [1]. We note that, for $\rho = 0$, we obtain the class $P_k$ defined by Pinchuk [2] and for $\rho = 0$, $k = 2$, we have the class $P$ of functions with positive real part. The case $k = 2$ gives us the class $P(\rho)$ of functions with positive real part greater than $\rho$.

Also we can write

$$p(z) = \frac{1}{2} \int_0^{2\pi} \frac{1 + (1 - 2\rho) ze^{-it}}{1 - ze^{-it}} \, d\mu(t), \quad (1.2)$$

where $\mu(t)$ is a function with bounded variation on $[0,2\pi]$ such that

$$\int_0^{2\pi} d\mu(t) = 2$$

and

$$\int_0^{2\pi} |d\mu(t)| < k, \quad (1.3)$$

From (1.1), we have the following.
THEOREM 1.1. Let $p \in P_k(\rho)$. Then
\[ p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z), \]
where $p_i \in P(\rho), i = 1, 2$.

We now prove.

THEOREM 1.2. The class $P_k(\rho)$ is a convex set.

PROOF. Let $H_1, H_2 \in P_k(\rho)$. We shall show that, for $\alpha, \beta > 0$
\[ H(z) = \frac{1}{\alpha + \beta} \left[ \alpha H_1(z) + \beta H_2(z) \right] \]
begins to $P_k(\rho)$.

From Theorem 1.1, we can write
\[ H(z) = \frac{1}{\alpha + \beta} \left[ \alpha \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z) \right] \]
\[ + \beta \left( \frac{k}{4} + \frac{1}{2} \right) p_3(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_4(z) \]
where $p_i \in P(\rho), i = 1, 2, 3, 4$.

Now, writing $p_i(z) = (1-\rho) h_i(z) + \rho, i=1,2,3,4$, see [3],
we have
\[ H(z) - \rho = \left( \frac{k}{4} + \frac{1}{2} \right) \left[ \frac{1}{\alpha + \beta} \left( \alpha h_1(z) + \beta h_3(z) \right) \right] - \left( \frac{k}{4} - \frac{1}{2} \right) \left[ \frac{1}{\alpha + \beta} \left( \alpha h_2(z) + \beta h_4(z) \right) \right] \]
\[ = \left( \frac{k}{4} + \frac{1}{2} \right) f_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) f_2(z), \]
where $f_1$ and $f_2 \in P$, since $P$ is a convex set, see [2] and this gives us the required result.

THEOREM 1.3. Let $p \in P_k(\rho)$ and be given by
\[ p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \]
Then

(i) \[ \frac{1}{2\pi} \int_{0}^{2\pi} |p(re^{i\theta})|^2 d\theta < \frac{1 + \left( k^2 (1-\rho)^2 - 1 \right) r^2}{1 - r^2} \]
and

(ii) \[ \frac{1}{2\pi} \int_{0}^{2\pi} |p'(re^{i\theta})| d\theta < \frac{k(1-\rho)}{1 - r^2} \]

PROOF. (i) Using Parseval's identity, we have
\[ \frac{1}{2\pi} \int_{0}^{2\pi} |p(re^{i\theta})|^2 d\theta = \sum_{n=0}^{\infty} |c_n|^2 r^{2n} \]
\[ = 1 + k^2 (1-\rho)^2 \sum_{n=1}^{\infty} r^{2n} = \frac{1 + \left( k^2 (1-\rho)^2 - 1 \right) r^2}{1 - r^2}, \]
where we have used an easily established sharp result $|c_n| \leq k(1-\rho)$, for all $n > 1$. 

(ii) By using Theorem 1.1, we can write

\[ p(z) - \rho = \left( \frac{k}{4} + \frac{1}{2} \right) (1-\rho) \eta_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) (1-\rho) \eta_2(z), \]

where \( \eta_1, \eta_2 \in P \).

