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*Research Article*

# Convergence Analysis for a System of Generalized Equilibrium Problems and a Countable Family of Strict Pseudocontractions

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We introduce a new iterative algorithm for a system of generalized equilibrium problems and a countable family of strict pseudocontractions in Hilbert spaces. We then prove that the sequence generated by the proposed algorithm converges strongly to a common element in the solutions set of a system of generalized equilibrium problems and the common fixed points set of an infinitely countable family of strict pseudocontractions.

## 1. Introduction

Let $H$ be a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and inducted norm $\|\cdot\|$. Let $C$ be a nonempty, closed, and convex subset of $H$. Let $\{f_k\}_{k\in\Lambda} : C\times C \to \mathbb{R}$ be a family of bifunctions, and let $\{A_k\}_{k\in\Lambda} : C \to H$ be a family of nonlinear mappings, where $\Lambda$ is an arbitrary index set. The system of generalized equilibrium problems is to find $\hat{x}\in C$ such that

$$f_k(\hat{x},y)+\langle A_k\hat{x},y-\hat{x}\rangle \geq 0, \quad \forall y\in C,\ k\in\Lambda. \tag{1.1}$$

If $\Lambda$ is a singleton, then (1.1) reduces to find $\hat{x}\in C$ such that

$$f(\hat{x},y)+\langle A\hat{x},y-\hat{x}\rangle \geq 0, \quad \forall y\in C. \tag{1.2}$$

The solutions set of (1.2) is denoted by $\mathrm{GEP}(f,A)$. If $f\equiv 0$, then the solutions set of (1.2) is denoted by $\mathrm{VI}(C,A)$, and if $A\equiv 0$, then the solutions set of (1.2) is denoted by $\mathrm{EP}(f)$.

The problem (1.2) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, and the Nash equilibrium problem in noncooperative games; see also [1, 2]. Some methods have been constructed to solve the system of equilibrium problems (see, e.g., [3–7]). Recall that a mapping $A : C \to H$ is said to be

(1) *monotone* if

$$\langle Ax - Ay, x - y\rangle \geq 0, \quad \forall x, y \in C, \tag{1.3}$$

(2) *$\alpha$-inverse-strongly monotone* if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y\rangle \geq \alpha\|Ax - Ay\|^2, \quad \forall x, y \in C. \tag{1.4}$$

It is easy to see that if $A$ is $\alpha$-inverse-strongly monotone, then $A$ is monotone and $1/\alpha$-Lipschitz.

For solving the equilibrium problem, let us assume that $f$ satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$,

(A2) $f$ is monotone, that is, $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$,

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} f(tz + (1-t)x, y) \leq f(x, y)$,

(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Throughout this paper, we denote the fixed points set of a nonlinear mapping $T : C \to C$ by $F(T) = \{x \in C : Tx = x\}$. Recall that $T$ is said to be a *$\kappa$-strict pseudocontraction* if there exists a constant $0 \leq \kappa < 1$ such that

$$\|Tx - Ty\|^2 \leq \|x - y\|^2 + \kappa\|(I-T)x - (I-T)y\|^2. \tag{1.5}$$

It is well known that (1.5) is equivalent to

$$\langle Tx - Ty, x - y\rangle \leq \|x - y\|^2 - \frac{1-\kappa}{2}\|(I-T)x - (I-T)y\|^2. \tag{1.6}$$

It is worth mentioning that the class of strict pseudocontractions includes properly the class of nonexpansive mappings. It is also known that every $\kappa$-strict pseudocontraction is $((1+\kappa)/(1-\kappa))$-Lipschitz; see [8].

In 1953, Mann [9] introduced the iteration as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Sx_n, \quad n \geq 0, \tag{1.7}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset [0,1]$. If $S$ is a nonexpansive mapping with a fixed point and the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\sum_{n=0}^{\infty}\alpha_n(1-\alpha_n) = \infty$, then the sequence $\{x_n\}$ defined by (1.7) converges weakly to a fixed point of $S$ (this is also valid in a uniformly convex Banach space with the Fréchet differentiable norm [10]).

In 1967, Browder and Petryshyn [11] introduced the class of strict pseudocontractions and proved existence and weak convergence theorems in a real Hilbert setting by using Mann iterative algorithm (1.7) with a constant sequence $\alpha_n = \alpha$ for all $n \geq 0$. Recently, Marino and Xu [8] and Zhou [12] extended the results of Browder and Petryshyn [11] to Mann's iteration process (1.7). Since 1967, the construction of fixed points for pseudocontractions via the iterative process has been extensively investigated by many authors (see, e.g., [13–22]).

Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonexpansive mapping, $f : C \times C \to \mathbb{R}$ a bifunction, and let $A : C \to H$ be an inverse-strongly monotone mapping.

In 2008, Moudafi [23] introduced an iterative method for approximating a common element of the fixed points set of a nonexpansive mapping $S$ and the solutions set of a generalized equilibrium problem GEP$(f, \mathrm{A})$ as follows: a sequence $\{x_n\}$ defined by $x_0 \in C$ and

$$\begin{gathered} f(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sy_n, \quad n \geq 1, \end{gathered} \tag{1.8}$$

where $\{\alpha_n\}_{n=0}^{\infty} \subset (0, 1)$ and $\{r_n\}_{n=0}^{\infty} \subset (0, \infty)$. He proved that the sequence $\{x_n\}$ generated by (1.8) converges weakly to an element in $\mathrm{GEP}(f, A) \cap F(S)$ under suitable conditions.

Due to the weak convergence, recently, S. Takahashi and W. Takahashi [24] introduced another modification iterative method of (1.8) for finding a common element of the fixed points set of a nonexpansive mapping and the solutions set of a generalized equilibrium problem in the framework of a real Hilbert space. To be more precise, they proved the following theorem.

