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Research Article

Delay-Dependent Asymptotic Stability of Cohen-Grossberg Models with Multiple Time-Varying Delays

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Dynamical behavior of a class of Cohen-Grossberg models with multiple time-varying delays is studied in detail. Sufficient delay-dependent criteria to ensure local and global asymptotic stabilities of the equilibrium of this network are derived by constructing suitable Lyapunov functionals. The obtained conditions are shown to be less conservative and restrictive than those reported in the known literature. Some numerical examples are included to demonstrate our results.

1. Introduction

A large class of neural networks, which can function as stable content addressable memories or CAMs [1, 2], had been proposed by Cohen and Grossberg. These Cohen-Grossberg networks were designed to include additive neural networks, later studied by Hopfield [3, 4], and shunting neural networks. In the original analysis, Cohen and Grossberg assumed that the weight matrix was symmetric. Meanwhile, the activation functions are assumed to be continuous, differentiable, monotonically increasing, and bounded, such as the sigmoid-type function. Usually, such systems have been investigated under the assumption of asymmetric connection weight and nonmonotonic activation function. However, monotonicity and differentiability of activation functions come form the experimental results of brain sciences, moreover, they have very strong biological background. On the other hand, realistic modeling of many large neural networks with nonlocal interaction inevitably requires connection delays to be taken into account, since they naturally arise as a consequence of finite information transmission and processing speeds among the neurons.

It is also important to incorporate time delay into the model equations of the network such as delayed cellular neural network, which can be used to solve problems like the processing of moving images [5, 6]. Ye et al. [7] introduced discrete delays into the Cohen-Grossberg model. Furthermore, their global stability needed to satisfy the requirements that the connection should possess certain amount of symmetry and the discrete delays were sufficiently small.

For the delayed Hopfield networks [7–14], cellular neural networks [5, 6], as well as BAM networks [15–18], some delay-independent criteria for the global asymptotic stability are established without assuming the monotonicity and the differentiability of the activation functions and also the symmetry of the connection. Wang and Zou [19] also studied the Cohen-Grossberg model with time delays. The global stability criteria of this type of neural networks were also obtained by constructing appropriate Lyapunov functionals, or Lyapunov functions combined with the Rezumikhin technique. All of these criteria are independent of the magnitudes of the delays, and therefore the delays are harmless in a network satisfying one of the criteria. Actually, the global exponential stability implies global asymptotic stability, and so the results leading to global exponential stability can provide relevant estimates on how fast such networks perform during real-time computations. Furthermore, Liao et al. [20, 21] studied this problem.

Generally, the stability criteria for time-delay systems can be classified into two categories, namely delay-independent criteria and delay-dependent criteria, depending on whether they contain the delay argument as a parameter. There have been a number of significant developments in searching the stability criteria for systems with constant delays [4, 6, 7, 9, 11, 12, 14–17]. Only a few of them are for neural networks with distributed delays; see, for instance, [1–5, 8, 10, 18]. To the best of the authors' knowledge, the delay-dependent criteria in the case of the delayed Cohen-Grossberg model are little studied yet. In this paper, we will present some new local and global asymptotic stabilities of the equilibrium of Cohen-Grossberg models with mulitiple delays. Our results essentially show that the equilibrium of the network remains globally asymptotically stable when the time delays are small enough. In order to prove our results, we construct the suitable Lyapunov functionals.

In this paper, the amplification functions need to be continuous, positive, and bounded. However, the self-signal functions are not assumed to be differentiable, but only need to satisfy condition (H_2), as stated in the next section. At the same time, we do not confine ourselves to the symmetric connections. The rest of this paper is organized as follows. In Section 2, the Cohen-Grossberg neural network with time-varying delays and some preliminary analyses are given. By constructing Lyapunov functionals, some global exponential stability criteria for the network are presented in Section 3. Finally, numerical example is given to illustrate our results and some conclusions are drawn in Section 4.