Therefore,

\[ p'(z) = \left( \frac{k}{4} + \frac{1}{2} \right) (1-\rho) \eta_1'(z) - \left( \frac{k}{4} - \frac{1}{2} \right) (1-\rho) \eta_2'(z) \tag{1.4} \]

Now, for all \( h \in P \), we have

\[ 2w'(z)h'(z) \frac{(1+w(z-\rho))}{2} \]

where \( w(z) \) is a Schwarz function [3], and

\[ \frac{1}{2\pi} \int_0^{2\pi} |h'(re^{i\theta})| d\theta = \frac{1}{2\pi} \int_0^{2\pi} \frac{2|w'(re^{i\theta})|}{|1+w(re^{i\theta})|^2} d\theta \leq \frac{2}{1-r^2}. \tag{1.5} \]

Hence, from (1.4) and (1.5), we have

\[ \frac{1}{2\pi} \int_0^{2\pi} |p'(re^{i\theta})| d\theta < \frac{k(1-\rho)}{1-r^2}, \]

which is the required result.

From Theorem 1.1 and the properties of the class \( P(\rho) \), we immediately have the following.

**THEOREM 1.4.** Let \( p \in P(\rho) \). Then

\[ \frac{1-k(1-\rho)r + (1-2\rho)r^2}{1-r^2} < \text{Re} p(z) < \frac{1+k(1-\rho)r + (1-2\rho)r^2}{1-r^2} \]

**THEOREM 1.5.** Let \( pcP_k(\rho) \). Then \( peP \) for \( |z| < r_0 \), where \( r_0 \) is given by

\[ r_0 = \frac{2/[k(1-\rho) + \sqrt{k^2(1-\rho)^2 - 4(1-2\rho)}], \rho \neq \frac{1}{2}} {1} \tag{1.6} \]

When \( \rho=0 \), we obtain the results proved in [2].

2. **THE CLASS \( V_k(\rho) \)**

**DEFINITION 2.1.** Let \( V_k(\rho) \) denote the class of analytic and locally univalent functions \( f \) in \( E \) with normalization \( f(0) = 0, f'(0) = 1 \) and satisfying the condition

\[ \frac{(zf'(z))^k}{f'(z)} \in P_k(\rho), \quad 0 < \rho < 1, \quad k > 2 \]

When \( \rho=0 \), we obtain the class \( V_k \) of functions with bounded boundary rotation. The class \( V_k(\rho) \) also generalizes the class \( C(\rho) \) of convex functions of order \( \rho \).

It can easily be seen [1] that \( f \in V_k(\rho) \) if and only if there
exists $F \in V_k$ such that

$$f'(z) = (F'(z))^{1-n} \quad (2.1)$$

In the following, we will study the distortion theorems for the class $V_k(\rho)$. We will use the hypergeometric functions

$$G(a,b;c,z) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(b)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n) \Gamma(b+n)}{\Gamma(c+n)} \frac{z^n}{n!}$$

$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 u^{a-1} (1-u)^{c-a-1} (1-zu)^{-b} du,$$

where $\Re a > 0$ and $\Re(c-a) > 0$. These functions are analytic for $z \in E [4]$. In addition, we define the functions

$$M_1(a,b;c,r) = \frac{2}{a} \left[ G(a,b;c,1) - r_1^{-a} G(a,b;c,-r_1) \right]$$

and

$$M_2(a,b;c,r) = \frac{2}{a} \left[ G(a,b;c,1) - r_1^{-a} G(a,b;c,-r_1) \right], \quad (2.2)$$

where $r_1 = \frac{1 - r}{1 + r}$.

**THEOREM 2.1.** Let $f \in V_k(\rho)$. Then, for $|z| = r (0 < r < 1)$, we have

$$M_2(a,b;c,r) < |f(z)| < M_1(a,b;c,r), \quad (2.3)$$

where

$$a = \left\{ \begin{array}{ll}
\left( \frac{k}{2} - 1 \right) (1 - \rho) + 1, \\
2 \rho
\end{array} \right. \quad (2.4)
$$

$$b = 2 \rho$$

$$c = \left( \frac{k}{2} - 1 \right) (1 - \rho) + 2$$

and $M_1, M_2$ are as defined in (2.2).

This result is sharp.

