ON GENERALIZATION OF CONTINUED FRACTION OF GAUSS

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ABSTRACT. In this paper we establish a continued fraction representation for the ratio of two basic bilateral hypergeometric series \( \psi_2 \)'s which generalize Gauss' continued fraction for the ratio of two \( F_1 \)'s.

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1. INTRODUCTION.

Gauss (see Wall [3] and also Jones and Thron [2], gave the following continued fraction involving the ratio of two Gaussian \( F_1 \)'s,

\[
\frac{F_{\{a, b+1; \frac{c}{c+1}\}}}{F_{\{a, b; \frac{c}{c+1}\}}} = 1 - \frac{a(c-b)z/(c(c+1))}{1 - \frac{(b+1)(c-a+1)z/(c+1)(c+2)}{1 - \frac{(a+1)(c-b+1)z/(c+2)(c+3)}{1 - \frac{(b+2)(c-a+2)z/(c+3)(c+4)}{1 - \vdots}}}}
\]

where

\[
\psi_{\{a, \beta; \gamma\}} = \sum_{n=0}^{\infty} \frac{[a]_n [\beta]^n z^n}{[\gamma]_n^n} \quad (|z| < 1)
\]

in which the symbol \([a]_n\) stands for \(a(a+1)(a+2)\ldots(a+n-1)\) and \([a]_0 = 1\).

In this paper we establish the continued fraction for the ratio

\[
\psi_{\{a, \beta; \gamma\}} / \psi_{\{a, \beta; \gamma\}} = \sum_{n=0}^{\infty} \frac{[a]_n [\beta]^n z^n}{[\delta]_n [\gamma]_n^n} \quad (|\delta/\alpha\beta| < |x| < 1, |q| < 1),
\]

where

\[
\psi_{\{a, \beta; \gamma\}} = \sum_{n=0}^{\infty} \frac{[a]_n [\beta]^n x^n}{[\delta]_n [\gamma]_n^n} \quad (|\delta/\alpha\beta| < |x| < 1, |q| < 1),
\]
where
\[ [\alpha]_n \equiv [\alpha; q]_n = (1 - \alpha)(1 - \alpha q)...(1 - \alpha q^{n-1}), [\alpha]_0 = 1. \]

The other notations appearing in this paper carry their usual meaning.

2. MAIN RESULT.

In this paper we establish the following result

\[ 2\psi_2 \left[ \alpha, \beta q; x \right] / 2\psi_2 \left[ \alpha, \beta; x \right] \]
\[ = \frac{1}{A_o + C_o} + \frac{1}{A_1 + C_1} + \frac{1}{A_2 + C_2} + \cdots, \]

where for \( i = 0, 1, 2, 3, \ldots \)

\[ A_i = \frac{(1 - \beta q^i)(\gamma q^{2i+1} - \delta)}{(1 - \gamma q^{2i+1})(\beta q^i + 1 - \delta)}, \]

\[ B_i = \frac{q^{i+1}(1 - \alpha q^i)(1 - \beta q^i)(\beta - \gamma q^i)}{(1 - \gamma q^{2i+1})(1 - \gamma q^{2i+1})(\beta q^i + 1 - \delta)}, \]

\[ C_i = \frac{(1 - \alpha q^i)(\gamma q^{2i+2} - \delta)}{(1 - \gamma q^{2i+1})(\alpha q^i + 1 - \delta)}, \]

and

\[ D_i = \frac{q^{i+1}(1 - \beta q^i)(1 - \alpha q^i)(\gamma q^{i+1})}{(1 - \gamma q^{2i+1})(1 - \gamma q^{2i+1})(\alpha q^i + 1 - \delta)}. \]

PROOF of (2.1). It is easy to see that the following relation is true (for nonnegative integral \( i \)),

\[ 2\psi_2 \left[ \alpha, \beta q^i; x \right] / 2\psi_2 \left[ \alpha, \beta q^i; x \right] \]
\[ = A_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^i; x \right] + x B_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^i; x \right] \]
\[ = D_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^i; x \right] + x C_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^i; x \right] \]