**Theorem 1.1** (see [24]). *Let $C$ be a closed convex subset of a real Hilbert space $H$, and let $f : C \times C \to \mathbb{R}$ be a bifunction satisfying (A1)–(A4). Let $A$ be an $\alpha$-inverse-strongly monotone mapping of $C$ into $H$, and let $S$ be a nonexpansive mapping of $C$ into itself such that $F(S) \cap \mathrm{GEP}(f, A) \neq \emptyset$. Let $u \in C$ and $x_1 \in C$, and let $\{y_n\} \subset C$ and $\{x_n\} \subset C$ be sequences generated by*

$$\begin{gathered} f(y_n, y) + \langle Ax_n, y - y_n\rangle + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \beta_n x_n + (1 - \beta_n)S[\alpha_n u + (1 - \alpha_n)y_n], \quad n \geq 1, \end{gathered} \tag{1.9}$$

*where $\{\alpha_n\}_{n=1}^{\infty} \subset [0, 1]$, $\{\beta_n\}_{n=1}^{\infty} \subset [0, 1]$ and $\{r_n\}_{n=1}^{\infty} \subset [0, 2\alpha]$ satisfy*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < c \leq \beta_n \leq d < 1$,

(iii) $0 < a \leq r_n \leq b < 2\alpha$,

(iv) $\lim_{n\to\infty}(r_n - r_{n+1}) = 0$.

*Then, $\{x_n\}$ converges strongly to $z = P_{F(S)\cap\mathrm{GEP}(f,A)}u$.*

Recently, Yao et al. [25] introduced a new modified Mann iterative algorithm which is different from those in the literature for a nonexpansive mapping in a real Hilbert space. To be more precise, they proved the following theorem.

**Theorem 1.2** (see [25]). *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$. Let $\{\alpha_n\}_{n=0}^{\infty}$, and let $\{\beta_n\}_{n=0}^{\infty}$ be two real sequences in $(0,1)$. For given $x_0 \in C$ arbitrarily, let the sequence $\{x_n\}$, $n \geq 0$, be generated iteratively by*

$$\begin{aligned} y_n &= P_C[(1-\alpha_n)x_n], \\ x_{n+1} &= (1-\beta_n)x_n + \beta_n S y_n. \end{aligned} \tag{1.10}$$

*Suppose that the following conditions are satisfied:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

*then, the sequence $\{x_n\}$ generated by* (1.10) *strongly converges to a fixed point of $S$.*

We know the following crucial lemmas concerning the equilibrium problem in Hilbert spaces.

**Lemma 1.3** (see [1]). *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$, let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \tag{1.11}$$

**Lemma 1.4** (see [26]). *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $f$ be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying (A1)–(A4). For $x \in H$ and $r > 0$, define the mapping $T_r^f : H \to 2^C$ as follows:*

$$T_r^f(x) = \left\{ z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0,\ \forall y \in C \right\}. \tag{1.12}$$

*Then, the following statements hold:*

(1) $T_r^f$ *is single-valued,*

(2) $T_r^f$ *is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\left\| T_r^f x - T_r^f y \right\|^2 \leq \left\langle T_r^f x - T_r^f y, x - y \right\rangle, \tag{1.13}$$

(3) $F(T_r^f) = \mathrm{EP}(f)$,

(4) $\mathrm{EP}(f)$ *is closed and convex.*

Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $r_k > 0$ for each $k \in \{1, 2, \ldots, M\}$. Let $\{f_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^{M} : C \to H$ be a family of $\alpha_k$-inverse-strongly monotone mappings, and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of $\kappa$-strict pseudocontractions. For each $k \in \{1, 2, \ldots, M\}$, denote the mapping $T_{r_k}^{f_k, A_k} : C \to C$ by $T_{r_k}^{f_k, A_k} := T_{r_k}^{f_k}(I - r_k A_k)$, where $T_{r_k}^{f_k} : H \to C$ is the mapping defined as in Lemma 1.4.

Motivated and inspired by Marino and Xu [8], Moudafi [23], S. Takahashi and W. Takahashi [24], and Yao et al. [25], we consider the following iteration: $x_1 \in C$ and

$$\begin{aligned} y_n &= P_C[(1 - \alpha_n)x_n], \\ u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\ x_{n+1} &= \beta_n x_n + (1 - \beta_n)[\gamma u_n + (1 - \gamma)T_n u_n], \quad n \geq 1, \end{aligned} \tag{1.14}$$

where $\{\alpha_n\}_{n=1}^{\infty} \subset (0, 1)$, $\{\beta_n\}_{n=1}^{\infty} \subset (0, 1)$ and $\gamma \in (0, 1)$.

In this paper, we first prove a path convergence result for a nonexpansive mapping and a system of generalized equilibrium problems. Then, we prove a strong convergence theorem of the iteration process (1.14) for a system of generalized equilibrium problems and a countable family of strict pseudocontractions in a real Hilbert space. Our results extend the main results obtained by Yao et al. [25] in several aspects.

## 2. Preliminaries

Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. For each $x \in H$, there exists a unique nearest point in $C$, denoted by $P_C x$, such that $\|x - P_C x\| = \min_{y \in C} \|x - y\|$. $P_C$ is called the metric projection of $H$ onto $C$. It is also known that for $x \in H$ and $z \in C$, $z = P_C x$ is equivalent to $\langle x - z, y - z \rangle \leq 0$ for all $y \in C$. Furthermore,

$$\|y - P_C x\|^2 + \|x - P_C x\|^2 \leq \|x - y\|^2, \tag{2.1}$$

for all $x \in H$, $y \in C$. In a real Hilbert space, we also know that

$$\|\lambda x + (1 - \lambda)y\|^2 = \lambda\|x\|^2 + (1 - \lambda)\|y\|^2 - \lambda(1 - \lambda)\|x - y\|^2, \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

In the sequel, we need the following lemmas.

**Lemma 2.1** (see [27, 28]). *Let $E$ be a real uniformly convex Banach space, and let $C$ be a nonempty, closed, and convex subset of $E$, and let $S : C \to C$ be a nonexpansive mapping such that $F(S) \neq \emptyset$, then $I - S$ is demiclosed at zero.*

**Lemma 2.2** (see [29]). *Let $\{x_n\}$ and $\{z_n\}$ be two sequences in a Banach space $E$ such that*

$$x_{n+1} = \beta_n x_n + (1 - \beta_n)z_n, \quad n \geq 1, \tag{2.3}$$

*where* $\{\beta_n\}_{n=1}^{\infty}$ *satisfies conditions:* $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. *If* $\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0$, *then* $\|x_n - z_n\| \to 0$ *as* $n \to \infty$.