2. Some preliminaries and network models

We consider Cohen-Grossberg neural networks with multiple time-varying delays, described by equations of theform

$$\dot{u}_i(t) = -a_i(u_i(t))\left[b_i(u_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(u_j(t-\tau_k(t))) + I_i\right], \quad i = 1,2,\dots,n, \tag{2.1}$$

where u_i denotes the state variable associated with the ith neuron, the function a_i represents an amplification function, and b_i is an arbitrary function; however we will require that b_i be sufficiently well behaved to keep the solutions of (2.1) bounded. The $t_{ij}^{(k)}$'s denote the interconnections which are associated with delay $\tau_k(t)$, $\tau_k(t)$ denotes the kth time delay for $k = 0,1,2,\dots,K$ such that $0 = \tau_0 < \tau_1 < \cdots < \tau_k$.

System (2.1) is said to be globally stable if for any solution $u(t)$, $\lim_{t\to\infty} u(t)$ exists. For the definitions of stability and asymptotic stability of an equilibrium of (2.1), refer to any of several standard texts (see, e.g., [22]).

In this paper, we assume that the Cohen-Grossberg neural networks (2.1) satisfy the following assumptions.

(H_1) The function a_i is bounded, positive, and continuous.

(H_2) The function b_i is continuous, and there exist positive constants $\underline{B_i}$ and $\overline{B_i}$, $i = 0,1,2,\dots,n$, such that

$$\begin{gathered} 0 < \underline{B_i} \le \frac{b_i(x_i) - b_i(y_i)}{x_i - y_i} \le \overline{B_i}, \quad \text{for } x_i \ne y_i,\ i = 1,2,\dots,n, \\ \lim_{u_i\to+\infty} b_i(u_i) = +\infty, \qquad \lim_{u_i\to-\infty} b_i(u_i) = -\infty. \end{gathered} \tag{2.2}$$

(H_3) $f_j \in C^1(\mathbb{R},\mathbb{R})$ is a sigmoidal function (so that $s_j'(x_j) \triangleq ds_j(x_j)/dx_j > 0$, $\lim_{x_j\to+\infty} f_j(x_j) = 1$, $\lim_{x_j\to-\infty} f_j(x_j) = -1$, and $\lim_{|x_j|\to\infty} s_j'(x_j) = 0$).

(H_4) $\tau_k : [0,+\infty) \to [0,+\infty)$ is continuous and $0 \le \tau_k(t) \le \overline{\tau}$.

The initial condition for system (2.1) is given as follows:

$$u_j(s) = \phi_j(s), \quad s \in [-\overline{\tau}, 0]. \tag{2.3}$$

Lemma 2.1. *If assumption (H_1)–(H_4) are satisfied for system (2.1)–(2.3), then any solution of (2.1) and (2.3) is bounded.*

Proof. We only need to consider system (2.1). We know by (H_1)–(H_4) that the terms $f_j(u_j(t))$ and $f_j(u_j(t-\tau_k(t)))$ are bounded for all $j = 1,2,\dots,n$. Furthermore, since $\lim_{u_i\to+\infty} b_i(u_i) = +\infty$ and $\lim_{u_i\to-\infty} b_i(u_i) = -\infty$, there must exist $M > 0$ such that

$$b_i(u_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(u_j(t-\tau_k(t))) + I_i > 0 \tag{2.4}$$

whenever $u_i(t) \ge M$ and

$$b_i(u_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(x_j(t-\tau_k(t))) + I_i < 0 \tag{2.5}$$

whenever $u_i(t) \le -M$ for all $i = 1,2,\dots,n$. Since $a_i(u_i(t))$ is positive by (H_1), it can be concluded that for any solution $u(t)$ of system (2.1), $\dot{u}_i(t) < 0$ whenever $u_i(t) \ge M$ and

$\dot{u}_i(t) > 0$ whenever $u_i(t) \le -M$ for all $i = 1,2,\dots,n$. We may assume that for the initial condition $\phi_j(s)$, $|\phi_j(s)| < M$, otherwise we just pick a larger M. Thus we can conclude that $\|u_i(t)\| \le M$ for all $t \ge 0$ and all $i = 1,2,\dots,n$. □

It is not difficult to show that under (H_1)–(H_4), the solution of (2.1) satisfying the initial condition (2.3) exists on $\mathbb{R}_+ \equiv [0,+\infty)$ (see, e.g., [22, 23]). Actually, note that from Lemma 2.2, it is clear that the solution of (2.1) is also unique.

