GENERALIZATION OF CERTAIN SUBCLASSES OF ANALYTIC FUNCTIONS

TADAYUKI SEKINE

Department of Mathematics
Science and Technology
Nihon University
1-8 Kanda Surugadai, Chiyoda-ku
Tokyo 101, Japan

(Received June 16, 1986 and in revised form September 18, 1986)

ABSTRACT. We introduce the subclass $T_j(n,m,a)$ of analytic functions with negative coefficients by the operator $D^n$. Coefficient inequalities and distortion theorems of functions in $T_j(n,m,a)$ are determined. Further, distortion theorems for fractional calculus of functions in $T_j(n,m,a)$ are obtained.

KEYWORDS AND PHRASES. Analytic functions, negative coefficients, coefficient inequalities, distortion theorem, fractional calculus.


1. INTRODUCTION.

Let $A_j$ denote the class of functions of the form

$$f(z) = z + \sum_{k=j+1}^{\infty} a_k z^k \quad (j \in \mathbb{N} = \{1, 2, 3, \ldots\}) \quad (1.1)$$

which are analytic in the unit disk $U = \{z: |z| < 1\}$.

For a function $f(z)$ in $A_j$, we define

$$D^0 f(z) = f(z), \quad (1.2)$$

$$D^1 f(z) = Df(z) = zf'(z), \quad (1.3)$$

and

$$D^n f(z) = D(D^{n-1} f(z)) \quad (n \in \mathbb{N}). \quad (1.4)$$

With the above operator $D^n$, we say that a function $f(z)$ belonging to $A_j$ is in the class $A_j(n,m,a)$ if and only if

$$\text{Re} \left[ \frac{D^{n+m} f(z)}{D^n f(z)} \right] > a \quad (n,m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}) \quad (1.5)$$

for some $a \ (0 \leq a < 1)$, and for all $z \in U$. 
We note that $A_1(0,1,\alpha) = S^*(\alpha)$ is the class of starlike functions of order $\alpha$, $A_1(1,1,\alpha) = K(\alpha)$ is the class of convex functions of order $\alpha$, and that $A_1(n,1,\alpha) = S_n(\alpha)$ is the class of functions defined by Salagean [1].

Let $T_j$ denote the subclass of $A_j$ consisting of functions of the form

$$f(z) = z - \sum_{k=j+1}^{\infty} a_k z^k \quad (a_k \geq 0; \ j \in \mathbb{N}).$$

Further, we define the class $T_j(n,m,\alpha)$ by

$$T_j(n,m,\alpha) = A_j(n,m,\alpha) \cap T_j.$$  \hspace{1cm} (1.7)

Then we observe that $T_1(0,1,\alpha) = T^*(\alpha)$ is the subclass of starlike functions of order $\alpha$ (Silverman [2]), $T_1(1,1,\alpha) = C(\alpha)$ is the subclass of convex functions of order $\alpha$ (Silverman [2]), and that $T_1(0,1,\alpha)$ and $T_1(1,1,\alpha)$ are the classes defined by Chatterjea [3].

2. DISTORTION THEOREMS.

We begin with the statement and the proof of the following result.

**Lemma 1.** Let the function $f(z)$ be defined by (1.6) with $j = 1$. Then $f(z) \in T_1(n,m,\alpha)$ if and only if

$$\sum_{k=2}^{\infty} k^n (k^n - 1) a_k \leq 1 - \alpha$$

for $n \in \mathbb{N}_0$, $m \in \mathbb{N}_0$, and $0 \leq \alpha < 1$. The result is sharp.

**Proof.** Assume that the inequality (2.1) holds and let $|z| = 1$. Then we have

$$\left| \frac{D^{n+m}f(z)}{D^n f(z)} - 1 \right| \leq \frac{\sum_{k=2}^{\infty} k^n (k^n - 1) a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k |z|^{k-1}}$$

$$= \frac{\sum_{k=2}^{\infty} k^n (k^n - 1) a_k}{1 - \sum_{k=2}^{\infty} k^n a_k}$$

$$\leq 1 - \alpha$$

which implies (1.5). Thus it follows from this fact that $f(z) \in T_1(n,m,\alpha)$.