**Lemma 2.3** (see [30]). *Assume that* $\{a_n\}_{n=1}^{\infty}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \ge 1, \tag{2.4}$$

*where* $\{\gamma_n\}_{n=1}^{\infty}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}_{n=1}^{\infty}$ *is a sequence in* $\mathbb{R}$ *such that*

(a) $\sum_{n=1}^{\infty}\gamma_n = \infty$; (b) $\limsup_{n\to\infty}\delta_n \le 0$ *or* $\sum_{n=1}^{\infty}|\gamma_n\delta_n| < \infty$.

*Then,* $\lim_{n\to\infty}a_n = 0$.

**Lemma 2.4** (see [31]). *Let* $C$ *be a nonempty, closed, and convex subset of a real Hilbert space* $H$. *Let the mapping* $A : C \to H$ *be* $\alpha$*-inverse-strongly monotone, and let* $r > 0$ *be a constant. Then, we have*

$$\|(I - rA)x - (I - rA)y\|^2 \le \|x - y\|^2 + r(r - 2\alpha)\|Ax - Ay\|^2, \tag{2.5}$$

*for all* $x, y \in C$. *In particular, if* $0 \le r \le 2\alpha$, *then* $I - rA$ *is nonexpansive.*

To deal with a family of mappings, the following conditions are introduced: let $C$ be a subset of a real Hilbert space $H$, and let $\{T_n\}_{n=1}^{\infty}$ be a family of mappings of $C$ such that $\bigcap_{n=1}^{\infty}F(T_n) \ne \emptyset$. Then, $\{T_n\}$ is said to satisfy the AKTT-*condition* [32] if for each bounded subset $B$ of $C$,

$$\sum_{n=1}^{\infty}\sup\{\|T_{n+1}z - T_nz\| : z \in B\} < \infty. \tag{2.6}$$

**Lemma 2.5** (see [32]). *Let* $C$ *be a nonempty and closed subset of a Hilbert space* $H$, *and let* $\{T_n\}$ *be a family of mappings of* $C$ *into itself which satisfies the* AKTT*-condition. Then, for each* $x \in C$, $\{T_nx\}$ *converges strongly to a point in* $C$. *Moreover, let the mapping* $T$ *be defined by*

$$Tx = \lim_{n\to\infty}T_nx, \quad \forall x \in C. \tag{2.7}$$

*Then, for each bounded subset* $B$ *of* $C$,

$$\limsup_{n\to\infty}\{\|Tz - T_nz\| : z \in B\} = 0. \tag{2.8}$$

The following results can be found in [33, 34].

**Lemma 2.6** (see [33, 34]). *Let* $C$ *be a closed, and convex subset of a Hilbert space* $H$. *Suppose that* $\{T_n\}_{n=1}^{\infty}$ *is a family of* $\kappa$*-strictly pseudocontractive mappings from* $C$ *into* $H$ *with* $\bigcap_{n=1}^{\infty}F(T_n) \ne \emptyset$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *is a real sequence in* $(0,1)$ *such that* $\sum_{n=1}^{\infty}\mu_n = 1$. *Then, the following conclusions hold:*

(1) $G := \sum_{n=1}^{\infty}\mu_nT_n : C \to H$ *is a* $\kappa$*-strictly pseudocontractive mapping,*

(2) $F(G) = \bigcap_{n=1}^{\infty}F(T_n)$.

**Lemma 2.7** (see [34]). *Let $C$ be a closed and convex subset of a Hilbert space $H$. Suppose that $\{S_i\}_{i=1}^{\infty}$ is a countable family of $\kappa$-strictly pseudocontractive mappings of $C$ into itself with $\bigcap_{i=1}^{\infty} F(S_i) \neq \emptyset$. For each $n \in \mathbb{N}$, define $T_n : C \to C$ by*

$$T_n x = \sum_{i=1}^{n} \mu_n^i S_i x, \quad x \in C, \tag{2.9}$$

*where $\{\mu_n^i\}$ is a family of nonnegative numbers satisfying*

(i) $\sum_{i=1}^{n} \mu_n^i = 1$ *for all* $n \in \mathbb{N}$,

(ii) $\mu^i := \lim_{n\to\infty} \mu_n^i > 0$ *for all* $i \in \mathbb{N}$,

(iii) $\sum_{n=1}^{\infty} \sum_{i=1}^{n} |\mu_{n+1}^i - \mu_n^i| < \infty$.

*Then,*

(1) *Each $T_n$ is a $\kappa$-strictly pseudocontractive mapping.*

(2) *$\{T_n\}$ satisfies* AKTT-*condition.*

(3) *If $T : C \to C$ is defined by*

$$Tx = \sum_{i=1}^{\infty} \mu^i S_i x, \quad x \in C, \tag{2.10}$$

*then $Tx = \lim_{n\to\infty} T_n x$ and $F(T) = \bigcap_{n=1}^{\infty} F(T_n) = \bigcap_{i=1}^{\infty} F(S_i)$.*

In the sequel, we will write $(\{T_n\}, T)$ satisfies the AKTT-condition if $\{T_n\}$ satisfies the AKTT-condition and $T$ is defined by Lemma 2.5 with $F(T) = \bigcap_{n=1}^{\infty} F(T_n)$.

## 3. Path Convergence Results

Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonexpansive mapping. Let $\{f_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^{M} : C \to H$ be a family of $\alpha_k$-inverse-strongly monotone mappings, and let $r_k \in (0, 2\alpha_k)$. For each $k \in \{1, 2, \ldots, M\}$, we denote the mapping $T_{r_k}^{f_k, A_k} : C \to C$ by

$$T_{r_k}^{f_k, A_k} := T_{r_k}^{f_k}(I - r_k A_k), \tag{3.1}$$

where $T_{r_k}^{f_k}$ is the mapping defined as in Lemma 1.4. For each $t \in (0, 1)$, we define the mapping $S_t : C \to C$ as follows:

$$S_t x = S T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_1}^{f_1, A_1} P_C[(1-t)x], \quad \forall x \in C. \tag{3.2}$$

By Lemmas 1.4(2) and 2.4, we know that $T_{r_k}^{f_k}$ and $I - r_k A_k$ are nonexpansive for each $k \in \{1, 2, \ldots, M\}$. So, the mapping $T_{r_k}^{f_k, A_k}$ is also nonexpansive for each $k \in \{1, 2, \ldots, M\}$.