**PROOF.** Using (2.1) and the well-known bounds for $|F'(z)|$ with $F \in V_k$, see [2], we have

$$\frac{\left( \frac{k}{2} - 1 \right) (1 - \rho)}{\left( \frac{k}{2} + 1 \right) (1 - \rho)} < |f'(z)| < \frac{\left( \frac{k}{2} - 1 \right) (1 - \rho)}{\left( \frac{k}{2} + 1 \right) (1 - \rho)} \quad (2.5)$$

Let $d_r$ denote the radius of the largest schlicht disk centered at the origin contained in the image of $|z| < r$ under $f(z)$. Then there is a point $z_0$, $|z_0| = r$, such that $|f(z_0)| = d_r$. The ray from 0 to $f(z_0)$ lies entirely in the image of $E$ and the inverse image of this ray is a curve in $|z| < r$.

Thus

$$d_r = |f(z_0)| = \int_C |f'(z)| \, |dz|$$
Let $\frac{1-t}{1+t} = \xi$. Then $\frac{-2}{(1+t)^2} dt = d\xi$.

So

$$|f(z_o)| > 2^{2\rho-1} \int_0^1 \left( \frac{k}{2} - 1 \right) (1-\rho) \xi^{(1+\xi)^{-2\rho}} d\xi$$

$$= \int_0^1 \frac{|1-|z||}{1+|z|} \left( \frac{k}{2} - 1 \right) (1-\rho) \xi^{(1+\xi)^{-2\rho}} d\xi$$

Put $\frac{1-t}{1+t} = \frac{1-r}{1+r} = r_1$ and $\xi = r_1 u$.

This gives

$$|f(z_o)| > \frac{b-1}{a} \left[ G(a, b; c, -1) - \xi G(a, b; c, -r_1) \right]$$

$$= M_2(a, b, c, r),$$

where $a, b, c$ and $M_2$ are respectively defined by (2.4) and (2.2).

Similarly we can calculate the lower bound for $|f(z)|$ and this establishes our result.

Equality is attained in (2.3) for the function $f_o \in V_k(\rho)$ defined by

$$f'_o(z) = \frac{(1 + \delta_1 z)^{\frac{k}{2}} - 1}{(1 - \delta_2 z)^{\frac{k}{2}} - 1} (1-\rho), \quad |\delta_1| = |\delta_2| = 1 \quad (2.6)$$

We now study the behaviour of the integral transform

$$f_o(z) = \int_0^z (f'(\xi))^a d\xi \quad (2.7)$$

for $f \in V_k(\rho)$.

This problem has been studied for the class of univalent normalized functions in $\mathbb{E}$ and for the close-to-convex functions, see [3]. We have
THEOREM 2.2. Let \( f \in \mathcal{V}_k(\rho), \ 0 < \rho < 1, \ k > 2 \) and let \( a, 0 < a < 1 \) be given. Then \( f \in \mathcal{V}_m \) for \( m < \frac{1}{2}(1-\rho)(k-2)+2 \).

PROOF. From (2.1), we have

\[
f'(z) = (F'(z))^{1-\rho}, \quad F \in \mathcal{V}_k
\]

Now

\[
f'_a(z) = (f'(z))^a = (F'(z))^a(1-\rho)
\]

\[
= \exp \int_{-\pi}^{\pi} -\log (1-\zeta e^{-it}) a(1-\rho) \, dm(t)
\]

\[
= \exp \int_{-\pi}^{\pi} -\log (1-\zeta e^{-it}) \, d\mu(t),
\]

where \( d\mu(t) = a(1-\rho) \, dm(t) + [1 - a(1-\rho)] \, dt \).

Also

\[
\int_{-\pi}^{\pi} d\mu(t) = a(1-\rho) \int_{-\pi}^{\pi} dm(t) + \frac{1-a(1-\rho)}{2} \int_{-\pi}^{\pi} dt = 2,
\]

and

\[
\int_{-\pi}^{\pi} |d\mu(t)| < a(1-\rho) \int_{-\pi}^{\pi} |dm(t)| + \frac{1-a(1-\rho)}{2} \int_{-\pi}^{\pi} dt
\]

\[
< a(1-\rho)k + 2[1 - a(1-\rho)].
\]

Hence the result.

We note that \( f_a \) is univalent for \( a < \frac{2}{(1-\rho)(k-2)} \), since \( \mathcal{V}_m \) consists of univalent functions for \( 2 < m < 4 \). Hence \( f_a \) is univalent even if \( f \) is not univalent provided \( a < \frac{2}{(1-\rho)(k-2)} \).

Using the standard technique, we can easily prove the following.