Conversely, assume that the function $f(z)$ is in the class $T_1(n,m,\alpha)$. Then

$$\text{Re} \left[ \frac{D^{n+m}f(z)}{D^n f(z)} \right] = \text{Re} \left[ \frac{1 - \sum_{k=2}^{\infty} k^{n+m} a_k z^{k-1}}{1 - \sum_{k=2}^{\infty} k^n a_k z^{k-1}} \right]$$

$$> \alpha$$  \hspace{1cm} (2.3)
for \( z \in U \). Choose values of \( z \) on the real axis so that \( D^{n+m}f(z)/D^n f(z) \) is real. Upon clearing the denominator in (2.3) and letting \( z \) through real values, we obtain

\[
1 - \sum_{k=2}^{\infty} k^{n+m} a_k \geq \alpha (1 - \sum_{k=2}^{\infty} k^n a_k) \tag{2.4}
\]

which gives (2.1). The result is sharp with the extremal function \( f(z) \) defined by

\[
f(z) = z - \frac{1 - \alpha}{k^n (k^m - \alpha)} z^k \quad (k \geq 2) \tag{2.5}
\]

**Remark 1.** In view of Lemma 1, \( T_l(n,m,a) \) when \( n \in N_0 \) and \( m \in N \) is the subclass of \( T^*(a) \) introduced by Silverman [2], and \( T_l(n,m,a) \) when \( n \in N \) and \( m \in N \) is the subclass of \( C(a) \) introduced by Silverman [2].

With the aid of Lemma 1, we prove

**Theorem 1.** Let the function \( f(z) \) be defined by (1.6). Then \( f(z) \in T_j(n,m,a) \) if and only if

\[
\sum_{k=j+1}^{\infty} k^n (k^m - a) a_k \leq 1 - \alpha \tag{2.6}
\]

for \( n \in N_0 \), \( m \in N_0 \) and \( 0 \leq \alpha < 1 \). The result is sharp for the function

\[
f(z) = z - \frac{1 - \alpha}{k^n (k^m - a)} z^k \quad (k \geq j + 1). \tag{2.7}
\]

**Proof.** Putting \( a_k = 0 \) \( (k = 2, 3, 4, \ldots, j) \) in Lemma 1, we can prove the assertion of Theorem 1.

**Corollary 1.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[
a_k \leq \frac{1 - \alpha}{k^n (k^m - a)} \quad (k \geq j + 1). \tag{2.8}
\]

The equality in (2.8) is attained for the function \( f(z) \) given by (2.7).

**Corollary 2.** \( T_j(n+1,m,a) \subset T_j(n,n,a) \) and \( T_j(n,m+1,a) \subset T_j(n,m,a) \).

**Remark 2.** Taking \( (j,n,m) = (1,0,1) \) and \( (j,n,m) = (1,1,1) \) in Theorem 1, we have the corresponding results by Silverman [2]. Taking \( (j,n,m) = (j,0,1) \) and \( (j,n,m) = (1,1,1) \) in Theorem 1, we have the corresponding results by Chatterjea [3].

**Theorem 2.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[
|D^i f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^{n-i} ((j + 1)^m - \alpha)} |z|^{j+1} \tag{2.9}
\]

and

\[
|D^i f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^{n-i} ((j + 1)^m - \alpha)} |z|^{j+1} \tag{2.10}
\]

for \( z \in U \), where \( 0 \leq i \leq n \). The equalities in (2.9) and (2.10) are attained for the
function \( f(z) \) given by

\[
f(z) = z - \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)} z^{j+1}
\]  
(2.11)

**PROOF.** Note that \( f(z) \in T^j_{n,m,\alpha} \) if and only if \( D^i f(z) \in T^j_{n-i,m,\alpha} \), and that

\[
D^i f(z) = z - \sum_{k=j+1}^{\infty} k^i a_k z^k.
\]  
(2.12)