Moreover, we can check easily that $S_t$ is a contraction. Then, the Banach contraction principle ensures that there exists a unique fixed point $x_t$ of $S_t$ in $C$, that is,

$$x_t = ST_{r_M}^{f_M,A_M}T_{r_{M-1}}^{f_{M-1},A_{M-1}}\cdots T_{r_1}^{f_1,A_1}P_C[(1-t)x_t], \quad t\in(0,1). \tag{3.3}$$

**Theorem 3.1.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $S : C \to C$ be a nonexpansive mapping. Let $\{f_k\}_{k=1}^M : C\times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \to H$ be a family of $\alpha_k$-inverse-strongly monotone mappings, and let $r_k \in (0,2\alpha_k)$. For each $k \in \{1,2,\ldots,M\}$, let the mapping $T_{r_k}^{f_k,A_k}$ be defined by (3.1). Assume that $F := (\bigcap_{k=1}^M \mathrm{GEP}(f_k,A_k)) \cap (\bigcap_{n=1}^\infty F(T_n)) \neq \emptyset$. For each $t \in (0,1)$, let the net $\{x_t\}$ be generated by (3.3). Then, as $t \to 0$, the net $\{x_t\}$ converges strongly to an element in $F$.*

*Proof.* First, we show that $\{x_t\}$ is bounded. For each $t \in (0,1)$, let $y_t = P_C[(1-t)x_t]$ and $u_t = T_{r_M}^{f_M,A_M}T_{r_{M-1}}^{f_{M-1},A_{M-1}}\cdots T_{r_1}^{f_1,A_1}y_t$. From (3.3), we have for each $p\in F$ that

$$\|x_t-p\| = \|Su_t - Sp\| \le \|u_t-p\| \le \|y_t-p\| \le (1-t)\|x_t-p\| + t\|p\|. \tag{3.4}$$

It follows that

$$\|x_t-p\| \le \|p\|. \tag{3.5}$$

Hence, $\{x_t\}$ is bounded and so are $\{y_t\}$ and $\{u_t\}$. Observe that

$$\|y_t-x_t\| \le t\|x_t\| \longrightarrow 0, \tag{3.6}$$

as $t\to 0$ since $\{x_t\}$ is bounded.

Next, we show that $\|u_t-x_t\| \to 0$ as $t\to 0$. Denote $\Theta^k = T_{r_k}^{f_k,A_k}T_{r_{k-1}}^{f_{k-1},A_{k-1}}\cdots T_{r_1}^{f_1,A_1}$ for any $k\in\{1,2,\ldots,M\}$ and $\Theta^0 = I$. We note that $u_t = \Theta^M y_t$ for each $t\in(0,1)$. From Lemma 2.4, we have for each $k\in\{1,2,\ldots,M\}$ and $p\in F$ that

$$\begin{aligned}
\left\|\Theta^k y_t - p\right\|^2 &= \left\|T_{r_k}^{f_k,A_k}\Theta^{k-1}y_t - T_{r_k}^{f_k,A_k}\Theta^{k-1}p\right\|^2 \\
&= \left\|T_{r_k}^{f_k}\left(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t\right) - T_{r_k}^{f_k}\left(\Theta^{k-1}p - r_kA_k\Theta^{k-1}p\right)\right\|^2 \\
&\le \left\|\left(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t\right) - \left(\Theta^{k-1}p - r_kA_k\Theta^{k-1}p\right)\right\|^2 \\
&\le \left\|\Theta^{k-1}y_t - p\right\|^2 + r_k(r_k-2\alpha_k)\left\|A_k\Theta^{k-1}y_t - A_kp\right\|^2.
\end{aligned} \tag{3.7}$$

It follows that

$$\begin{aligned}\|u_t - p\|^2 &= \left\|\Theta^M y_t - p\right\|^2 \\ &\le \|y_t - p\|^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_t - A_i p\right\|^2 \\ &= \|P_C[(1-t)x_t] - p\|^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_t - A_i p\right\|^2 \\ &\le (\|x_t - p\| + t\|x_t\|)^2 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_t - A_i p\right\|^2 \\ &\le \|x_t - p\|^2 + tM_1 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_t - A_i p\right\|^2,\end{aligned} \tag{3.8}$$

where $M_1 = \sup_{0<t<1}\{2\|x_t - p\|\|x_t\| + t\|x_t\|^2\}$. So, we have

$$\begin{aligned}\|x_t - p\|^2 &\le \|u_t - p\|^2 \\ &\le \|x_t - p\|^2 + tM_1 + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_t - A_i p\right\|^2,\end{aligned} \tag{3.9}$$

which implies that

$$\lim_{t\to 0}\left\|A_k\Theta^{k-1}y_t - A_k p\right\| = 0, \tag{3.10}$$

for each $k \in \{1,2,\ldots,M\}$. Since $T_{r_k}^{f_k}$ is firmly nonexpansive for each $k \in \{1,2,\ldots,M\}$, we have for each $p \in F$ and $k \in \{1,2,\ldots,M\}$ that

$$\begin{aligned}\left\|\Theta^k y_t - p\right\|^2 &= \left\|T_{r_k}^{f_k,A_k}\Theta^{k-1}y_t - T_{r_k}^{f_k,A_k}\Theta^{k-1}p\right\|^2 \\ &= \left\|T_{r_k}^{f_k}\left(\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t\right) - T_{r_k}^{f_k}\left(\Theta^{k-1}p - r_kA_k\Theta^{k-1}p\right)\right\|^2 \\ &\le \left\langle \Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp), \Theta^k y_t - p\right\rangle \\ &= \frac{1}{2}\Bigg(\left\|\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp)\right\|^2 + \left\|\Theta^k y_t - p\right\|^2 \\ &\qquad - \left\|\Theta^{k-1}y_t - r_kA_k\Theta^{k-1}y_t - (p - r_kA_kp) - \left(\Theta^k y_t - p\right)\right\|^2\Bigg)\end{aligned}$$

$$\le \frac{1}{2}\left(\left\|\Theta^{k-1}y_t - p\right\|^2 + \left\|\Theta^k y_t - p\right\|^2 - \left\|\Theta^{k-1}y_t - \Theta^k y_t - r_k\left(A_k\Theta^{k-1}y_t - A_k p\right)\right\|^2\right)$$
$$\le \frac{1}{2}\left(\left\|\Theta^{k-1}y_t - p\right\|^2 + \left\|\Theta^k y_t - p\right\|^2 - \left\|\Theta^{k-1}y_t - \Theta^k y_t\right\|^2 \right.$$
$$\left. +2r_k\left\|\Theta^{k-1}y_t - \Theta^k y_t\right\|\left\|A_k\Theta^{k-1}y_t - A_k p\right\|\right). \tag{3.11}$$