THEOREM 2.3. Let \( g, h \in \mathcal{V}_k(\rho) \) and let \( \alpha > 0, \beta > 0 \) and \( \alpha + \beta < 1 - \rho \). Then

\[
H(z) = \int_0^z (g'(t))^\alpha (h'(t))^\beta \, dt
\]

is convex of order \( \alpha_1 = (1 - \frac{\alpha + \beta}{1-\rho}) \) for \( |z| < r_1 \),

where

\[
r_1 = \frac{1}{2} \left[ k - \sqrt{k^2 - 4} \right] \tag{2.8}
\]

The result is sharp when

\[
g'(z) = h'(z) = \left[ \frac{(1-z)^{(k_2-1)(1-\rho)}}{(1+z)^{(k_2+1)(1-\rho)}} \right].
\]
We now prove the following.

**THEOREM 2.4.** Let \( f: f(z) = z + \sum_{n=2}^{\infty} n a_n z^n \in V_k(\rho) \). Then, for all \( n > 3 \),

\[
|a_n| < \left| k^2(1-\rho)^2 + k(1-\rho) \right| z^{-2\rho} \left( \frac{2n}{3} \right)^{(1-\rho)(\frac{k}{2} + 1) - 2}
\]

The function \( f_0 \) defined by (2.6) shows that the exponent \( [(1-\rho)(\frac{k}{2} + 1) - 2] \) is best possible.

**PROOF.** By definition, we have

\[
(zf'(z))' = f'(z) p(z), \quad p \in P_k(\rho).
\]

Set

\[
F(z) = (z(f'(z))')',
\]

\[
= f'(z) \left[ p^2(z) + zp'(z) \right].
\]

For \( z = re^{i\theta} \), we have

\[
n^3 |a_n| < \frac{1}{2\pi} \int_0^{2\pi} |f'(z)| |p^2(z) + zp'(z)| \, d\theta
\]

Using (2.5) and theorem 1.3, we obtain

\[
n^3 |a_n| < \frac{1}{r^{n-3}} \frac{(1-\rho)(\frac{k-2}{2})}{(1-\rho)(\frac{k+2}{2})} \left[ 1 + \frac{[k^2(1-\rho)^2 - 1]r^2 + k(1-\rho)]}{1 - r^2} \right]
\]

\[
= \frac{1}{r^{n-3}} \frac{(1-\rho)(\frac{k-2}{2})}{(1-\rho)(\frac{k+2}{2})+1} \left[ 1+k(1-\rho) + [k^2(1-\rho)^2 - 1]r^2 \right]
\]

Let \( r = 1 - \frac{3}{n} \), \( n > 3 \). Then

\[
n^3 |a_n| < \left| k^2(1-\rho)^2 + k(1-\rho) \right| e^3 \left( 2 - \frac{3}{n} \right) \left( 1-\rho\right)(\frac{k-2}{2}) \left( \frac{n}{3} \right) \left( 1-\rho\right)(\frac{k+2}{2})+1
\]

\[
= \left| k^2(1-\rho)^2 + k(1-\rho) \right| e^3 \left( \frac{n}{3} \right) \left[ (1-\rho)(\frac{k+2}{2}) - 2 \right] \left( 2 - \frac{3}{n} \right) \left( 1-\rho\right)(\frac{k}{2} - 1) - 1
\]

Thus, for \( n \geq 3 \),

\[
|a_n| < \left| k^2(1-\rho)^2 + k(1-\rho) \right| (2)^{-2\rho} \left( \frac{2n}{3} \right)
\]

**THEOREM 2.5.** Let \( f \in V_k(\rho), \ \rho \neq \frac{1}{2} \). Then \( f \) maps \( |z| < r_0 \) onto a convex domain where \( r_0 \) is given by (1.6). The function \( f_0 \), defined by (2.6)

shows that this result is sharp.
The proof is straightforward and follows immediately from the definition and Theorem 1.5.

Furthermore it can easily be shown that if $f \in V_k(\rho)$, then $f$ is convex of order $\rho$ for $|z| < r_1$ where $r_1$ is given by (2.8).

3. THE CLASS $T_k(\rho)$.

A class $T_k$ of analytic functions related with the class $V_k$ has been introduced and studied in [5]. We now define the following.