Using Theorem 1, we know that

\[
(j + 1)^{n-i}((j + 1)^m - \alpha) \sum_{k=j+1}^{\infty} k^i a_k \leq 1 - \alpha,
\]  
(2.13)

that is, that

\[
\sum_{k=j+1}^{\infty} k^i a_k \leq \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)}.
\]  
(2.14)

It follows from (2.12) and (2.14) that

\[
|D^i f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)} |z|^{j+1}
\]  
(2.15)

and

\[
|D^i f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)} |z|^{j+1}.
\]  
(2.16)

Finally, we note that the equalities in (2.9) and (2.10) are attained for the function \( f(z) \) defined by

\[
D^i f(z) = z - \frac{1 - \alpha}{(j + 1)^{n-i}((j + 1)^m - \alpha)} z^{j+1}.
\]  
(2.17)

This completes the proof of Theorem 2.

**COROLLARY 3.** Let the function \( f(z) \) defined by (1.6) be in the class \( T^j_{n,m,\alpha} \). Then

\[
|f(z)| \geq |z| - \frac{1 - \alpha}{(j + 1)^{n}((j + 1)^m - \alpha)} |z|^{j+1}
\]  
(2.18)

and

\[
|f(z)| \leq |z| + \frac{1 - \alpha}{(j + 1)^{n}((j + 1)^m - \alpha)} |z|^{j+1}
\]  
(2.19)

for \( z \in U \). The equalities in (2.18) and (2.19) are attained for the function \( f(z) \) given by (2.11).

**PROOF.** Taking \( i = 0 \) in Theorem 2, we can easily show (2.18) and (2.19).
COROLLARY 4. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|f'(z)| \geq 1 - \frac{1 - \alpha}{(j + 1)^{n-1}\{(j + 1)^m - \alpha\}}|z|^j$$

(2.20)

and

$$|f'(z)| \leq 1 + \frac{1 - \alpha}{(j + 1)^{n-1}\{(j + 1)^m - \alpha\}}|z|^j$$

(2.21)

for $z \in U$. The equalities in (2.20) and (2.21) are attained for the function $f(z)$ given by (2.11).

PROOF. Note that $Df(z) = zf'(z)$. Hence, making $i = 1$ in Theorem 2, we have the corollary.

REMARK 3. Taking $(j,n,m) = (1,0,1)$ and $(j,n,m) = (1,1,1)$ in Corollary 3 and Corollary 4, we have distortion theorems due to Silverman [2].

3. DISTORTION THEOREMS FOR FRACTIONAL CALCULUS.

In this section, we use the following definitions of fractional calculus by Owa [4].

DEFINITION 1. The fractional integral of order $\lambda$ is defined by

$$D^{-\lambda}f(z) = \frac{1}{\Gamma(\lambda)} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,$$  

(3.1)

where $\lambda > 0$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\xi)^{\lambda-1}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

DEFINITION 2. The fractional derivative of order $\lambda$ is defined by

$$D^\lambda f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \int_0^z \frac{f(\xi)}{(z-\xi)^{1-\lambda}} d\xi,$$  

(3.2)

where $0 < \lambda < 1$, $f(z)$ is an analytic function in a simply connected region of the $z$-plane containing the origin and the multiplicity of $(z-\xi)^{-\lambda}$ is removed by requiring $\log(z-\xi)$ to be real when $(z-\xi) > 0$.

DEFINITION 3. Under the hypotheses of Definition 2, the fractional derivative of order $(n+\lambda)$ is defined by

$$D^{n+\lambda} f(z) = \frac{d^n}{dz^n} D^\lambda f(z),$$

(3.3)

where $0 < \lambda < 1$ and $n \in \{0,1,2,3,\ldots\}$.

THEOREM 3. Let the function $f(z)$ defined by (1.6) be in the class $T_j(n,m,\alpha)$. Then

$$|D^{-\lambda}(D^zf(z))| \geq \frac{|z|^{1+\lambda}}{\Gamma(2+\lambda)} \left(1 - \frac{\Gamma(i+2)\Gamma(j+\lambda),(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^{n-i}\{(j+1)^m-\alpha\}}|z|^j\right)$$

(3.4)
The equalities in (3.4) and (3.5) are attained for the function \( f(z) \) given by (2.11).