This implies that

$$\begin{aligned}\left\|\Theta^k y_t - p\right\|^2 &\le \left\|\Theta^{k-1}y_t - p\right\|^2 - \left\|\Theta^{k-1}y_t - \Theta^k y_t\right\|^2 \\ &\quad + 2r_k\left\|\Theta^{k-1}y_t - \Theta^k y_t\right\|\left\|A_k\Theta^{k-1}y_t - A_k p\right\| \\ &\le \left\|\Theta^{k-1}y_t - p\right\|^2 - \left\|\Theta^{k-1}y_t - \Theta^k y_t\right\|^2 + M_2\left\|A_k\Theta^{k-1}y_t - A_k p\right\|,\end{aligned} \tag{3.12}$$

where $M_2 = \max_{1\le k\le M}\sup_{0<t<1}\{2r_k\|\Theta^{k-1}y_t - \Theta^k y_t\|\}$. This shows that

$$\begin{aligned}\|u_t - p\|^2 &= \left\|\Theta^M y_t - p\right\|^2 \\ &\le \|y_t - p\|^2 - \sum_{i=1}^{M}\left\|\Theta^{i-1}y_t - \Theta^i y_t\right\|^2 + M_2\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_t - A_i p\right\| \\ &\le \|x_t - p\|^2 + tM_1 - \sum_{i=1}^{M}\left\|\Theta^{i-1}y_t - \Theta^i y_t\right\|^2 + M_2\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_t - A_i p\right\|.\end{aligned} \tag{3.13}$$

Hence,

$$\begin{aligned}\|x_t - p\|^2 &\le \|u_t - p\|^2 \\ &\le \|x_t - p\|^2 + tM_1 - \sum_{i=1}^{M}\left\|\Theta^{i-1}y_t - \Theta^i y_t\right\|^2 + M_2\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_t - A_i p\right\|.\end{aligned} \tag{3.14}$$

From (3.10), we obtain

$$\sum_{i=1}^{M}\left\|\Theta^{i-1}y_t - \Theta^i y_t\right\| \longrightarrow 0, \tag{3.15}$$

as $t \to 0$. So, we can conclude that

$$\lim_{t\to 0}\left\|\Theta^{k-1}y_t - \Theta^k y_t\right\| = 0, \tag{3.16}$$

for each $k \in \{1, 2, \ldots, M\}$. Observing

$$\begin{aligned} \|u_n - y_t\| &= \left\|\Theta^M y_t - y_t\right\| \\ &\le \left\|\Theta^M y_t - \Theta^{M-1} y_t\right\| + \left\|\Theta^{M-1} y_t - \Theta^{M-2} y_t\right\| + \cdots + \left\|\Theta^1 y_t - y_t\right\|, \end{aligned} \tag{3.17}$$

it follows by (3.16) that

$$\lim_{t \to 0} \|u_t - y_t\| = 0. \tag{3.18}$$

From (3.6) and (3.18), we have

$$\lim_{t \to 0} \|u_t - x_t\| = 0. \tag{3.19}$$

Hence,

$$\|x_t - Sx_t\| = \|Su_t - Sx_t\| \le \|u_t - x_t\| \longrightarrow 0, \tag{3.20}$$

as $t \to 0$.

Next, we show that $\{x_t\}$ is relatively norm compact as $t \to 0$. Let $\{t_n\} \subset (0, 1)$ be a sequence such that $t_n \to 0$ as $n \to \infty$. Put $x_n := x_{t_n}$. From (3.20), we obtain

$$\lim_{n \to \infty} \|x_n - Sx_n\| = 0. \tag{3.21}$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $\{x_n\}$ converges weakly to $x^* \in C$. Applying Lemma 2.1 to (3.21), we can conclude that $x^* \in F(S)$.

Next, we show that $x^* \in \bigcap_{k=1}^{M} \mathrm{GEP}(f_k, A_k)$. Note that $\Theta^k y_n = T_{r_k}^{f_k, A_k} \Theta^{k-1} y_n = T_{r_k}^{f_k} (\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n)$ for each $k \in \{1, 2, \ldots, M\}$. Hence, for each $y \in C$ and $k \in \{1, 2, \ldots, M\}$, we obtain

$$f_k\left(\Theta^k y_n, y\right) + \frac{1}{r_k} \left\langle y - \Theta^k y_n, \Theta^k y_n - \left(\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n\right) \right\rangle \ge 0. \tag{3.22}$$

From (A2), we have

$$\frac{1}{r_k} \left\langle y - \Theta^k y_n, \Theta^k y_n - \left(\Theta^{k-1} y_n - r_k A_k \Theta^{k-1} y_n\right) \right\rangle \ge f_k\left(y, \Theta^k y_n\right), \quad \forall y \in C. \tag{3.23}$$

Therefore,

$$\left\langle y - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} + A_k \Theta^{k-1} y_{n_j} \right\rangle \ge f_k\left(y, \Theta^k y_{n_j}\right), \quad \forall y \in C. \tag{3.24}$$

For each $t \in (0,1)$ and $y \in C$, put $z_t = ty + (1-t)x^*$. Then, we have $z_t \in C$. From (3.24), we get that

$$\begin{aligned}\left\langle z_t - \Theta^k y_{n_j}, A_k z_t \right\rangle &\geq \left\langle z_t - \Theta^k y_{n_j}, A_k z_t \right\rangle \\ &\quad - \left\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} + A_k \Theta^{k-1} y_{n_j} \right\rangle + f_k\left(z_t, \Theta^k y_{n_j}\right) \\ &= \left\langle z_t - \Theta^k y_{n_j}, A_k z_t - A_k \Theta^k y_{n_j} \right\rangle + \left\langle z_t - \Theta^k y_{n_j}, A_k \Theta^k y_{n_j} - A_k \Theta^{k-1} y_{n_j} \right\rangle \\ &\quad - \left\langle z_t - \Theta^k y_{n_j}, \frac{\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}}{r_k} \right\rangle + f_k\left(z_t, \Theta^k y_{n_j}\right).\end{aligned} \tag{3.25}$$