(2.2)

Now, interchanging \( \alpha \) and \( \beta \) in (2.2) and then replacing \( \beta \) by \( \beta q \) and \( \gamma \) by \( \gamma q \) in it, we get

\[ 2\psi_2 \left[ \alpha, \beta q^i; x \right] / 2\psi_2 \left[ \alpha, \beta q^{i+1}; x \right] \]
\[ = C_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^{i+1}; x \right] + x D_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^{i+2}; x \right] \]
\[ = D_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^{i+1}; x \right] + x C_i \frac{1}{2\psi_2} \left[ \alpha, \beta q^{i+2}; x \right] \]

(2.3)
Now from (2.2) for \( i = 0 \), we get

\[
2^{\psi_2} \left[ \frac{a, b; x}{\delta, \gamma q} \right] / 2^{\psi_2} \left[ a, b; x \right]
\]

\[
= A_0 + \frac{x B_0}{2^{\psi_2} \left[ a, b; x \right]} + \frac{2^{\psi_2} \left[ aq, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^2 \right]} + \frac{2^{\psi_2} \left[ a, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^2 \right]}
\]

from (2.3) with \( i = 0 \)

\[
= A_0 + \frac{x B_0}{2^{\psi_2} \left[ aq, b; x \right]} + \frac{2^{\psi_2} \left[ aq, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^3 \right]} + \frac{2^{\psi_2} \left[ aq, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^3 \right]}
\]

from (2.2) with \( i = 1 \)

\[
= A_0 + \frac{x B_0}{2^{\psi_2} \left[ aq, b; x \right]} + \frac{2^{\psi_2} \left[ aq, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^4 \right]} + \frac{2^{\psi_2} \left[ aq, b; x \right]}{2^{\psi_2} \left[ \delta, \gamma q^4 \right]}
\]

(by repeated application of (2.2) and (2.3)). This proves (2.1).

3. SPECIAL CASES.

Here we shall reduce certain interesting special cases of (2.1). If in (2.1) we take \( \delta = q \), we get

\[
2^{\psi_1} \left[ \frac{a, b; x}{\gamma q} \right] / 2^{\psi_1} \left[ a, b; x \right]
\]

\[
= \frac{1}{1 + \frac{x u_0}{1 + \frac{x v_0}{1 + \frac{x n_1}{1 + \frac{x u_1}{1 + \frac{x u_2}{1 + \cdots}}}}}}
\]

where for \( i = 0, 1, 2, \ldots \)

\[
u_i = q^i (1-aq^i)(\gamma q^{i-1} - \beta)/(1- \gamma q^{2i})(1- \gamma q^{2i+1})
\]

and

\[
u_i = q^i (1-bq^{i+1})(\gamma q^{i+1} - \alpha)/(1- \gamma q^{2i+1})(1- \gamma q^{2i+2}).
\]

If \( q + 1 \) in (3.1), we get (1.1), the continued fraction of Gauss.

If in (3.1) we take \( b = 1 \) and replace \( \gamma \) by \( \gamma / q \), we get,
\[ 2^\Phi_2 \left[ \begin{array}{c} a, q; x \\ \gamma \end{array} \right] \]

\[ = \frac{1}{1} + \frac{x \mu_0}{1} + \frac{x \nu_0}{1} + \frac{x \mu_1}{1} + \frac{x \nu_1}{1} + \frac{x \mu_2}{1} + \ldots , \]  

(3.2)

where for \( i = 0, 1, 2, \ldots \)

\[ u_i = -q^i (1-aq^i)(1-q^{i-1} - 1)(1-q) \]

and

\[ v_i = -q^i (1+q^i)/(1-q^{2i-1})(1-q^{2i+1}) . \]