It is also easy to show that (2.1) has always an equilibrium u_j^*, $i = 1,2,\dots,n$. That is, there exist u_j^*, $i = 1,2,\dots,n$, such that

$$b_i(u_i^*) = \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(u_j^*) + I_i, \quad i = 1,2,\dots,n. \tag{2.6}$$

By using the strict monotonicity property of b_i, there exist positive numbers $\overline{b_i} > 0$, $i = 1,2,\dots,n$, such that

$$b_i(u_i^*) = \overline{b_i}u_i^*, \quad i = 1,2,\dots,n. \tag{2.7}$$

Thus,

$$u_i^* = \overline{b_i}^{-1}\left\{\sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(u_j^*) + I_i\right\}, \quad i = 1,2,\dots,n. \tag{2.8}$$

In fact, let us consider the map $P = (P_1,P_2,\dots,P_n)$ on the compact convex set Ω, where

$$\begin{gathered} P_i(u_1,u_2,\dots,u_n) = \overline{b_i}^{-1}\left\{\sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} f_j(u_j^*) + I_i\right\}, \quad i = 1,2,\dots,n, \\ \Omega = \{(u_1,u_2,\dots,u_n) \mid |u_i| \le N_{i0}\}, \\ N_{i0} = \frac{\sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| F_j + |I_i|}{|\overline{b_i}|}, \quad |f_j(u_j^*)| \le F_j, \quad j = 1,2,\dots,n. \end{gathered} \tag{2.9}$$

It follows from (H_1) that P is a continuous map Ω into itself. Thus, it follows from Brouwer's fixed point theorem (see, e.g., [22]) that P has at least one fixed point $(u_1^*, u_2^*,\dots,u_n^*)$ in Ω, that is,

$$(u_1^*,u_2^*,\dots,u_n^*) = P(u_1^*,u_2^*,\dots,u_n^*). \tag{2.10}$$

This shows that $(u_1^*,u_2^*,\dots,u_n^*)$ satisfies (2.6).

Lemma 2.2 is immediate.

Lemma 2.2. *If* (H_1)–(H_4) *are satisfied, then for any solution of (2.1),*

$$\limsup_{t\to\infty} |u_i(t)| \le N_i(\le N_{i0}), \quad i=1,2,\dots,n, \tag{2.11}$$

where the positive constants N_i, $i=1,2,\dots,n$, *satisfy*

$$N_i = \frac{\sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \overline{f}_i(N_i) + |I_i|}{|\overline{b}_i|}, \qquad \overline{f}_i(N_i) = \max\{f_i(N_i), -f_i(-N_i)\}, \quad i=1,2,\dots,n. \tag{2.12}$$

Proof. It is clear from (H_1)–(H_4) and (2.1) that

$$\limsup_{t\to\infty} |u_i(t)| \le N_{i0}, \quad i=1,2,\dots,n. \tag{2.13}$$

Thus, for sufficiently small $\eta>0$ and sufficiently large $T_0>0$, such that for $t\ge T_0$,

$$|u_i(t-\tau)| \le N_{i0}+\eta, \quad i=1,2,\dots,n, \tag{2.14}$$

which together with (H_3) and (2.1) yield that for $t\ge T_0$,

$$\dot u_i(t) \le a_i(u_i(t))\left[-|d_i(u_i(t))| + \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}|\overline{f_j}(N_{i0}+\eta) + |I_i|\right], \quad i=1,2,\dots,n. \tag{2.15}$$

Note that one can take $\eta\to 0$ as $t\to+\infty$, we have

$$\limsup_{t\to\infty} |u_i(t)| \le N_{i1}, \quad i=1,2,\dots,n, \tag{2.16}$$

where

$$N_{i1} = \frac{\sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \overline{f}_i(N_{i0}) + |I_i|}{|\overline{b}_i|} \le N_{i0}, \quad i=1,2,\dots,n. \tag{2.17}$$