We note that $\|A_k \Theta^k y_{n_j} - A_k \Theta^{k-1} y_{n_j}\| \leq (1/\alpha_k)\|\Theta^k y_{n_j} - \Theta^{k-1} y_{n_j}\| \to 0$, $\Theta^k y_{n_j} \rightharpoonup x^*$ as $j \to \infty$, and $\{A_k\}_{k=1}^M$ is a family of monotone mappings. It follows from (3.25) that

$$\langle z_t - x^*, A_k z_t \rangle \geq f_k(z_t, x^*). \tag{3.26}$$

So, by (A1), (A4) and (3.26), we have for each $y \in C$ and $k \in \{1, 2, \ldots, M\}$ that

$$\begin{aligned}0 = f_k(z_t, z_t) &\leq t f_k(z_t, y) + (1-t) f_k(z_t, x^*) \\ &\leq t f_k(z_t, y) + (1-t)\langle z_t - x^*, A_k z_t \rangle \\ &= t f_k(z_t, y) + t(1-t)\langle y - x^*, A_k z_t \rangle.\end{aligned} \tag{3.27}$$

This implies that

$$f_k(z_t, y) + (1-t)\langle y - x^*, A_k z_t \rangle \geq 0, \quad \forall y \in C. \tag{3.28}$$

Letting $t \to 0$ in (3.28), it follows from (A3) that

$$f_k(x^*, y) + \langle y - x^*, A_k x^* \rangle \geq 0, \quad \forall y \in C. \tag{3.29}$$

Hence $x^* \in \bigcap_{k=1}^{M} \mathrm{GEP}(f_k, A_k)$; consequently, $x^* \in F$. Further, we see that

$$\begin{aligned}
\|x_t - x^*\|^2 &= \|Su_t - x^*\|^2 \\
&\leq \|u_t - x^*\|^2 \\
&\leq \|y_t - x^*\|^2 \\
&\leq \|x_t - x^* - tx_t\|^2 \\
&= \|x_t - x^*\|^2 - 2t\langle x_t, x_t - x^*\rangle + t^2\|x_t\|^2 \\
&= \|x_t - x^*\|^2 - 2t\langle x_t - x^*, x_t - x^*\rangle - 2t\langle x^*, x_t - x^*\rangle + t^2\|x_t\|^2.
\end{aligned} \tag{3.30}$$

So, we have

$$\|x_t - x^*\|^2 \leq \langle x^*, x^* - x_t\rangle + \frac{t}{2}\|x_t\|^2. \tag{3.31}$$

In particular,

$$\|x_n - x^*\|^2 \leq \langle x^*, x^* - x_n\rangle + \frac{t_n}{2}\|x_n\|^2. \tag{3.32}$$

Since $x_n \rightharpoonup x^*$, we have $x_n \to x^*$ as $n \to \infty$. By using the same argument as in the proof of Theorem 3.1 of [25], we can show that $x_t \to x^* \in F$ as $t \to 0$. This completes the proof. □

## 4. Strong Convergence Results

**Theorem 4.1.** *Let $C$ be a nonempty, closed and convex subset of a real Hilbert space $H$. Let $\{f_k\}_{k=1}^{M} : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^{M} : C \to H$ be a family of $\alpha_k$-inverse-strongly monotone mappings and let $\{T_n\}_{n=1}^{\infty} : C \to C$ be a countable family of $\kappa$-strict pseudocontractions for some $0 < \kappa < 1$ such that $F := (\bigcap_{k=1}^{M} \mathrm{GEP}(f_k, A_k)) \cap (\bigcap_{n=1}^{\infty} F(T_n)) \neq \emptyset$. Assume that $\{\alpha_n\}_{n=1}^{\infty} \subset (0,1)$, $\{\beta_n\}_{n=1}^{\infty} \subset (0,1)$, $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \ldots, M\}$ satisfy the following conditions:*

(i) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(ii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$.

*Suppose that $(\{T_n\}, T)$ satisfies the* AKTT*-condition. Then, $\{x_n\}$ generated by* (1.14) *converges strongly to an element in $F$.*

*Proof.* For each $n \in \mathbb{N}$, define $S_n : C \to C$ by $S_n x = \gamma x + (1-\gamma)T_n x$, $x \in C$. Then, $F(S_n) = F(T_n) = F(T)$, since $\gamma \in (0,1)$. Moreover, we know that $\{S_n\}$ satisfies the AKTT-condition, since $\{T_n\}$ satisfies the AKTT-condition. By Lemma 2.5, we can define the mapping $S : C \to C$ by $Sx = \lim_{n\to\infty} S_n x$ for $x \in C$. Hence, $Sx = \gamma x + (1-\gamma)Tx$, $x \in C$, since $T_n x \to Tx$ for

$x \in C$. Further, we know that $S_n$ is nonexpansive for each $n \in \mathbb{N}$. Indeed, for each $x, y \in C$ and $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|S_n x - S_n y\|^2 &= \|\gamma x + (1-\gamma)T_n x - \gamma y - (1-\gamma)T_n y\|^2 \\
&= \|\gamma(x-y) + (1-\gamma)(T_n x - T_n y)\|^2 \\
&= \gamma\|x-y\|^2 + (1-\gamma)\|T_n x - T_n y\|^2 - \gamma(1-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&\le \gamma\|x-y\|^2 + (1-\gamma)\|x-y\|^2 + (1-\gamma)\kappa\|(I-T_n)x - (I-T_n)y\|^2 \\
&\quad - \gamma(1-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&= \|x-y\|^2 + (1-\gamma)(\kappa-\gamma)\|(I-T_n)x - (I-T_n)y\|^2 \\
&\le \|x-y\|^2.
\end{aligned} \tag{4.1}$$

Hence, $S_n$ is nonexpansive for each $n \in \mathbb{N}$ and so is $S$.