**Proof.** It is easy to see that

\[
\Gamma(2 + \lambda) z^{-\lambda} D_z^{-\lambda} (D^i f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k + 1)\Gamma(2 + \lambda)(1 - a)}{\Gamma(k + 1 + \lambda)(j + 1)n^{-i}\{(j + 1)^m - a}\} |z|^{j+1} k^i a_k.
\]

Since the function

\[
\phi(k) = \frac{\Gamma(k + 1 + \lambda)}{\Gamma(k + 1 + \lambda)} (k \geq j + 1)
\]

is decreasing in \( k \), we have

\[
0 < \phi(k) \leq \phi(j + 1) = \frac{\Gamma(j + 2)\Gamma(2 + \lambda)}{\Gamma(j + 2 + \lambda)}.
\]

Therefore, by using (2.14) and (3.8), we can see that

\[
|\Gamma(2 + \lambda) z^{-\lambda} D_z^{-\lambda} (D^i f(z))| \geq |z| - \phi(j + 1) |z|^{j+1} \sum_{k=j+1}^{\infty} k^i a_k
\]

which implies (3.4), and that

\[
|\Gamma(2 + \lambda) z^{-\lambda} D_z^{-\lambda} (D^i f(z))| \leq |z| + \frac{\Gamma(j + 2)\Gamma(2 + \lambda)(1 - a)}{\Gamma(j + 2 + \lambda)(j + 1)n^{-i}\{(j + 1)^m - a}\} |z|^{j+1}
\]

which shows (3.5). Furthermore, note that the equalities in (3.4) and (3.5) are attained for the function \( f(z) \) defined by

\[
D_z^{-\lambda} (D^i f(z)) = \frac{z^{1+\lambda}}{\Gamma(2 + \lambda)} \left( 1 - \frac{\Gamma(j + 2)\Gamma(2 + \lambda)(1 - a)}{\Gamma(j + 2 + \lambda)(j + 1)n^{-i}\{(j + 1)^m - a}\} z^j \right)
\]

or (2.17). Thus we complete the assertion of Theorem 3.

Taking \( i = 0 \) in Theorem 3, we have

**Corollary 5.** Let the function \( f(z) \) by (1.6) be in the class \( T_j(n,m,a) \). Then
\[ |D_z^\lambda f(z)| \geq |z|^{1+\lambda} \left( 1 - \frac{\Gamma(1+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^n((j+1)^m - \alpha)} |z|^j \right) \quad (3.12) \]

and

\[ |D_z^\lambda f(z)| \leq |z|^{1+\lambda} \left( 1 + \frac{\Gamma(1+2)\Gamma(2+\lambda)(1-\alpha)}{\Gamma(j+2+\lambda)(j+1)^n((j+1)^m - \alpha)} |z|^j \right) \quad (3.13) \]

for \( \lambda > 0 \) and \( z \in U \). The equalities in (3.12) and (3.13) are attained for the function \( f(z) \) given by (2.11).

Finally, we prove

**THEOREM 4.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,a) \). Then

\[ |D_z^\lambda(D^i f(z))| \geq |z|^{1-\lambda} \left( 1 - \frac{\Gamma(1+i)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-1}((j+1)^m - \alpha)} |z|^j \right) \quad (3.14) \]

and

\[ |D_z^\lambda(D^i f(z))| \leq |z|^{1-\lambda} \left( 1 + \frac{\Gamma(1+i)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-1}((j+1)^m - \alpha)} |z|^j \right) \quad (3.15) \]

for \( 0 \leq \lambda < 1 \), \( 0 \leq i \leq n-1 \), and \( z \in U \). The equalities in (3.14) and (3.15) are attained for the function \( f(z) \) given by (2.11).