Now, if in (3.2) we let \( q \to 1 \), we get the following known result [2]

\[ F \left[ \begin{array}{c} a, q; x \\ \gamma \end{array} \right] \]

\[ = \frac{1}{1} - \frac{x \xi_0}{1} + \frac{x \eta_0}{1} - \frac{x \xi_1}{1} + \frac{x \eta_1}{1} - \frac{x \xi_2}{1} + \ldots , \]  

(3.3)

where for \( i = 0, 1, 2, \ldots \)

\[ \xi_1 = (a+1)(y+1)/(y+2i-1)(y+2i) \]

and

\[ \eta_1 = (i+1)(y-a+1)/(y+2i)(y+2i+1) . \]

If we put \( y = 0 \) in (3.2) and replace \( x \) by \( xq/a \) and then let \( a \to \infty \), we get the following interesting result

\[ \frac{1}{1} + \frac{x q}{1} + \frac{x(q-1)}{1} + \frac{x^2 (q-1)}{1} + \frac{x^3 (q-1)}{1} + \ldots , \]  

(3.4)

If we take \( y = q \) in (3.2) we get a continued fraction representation for \( F \left[ a; -; x \right] \) which, when \( q \to 1 \), yields the continued fraction representation for general binomial \( (1-x)^{-a} \).

Again, if we take \( a = q, \gamma = q^2 \) and replace \( x \) by \(-x\) in (3.2), we get a continued fraction representation for \( F \left[ q; q; q^2; -x \right] \) which, when \( q \to 1 \) yields the continued fraction representation for

\[ \frac{1}{x} \log (1+x) = F \left[ \begin{array}{c} 1,1; -x \\ 2 \end{array} \right] . \]

Similarly, we can get the continued fraction representation for

\[ \log \left( \frac{2x}{1-x} \right) = 2x F \left[ \begin{array}{c} 1/2, 1; x \\ 3/2 \end{array} \right] . \]
Further, if we take \( q = 0 \) in (3.1), we get the following result after some simplification,

\[
\frac{\phi_1 \left[ \frac{\beta; x}{q} \right]}{\phi_1 \left[ \frac{\beta_q; x}{q} \right]} = 1 + \frac{x_{\nu_0}}{1} + \frac{x_{\nu_1}}{1} + \frac{x_{\nu_2}}{1} + \cdots ,
\]

where for \( i = 0, 1, 2, \ldots \)

\[
\mu_i = q^i (q - 1)^2 (1 - q^{2i+1})
\]

and

\[
\nu_i = q^{2i+1} (1 - q^{i+1}) (1 - q^{2i+1}) (1 - q^{2i+2}) .
\]

The above (3.5) is the \( q \)-analogue of a known result [2].

Again, setting \( \beta = 1 \) in (3.5) we get the continued fraction representation for \( \phi_1 \left[ \frac{\beta; x}{q} \right] \) from which one can, for \( \gamma = 1 \), deduce the corresponding continued fraction expression for \( q \)-exponential function \( e_q(x) \) which in turn yields the continued fraction representation for exponential function \( e^z \) when \( q = 1 \) [2].

A number of other interesting special cases could also be deduced. The reader is referred to Wall [1] and Jones [2].

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REFERENCES

This subject has been extensively studied in the past years for one-, two-, and three-dimensional space. Additionally, such dynamical systems can exhibit a very important and still unexplained phenomenon, called as the Fermi acceleration phenomenon. Basically, the phenomenon of Fermi acceleration (FA) is a process in which a classical particle can acquire unbounded energy from collisions with a heavy moving wall. This phenomenon was originally proposed by Enrico Fermi in 1949 as a possible explanation of the origin of the large energies of the cosmic particles. His original model was then modified and considered under different approaches and using many versions. Moreover, applications of FA have been of a large broad interest in many different fields of science including plasma physics, astrophysics, atomic physics, optics, and time-dependent billiard problems and they are useful for controlling chaos in Engineering and dynamical systems exhibiting chaos (both conservative and dissipative chaos).

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