DEFINITION 3.1. Let $f$ with $f(0) = 0$, $f'(0) = 1$ be analytic in $E$. Then $f \in T_k(\rho)$, $k > 2$, $0 < \rho < 1$, if there exists a function $g \in V_k(\rho)$ such that

$$\frac{f'(z)}{g'(z)} \in P \text{ for } z \in E.$$

Note that $T_k(0) = T_k$ and $T_2(0)$ is the class of close-to-convex functions.

THEOREM 3.1. Let $f \in T_k(\rho)$. Then

$$|f(z)| > M_2(a+1, b; c+1, r),$$

where $M_2(a, b; c, r)$ is defined by (2.2) and $a, b, c$ are given by (2.4). This result is sharp.

PROOF. Since $f \in T_k(\rho)$, we can write

$$f'(z) = g'(z) h(z), \ g \in V_k(\rho), \ h \in P.$$  

It is well-known that for $h \in P$

$$|h(z)| > \frac{1 - |z|}{1 + |z|} \quad (3.1)$$

Thus, using (3.1) and (2.5), we have

$$|f'(z)| > \frac{(1 - |z|)^{k/2 - 1}(1 - \rho) + 1}{(1 + |z|)^{k/2 + 1}(1 - \rho) + 1}$$

Proceeding in the same way as in Theorem 2.1, we obtain the required result.

REMARK 3.1. When $\rho = 0$, $f \in T_k$ and since in this case $b = 0 < 1$, $c = 1 + a - b$, we have $G(a, b; c, -1) = 1$. Letting $r + 1$, with $\rho = 0$, in Theorem 3.1, we see that the image of $E$ under functions $f$ in $T_k$ contains the schlicht disk $|z| < \frac{1}{k+2}$.

We now give a necessary condition for a function $f$ to belong to the class $T_k(\rho)$.

THEOREM 3.2. Let $f \in T_k(\rho)$. Then, with $z = re^{i\theta}$ and $0 < \theta_1 < \theta_2 < 0 < \rho < 1,$
PROOF. We can write
\[ f'(z) = (g_1'(z))^{1-\rho} (h_1(z))^{1-\rho}, \text{ for some } g_1 \in V_k, \ h_1 \in P. \]
So
\[ f'(z) = (g_1'(z) h_1(z))^{1-\rho} = (f_1'(z))^{1-\rho}, \quad (3.3) \]
for some \( f_1 \in T_k. \)
Hence
\[ (zf'(z))' = (1-\rho) \left( \frac{(zf_1'(z))'}{f_1'(z)} \right) + \rho. \]
The required result follows on noting that, for \( 0_1 < 0_2, \ f_1 \in T_k \)
\[ \oint_{0_1}^{0_2} \frac{(zf_1'(z))'}{f_1'(z)} \, d\theta > -k \frac{\pi}{2}, \text{ see } [5]. \]

REMARK 3.2. In [1], Goodman introduced the class \( K(\beta) \) of normalized
analytic functions which are close-to-convex of order \( \beta > 0 \) and showed that
if \( f \) is analytic in \( E \) and \( f'(z) \neq 0 \), then for \( \beta > 0, \ f \in K(\beta) \)
if for \( z = re^{i\theta} \) and \( 0_1 < 0_2 \)
\[ \oint_{0_1}^{0_2} \frac{(zf'(z))'}{f'(z)} \, d\theta > -\beta \pi. \]
When \( 0 < \beta < 1, \ K(\beta) \) consists of univalent functions, whilst if
\( \beta > 1, \ f \) need not even be finitely-valent.

We note that Theorem 3.2 shows that
\[ T_k(\rho) \subseteq K(\frac{k(1-\rho)}{2}). \]
Hence \( T_k(\rho) \) consists entirely of univalent functions if \( 2 < k < \frac{2}{1-\rho} \). It
also follows easily from the definition that the class \( T_k(\rho) \) forms a sub-
set of a linear-invariant family of order \( \frac{k(1-\rho)+1}{2} \).

Using the method of Clunie and Pommerenke as modified by Thomas [7],
we can easily prove the following:

THEOREM 3.3. Denote by \( L(r,f) \) the length of the image of the circle
\( |u|=r \) under \( f \) and by \( M(r) = \max |f(re^{i\theta})| \). Then, for \( 0 < r < 1, \)
\[ L(r) < A(k,\rho) M(r) \log \frac{1}{1-r}, \]
where \( A(k,\rho) \) is a constant depending only on \( k \) and \( \rho \).