Next, we show that $\{x_n\}$ is bounded. Denote $\Theta^k = T_{r_k}^{f_k,A_k} T_{r_{k-1}}^{f_{k-1},A_{k-1}} \cdots T_{r_1}^{f_1,A_1}$ for any $k \in \{1, 2, \dots, M\}$ and $\Theta^0 = I$. We note that $u_n = \Theta^M y_n$. From (1.14), we have for each $p \in F$ that

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\beta_n x_n + (1-\beta_n)S_n u_n\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|S_n u_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|u_n - p\| \\
&= \beta_n\|x_n - p\| + (1-\beta_n)\left\|\Theta^M y_n - p\right\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)\|y_n - p\| \\
&\le \beta_n\|x_n - p\| + (1-\beta_n)[(1-\alpha_n)\|x_n - p\| + \alpha_n\|p\|] \\
&= (1-\alpha_n(1-\beta_n))\|x_n - p\| + \alpha_n(1-\beta_n)\|p\| \\
&\le \max\{\|x_n - p\|, \|p\|\}.
\end{aligned} \tag{4.2}$$

Hence, by induction, $\{x_n\}$ is bounded and so are $\{y_n\}$ and $\{u_n\}$.

Next, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{4.3}$$

Since $u_n = \Theta^M y_n$ and $u_{n+1} = \Theta^M y_{n+1}$,

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \left\|\Theta^M y_{n+1} - \Theta^M y_n\right\| \\
&\le \|y_{n+1} - y_n\|.
\end{aligned} \tag{4.4}$$

Set $z_n = S_n u_n$, $n \in \mathbb{N}$. So, we have from (1.14) and (4.4) that

$$\begin{aligned}
\|z_{n+1} - z_n\| &= \|S_{n+1}u_{n+1} - S_n u_n\| \\
&\le \|S_{n+1}u_{n+1} - S_{n+1}u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|u_{n+1} - u_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|y_{n+1} - y_n\| + \|S_{n+1}u_n - S_n u_n\| \\
&\le \|(1-\alpha_{n+1})x_{n+1} - (1-\alpha_n)x_n\| + \sup_{z\in\{u_n\}} \|S_{n+1}z - S_n z\| \\
&\le \|x_{n+1} - x_n\| + \alpha_{n+1}\|x_{n+1}\| + \alpha_n\|x_n\| + \sup_{z\in\{u_n\}} \|S_{n+1}z - S_n z\|.
\end{aligned} \tag{4.5}$$

Since $\{S_n\}$ satisfies the AKTT-condition and $\lim_{n\to\infty}\alpha_n = 0$, it follows that

$$\limsup_{n\to\infty}(\|z_{n+1} - z_n\| - \|x_{n+1} - x_n\|) \le 0. \tag{4.6}$$

So, by Lemma 2.2 and (ii), we obtain

$$\lim_{n\to\infty}\|z_n - x_n\| = 0. \tag{4.7}$$

Hence,

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0. \tag{4.8}$$

Observe that

$$\|y_n - x_n\| = \|P_C[(1-\alpha_n)x_n] - P_C x_n\| \le \alpha_n\|x_n\| \longrightarrow 0, \tag{4.9}$$

as $n \to \infty$. Similar to the proof of Theorem 3.1, we obtain for each $p \in F$ that

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \alpha_n M_1' + \sum_{i=1}^{M} r_i(r_i - 2\alpha_i)\left\|A_i\Theta^{i-1}y_n - A_i p\right\|^2, \tag{4.10}$$

$$\|u_n - p\|^2 \le \|x_n - p\|^2 + \alpha_n M_1' - \sum_{i=1}^{M}\left\|\Theta^{i-1}y_n - \Theta^i y_n\right\|^2 + M_2'\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_n - A_i p\right\|, \tag{4.11}$$

for some $M_1' > 0$ and $M_2' > 0$. Then, from (4.10), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|S_nu_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\\ &\quad\times\left(\|x_n-p\|^2+\alpha_nM_1'+\sum_{i=1}^{M}r_i(r_i-2\alpha_i)\left\|A_i\Theta^{i-1}y_n-A_ip\right\|^2\right)\\ &\le \|x_n-p\|^2+\alpha_nM_1'+(1-\beta_n)\sum_{i=1}^{M}r_i(r_i-2\alpha_i)\left\|A_i\Theta^{i-1}y_n-A_ip\right\|^2,\end{aligned}\tag{4.12}$$

which implies that

$$(1-\beta_n)\sum_{i=1}^{M}r_i(2\alpha_i-r_i)\left\|A_i\Theta^{i-1}y_n-A_ip\right\|^2\le\|x_{n+1}-p\|^2-\|x_n-p\|^2+\alpha_nM_1'.\tag{4.13}$$

So, from (4.8), (i), (ii) and $0<r_k<2\alpha_k$ for each $k=1,2,\ldots,M$, we have

$$\lim_{n\to\infty}\left\|A_k\Theta^{k-1}y_n-A_kp\right\|=0,\tag{4.14}$$

for each $k\in\{1,2,\ldots,M\}$. Similarly, from (4.11), we have

$$\begin{aligned}\|x_{n+1}-p\|^2 &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|S_nu_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\|u_n-p\|^2\\ &\le \beta_n\|x_n-p\|^2+(1-\beta_n)\\ &\quad\times\left(\|x_n-p\|^2+\alpha_nM_1'-\sum_{i=1}^{M}\left\|\Theta^{i-1}y_n-\Theta^iy_n\right\|^2+M_2'\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_n-A_ip\right\|\right)\\ &\le \|x_n-p\|^2+\alpha_nM_1'-(1-\beta_n)\sum_{i=1}^{M}\left\|\Theta^{i-1}y_n-\Theta^iy_n\right\|^2+M_2'\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_n-A_ip\right\|.\end{aligned}\tag{4.15}$$

This implies that

$$\begin{aligned}&(1-\beta_n)\sum_{i=1}^{M}\left\|\Theta^{i-1}y_n-\Theta^iy_n\right\|^2\\ &\quad\le\|x_n-p\|^2-\|x_{n+1}-p\|^2+\alpha_nM_1'+M_2'\sum_{i=1}^{M}\left\|A_i\Theta^{i-1}y_n-A_ip\right\|.\end{aligned}\tag{4.16}$$

From (i), (ii), (4.8), and (4.14), it follows that

$$\lim_{n\to\infty}\left\|\Theta^{k-1}y_n-\Theta^k y_n\right\|=0, \tag{4.17}$$

for each $k\in\{1,2,\dots,M\}$.