**PROOF.** A simple computation gives that

\[ \Gamma(2-\lambda)z^\lambda D_z^\lambda(D^i f(z)) = z - \sum_{k=j+1}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} a_k z^k. \quad (3.16) \]

Note that the function

\[ \psi(k) = \frac{\Gamma(k)\Gamma(2-\lambda)}{\Gamma(k+1-\lambda)} \quad (k \geq j+1) \quad (3.17) \]

is decreasing in \( k \). It follows from this fact that

\[ 0 < \psi(k) \leq \psi(j+1) = \frac{\Gamma(j+1)\Gamma(2-\lambda)}{\Gamma(j+2-\lambda)}. \quad (3.18) \]

Consequently, with the aid of (2.14) and (3.18), we have

\[ |\Gamma(2-\lambda)z^\lambda D_z^\lambda(D^i f(z))| \geq |z| - \psi(j+1) |z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1}a_k \]

\[ \geq |z| - \frac{\Gamma(j+1)\Gamma(2-\lambda)(1-\alpha)}{\Gamma(j+2-\lambda)(j+1)^{n-1}((j+1)^m - \alpha)} |z|^{j+1} \quad (3.19) \]

and

\[ |\Gamma(2-\lambda)z^\lambda D_z^\lambda(D^i f(z))| \leq |z| + \psi(j+1) |z|^{j+1} \sum_{k=j+1}^{\infty} k^{i+1}a_k \]
Thus (3.14) and (3.15) follow from (3.19) and (3.20), respectively. Further, since the equalities in (3.19) and (3.20) are attained for the function \( f(z) \) defined by

\[
D_z^\lambda (D_z^i f(z)) = \frac{1}{\Gamma(2 - \lambda)} \left[ 1 - \frac{\Gamma(j + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)^{n-1}((j + 1)^m - \alpha)} z^j \right],
\]

that is, by (2.17), this completes the proof of Theorem 4.

Making \( i = 0 \) in Theorem 4, we have

**COROLLARY 6.** Let the function \( f(z) \) defined by (1.6) be in the class \( T_j(n,m,\alpha) \).

Then

\[
|D_z^\lambda f(z)| \geq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left[ 1 - \frac{\Gamma(j + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)^{n-1}((j + 1)^m - \alpha)} |z|^j \right]
\]

and

\[
|D_z^\lambda f(z)| \leq \frac{|z|^{1-\lambda}}{\Gamma(2 - \lambda)} \left[ 1 + \frac{\Gamma(j + 1)\Gamma(2 - \lambda)\cdot(1 - \alpha)}{\Gamma(j + 2 - \lambda)(j + 1)^{n-1}((j + 1)^m - \alpha)} |z|^j \right]
\]

for \( 0 \leq \lambda < 1 \) and \( z \in U \). The equalities in (3.22) and (3.23) are attained for the function \( f(z) \) given by (2.11).

**ACKNOWLEDGEMENT.** The author wishes to thank the referee for his helpfull comments.

**REFERENCES**

Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

Authors should follow the Mathematical Problems in Engineering manuscript format described at http://www.hindawi.com/journals/mpe/. Prospective authors should submit an electronic copy of their complete manuscript through the journal Manuscript Tracking System at http://mts.hindawi.com/ according to the following timetable:

<table>
<thead>
<tr>
<th>Manuscript Due</th>
<th>December 1, 2008</th>
</tr>
</thead>
<tbody>
<tr>
<td>First Round of Reviews</td>
<td>March 1, 2009</td>
</tr>
<tr>
<td>Publication Date</td>
<td>June 1, 2009</td>
</tr>
</tbody>
</table>

Guest Editors

José Roberto Castilho Piqueira, Telecommunication and Control Engineering Department, Polytechnic School, The University of São Paulo, 05508-970 São Paulo, Brazil; piqueira@lac.usp.br

Elbert E. Neher Macau, Laboratório Associado de Matemática Aplicada e Computação (LAC), Instituto Nacional de Pesquisas Espaciais (INPE), São José dos Campos, 12227-010 São Paulo, Brazil; elbert@lac.inpe.br

Celso Grebogi, Center for Applied Dynamics Research, King’s College, University of Aberdeen, Aberdeen AB24 3UE, UK; grebogi@abdn.ac.uk