Let \( p_{a,1} \) denote the class of functions \( p(z) \) in \( E \) given by
\[ p(z) = 1 + c_1 z + c_2 z^2 + \cdots \]

which satisfy the inequality
\[ \left| p(z) - \frac{1}{2a} \right| < \frac{1}{2a}, \quad 0 < a < 1 \]

The class \( P_{\alpha,1} \) has been introduced in [8] and it is shown there that, for \( p \in P_{\alpha,1} \), \( |z| = r < 1 \).

\[ \frac{|p'(z)|}{|p(z)|} < \frac{(1 + c)}{(1 + cr)(1 - r)}, \quad \text{(3.4)} \]

where \( c = 1 - 2a \)

We now prove the following.

**THEOREM 3.4.** Let \( g \in V_k(\rho) \) and let \( \frac{f'(z)}{g'(z)} \in P_{\alpha,1} \). Then \( f \) is a convex function of order \( \rho \) for \( |z| < r \) where \( r \in (0,1) \) is the least positive root of the equation

\[ (1-\rho)c_3 - [(\rho+c) + ck(1-\rho)]x^2 + [\rho(k-c) - (1+k)]x + (1-\rho) = 0 \]

**PROOF.** We can write

\[ f'(z) = (g'_1(z))^{1-\rho} p(z), \quad g_1 \in V_k, \quad p \in P_{\alpha,1} \]

So

\[ \text{Re} \left[ \frac{(zf'(z))'}{f'(z)} - \rho \right] > (1-\rho) \text{Re} \left[ \frac{(xg'_1(z))'}{g'_1(z)} \right] - \frac{|zp'(z)|}{p(z)} \]

Using Theorem 1.4 with \( \rho = 0 \) and (3.4), we have the required result.

Furthermore, if

\[ T(r) = (1-\rho)cr^3 - [(\rho+c) + ck(1-\rho)]r^2 + [\rho(k-c) - (1+k)]r + (1-\rho), \]

then we note that

\[ T(0) = (1-\rho) > 0 \]
\[ T(1) = -2\rho c - 2\rho - ck(1-\rho) - k(1-\rho) < 0 \]

Thus \( r \in (0,1) \).

**COROLLARY 3.1.** When \( \alpha = 0, c = 1 \) and \( \rho = 0 \), \( f \in T_k \). Thus \( f \) maps \( |z| < r = \frac{1}{2}(k+2) - \sqrt{k^2 + 4k} \) onto a convex domain and this result is sharp, see [5].

**COROLLARY 3.2.** When \( \rho = 0, \alpha = \frac{1}{2} \), and then we have \( \frac{|f'(z)|}{g'(z)} = 1 \) for \( g \in V_k \). Then \( f \) is convex for \( |z| < r = \frac{1}{k+1} \). For \( k=4 \), \( V_k \) consists of univalent functions and in this case \( r = \frac{1}{5} \). This result is proved in [8]. For \( \alpha = 0, k = 4 \) and \( \rho = 0 \), we obtain the known result \( r = 3 - 2\sqrt{2} \) of Ratti [9] and when \( k = 2 \), we have the well-known result giving us the radius of convexity for close-to-convex functions.
Finally we have

**THEOREM 3.5.** Let \( f \in V_k(\rho) \) and let

\[
F(z) = \frac{1}{1+m} z^{1-m} \left| z^m f(z) \right|', \quad m = 1, 2, 3, \ldots
\]

Then \( F \in T_k(\rho) \) for all \( |z| < r_2 \), where, for \( (1-2\rho-m) \neq 0, 0 < \rho < 1, \)

\[
r_2 = \frac{2(1+m)}{(1-\rho)k + \sqrt{(1-\rho)^2k^2 - 4(1-2\rho-m)(1+m)}},
\]

The proof is straightforward when we note that

\[
\text{Re} \frac{F'(z)}{f'(z)} = \frac{1}{1+m} \left[ \text{Re} \left( \frac{zf'(z)}{f'(z)} \right) + m \right]
\]

and then use theorem 1.4.

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**REFERENCES**


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