Next, we show that

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{4.18}$$

Observing

$$\begin{aligned}\|u_n-y_n\|&=\left\|\Theta^M y_n-y_n\right\|\\&\leq\left\|\Theta^M y_n-\Theta^{M-1}y_n\right\|+\left\|\Theta^{M-1}y_n-\Theta^{M-2}y_n\right\|+\cdots+\left\|\Theta^1 y_n-y_n\right\|,\end{aligned} \tag{4.19}$$

it follows, by (4.17), that

$$\lim_{n\to\infty}\|u_n-y_n\|=0. \tag{4.20}$$

From (4.9) and (4.20), we have

$$\lim_{n\to\infty}\|u_n-x_n\|=0. \tag{4.21}$$

We see that

$$\begin{aligned}\|x_n-Sx_n\|&\leq\|x_n-S_nu_n\|+\|S_nu_n-S_nx_n\|+\|S_nx_n-Sx_n\|\\&\leq\|x_n-S_nu_n\|+\|u_n-x_n\|+\sup_{z\in\{x_n\}}\|S_nz-Sz\|.\end{aligned} \tag{4.22}$$

So, by (4.7), (4.21), and Lemma 2.5, we have

$$\lim_{n\to\infty}\|x_n-Sx_n\|=0. \tag{4.23}$$

Let the net $\{x_t\}$ be defined by (3.3). By Theorem 3.1, we have $x_t\to x^*\in F$ as $t\to 0$. Moreover, by proving in the same manner as in Theorem 3.2 of [25], we can show that

$$\limsup_{n\to\infty}\langle x^*,x^*-x_n\rangle\leq 0. \tag{4.24}$$

Finally, we show that $x_n \to x^* \in F$ as $n \to \infty$. From (1.14), we have

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|S_n u_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|u_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\|(1-\alpha_n)(x_n - x^*) - \alpha_n x^*\|^2 \\
&\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n) \\
&\quad \times \left((1-\alpha_n)\|x_n - x^*\|^2 - 2\alpha_n(1-\alpha_n)\langle x^*, x_n - x^*\rangle + \alpha_n^2\|x^*\|^2\right) \\
&= (1 - \alpha_n(1-\beta_n))\|x_n - x^*\|^2 \\
&\quad + \alpha_n(1-\beta_n)\left(2(1-\alpha_n)\langle x^*, x^* - x_n\rangle + \alpha_n\|x^*\|^2\right).
\end{aligned} \tag{4.25}$$

By (i) and (4.24), it follows from Lemma 2.3 that $x_n \to x^* \in F$. This completes the proof. □

As a direct consequence of Lemmas 2.6 and 2.7 and Theorem 4.1, we obtain the following result.

**Theorem 4.2.** *Let $C$ be a nonempty, closed, and convex subset of a real Hilbert space $H$. Let $\{f_k\}_{k=1}^M : C \times C \to \mathbb{R}$ be a family of bifunctions, let $\{A_k\}_{k=1}^M : C \to H$ be a family of $\alpha_k$-inverse-strongly monotone mappings, and let $\{S_i\}_{i=1}^\infty$ be a sequence of $\kappa_i$-strict pseudocontractions of $C$ into itself such that $F := (\bigcap_{k=1}^M \mathrm{GEP}(f_k, A_k)) \cap (\bigcap_{i=1}^\infty F(S_i)) \neq \emptyset$ and $\sup\{\kappa_i : i \in \mathbb{N}\} = \kappa > 0$. Assume that $\gamma \in (\kappa, 1)$ and $r_k \in (0, 2\alpha_k)$ for each $k \in \{1, 2, \dots, M\}$. Define the sequence $\{x_n\}$ by $x_1 \in C$ and*

$$\begin{aligned}
y_n &= P_C[(1-\alpha_n)x_n], \\
u_n &= T_{r_M}^{f_M, A_M} T_{r_{M-1}}^{f_{M-1}, A_{M-1}} \cdots T_{r_2}^{f_2, A_2} T_{r_1}^{f_1, A_1} y_n, \\
x_{n+1} &= \beta_n x_n + (1-\beta_n)\left[\gamma u_n + (1-\gamma)\sum_{i=1}^n \mu_n^i S_i u_n\right], \quad n \ge 1,
\end{aligned} \tag{4.26}$$

*where $\{\alpha_n\}_{n=1}^\infty$ and $\{\beta_n\}_{n=1}^\infty$ are real sequences in $(0,1)$ which satisfy (i)-(ii) of Theorem 4.1 and $\{\mu_n^i\}$ is a real sequence which satisfies (i)–(iii) of Lemma 2.7. Then, $\{x_n\}$ converges strongly to an element in $F$.*

*Remark 4.3.* Theorems 4.1 and 4.2 extend the main results in [25] from a nonexpansive mapping to an infinite family of strict pseudocontractions and a system of generalized equilibrium problems.

*Remark 4.4.* If we take $A_k \equiv 0$ and $f_k \equiv 0$ for each $k = 1, 2, \dots, M$, then Theorems 3.1, 4.1, and 4.2 can be applied to a system of equilibrium problems and to a system of variational inequality problems, respectively.

*Remark 4.5.* Let $S_1, S_2, \ldots$ be an infinite family of nonexpansive mappings of $C$ into itself, and let $\xi_1, \xi_2, \ldots$ be real numbers such that $0 < \xi_i < 1$ for all $i \in \mathbb{N}$. Moreover, let $W_n$ and $W$ be the $W$-mappings [35] generated by $S_1, S_2, \ldots, S_n$ and $\xi_1, \xi_2, \ldots, \xi_n$ and $S_1, S_2, \ldots$ and $\xi_1, \xi_2, \ldots$. Then, we know from [7, 35] that $(\{W_n\}, W)$ satisfies the AKTT-condition. Therefore, in Theorem 4.1, the mapping $T_n$ can be also replaced by $W_n$.

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