By repeating the above procedure, we can obtain positive sequences $\{N_{i,k}\}$ such that

$$N_{i,k+1} = \frac{\sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \overline{f}_i(N_{i,k}) + |I_i|}{|\overline{b}_i|} \le N_{i,k}, \quad i=1,2,\dots,n, \qquad \limsup_{t\to\infty} |u_i(t)| \le N_{i,k}, \quad i=1,2,\dots,n,\ k=1,2,\dots. \tag{2.18}$$

Let N_i denote the limits of $\{N_{i,k}\}$ as $k \to +\infty$, respectively. Then, we have

$$N_i = \left\{ \frac{\sum_{k=0}^{K} \sum_{j=1}^{n} |t_{ij}^{(k)}| \overline{f_i}(N_{i,k}) + |I_i|}{|\overline{b_i}|} \right\}, \quad i = 1,2,\dots,n,$$
$$\limsup_{t\to\infty} |u_i(t)| \le N_i. \tag{2.19}$$

This shows that Lemma 2.2 holds. □

By Lemma 2.2, we see that for any sufficiently small positive constant ε, there exists a sufficiently large time, $T = T(\varepsilon) > 0$, such that for $t \ge T$,

$$|u_i(t)| \le N_i + \varepsilon, \quad i = 1,2,\dots,n. \tag{2.20}$$

Define positive constants $p_{i,\varepsilon}$ and $q_{i,\varepsilon}$, $i = 1,2,\dots,n$, as follows:

$$p_{i,\varepsilon} \equiv \min_{-(N_i+\varepsilon)\le w\le N_i+\varepsilon} f_i'(w) \le \max_{-(N_i+\varepsilon)\le w\le N_i+\varepsilon} g_i'(w) \equiv q_{i,\varepsilon}, \quad i = 1,2,\dots,n. \tag{2.21}$$

Let p_i and q_i, $i = 1,2,\dots,n$, denote the limits of $p_{i\varepsilon}$ and $q_{i\varepsilon}$, respectively, as $\varepsilon \to 0$.

Remark 2.3. It is easy to show that the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of (2.1) is also unique if (H_1)–(H_4) and the following (H_5) are satisfied.

The following well-known Barbalat lemma (see, e.g., [23]) will also be used.

Lemma 2.4. *Let f be a nonnegative function defined on $\mathbb{R}_+$ such that f is integrable and uniformly continuous on $\mathbb{R}_+$. Then* $\lim_{t\to+\infty} f(t) = 0$.

3. Stability analysis

In this section, we will consider the stability of the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1).

Let us first consider the case $t_{ij}^{(k)} \ne 0$, for some $i, j = 1,2,\dots,n$, $k = 1,2,\dots,K$. We further assume the following hypothesis.

(H_5) There exist positive constants λ_i, $i = 1,2,\dots,n$, such that the matrix:

$$R = \begin{pmatrix} \eta_1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & \eta_2 & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & \eta_n \end{pmatrix} \tag{3.1}$$

is negative definite, that is,

$$(-1)^i \begin{pmatrix} \eta_1 & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & \eta_2 & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & \eta_n \end{pmatrix} > 0, \quad i = 1,2,\dots,n, \tag{3.2}$$

where

$$\begin{aligned} \eta_i = {} & \lambda_i\left(-\frac{\underline{B_i}}{q_{i\varepsilon}} + \left(\sum_{k=0}^{K} t_{ii}^{(k)}\right)\right) + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\frac{\lambda_i q_{j\varepsilon}\overline{B_j}}{p_{j\varepsilon}}\left|t_{ij}^{(k)}\right| + \frac{\lambda_j q_{i\varepsilon}\underline{B_i}}{p_{i\varepsilon}}\left|t_{ji}^{(k)}\right|\right] \\ & + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\lambda_i q_{j\varepsilon}\left|t_{ij}^{(k)}\right|\left(\sum_{k=0}^{K}\sum_{l=1}^{n}\left|t_{jl}^{(k)}\right|\right) + q_{j\varepsilon}\left(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l\left|t_{lj}^{(k)}\right|\left|t_{ji}^{(k)}\right|\right)\right] \end{aligned} \tag{3.3}$$

$$r_{ij} = \frac{1}{2}\sum_{k=0}^{K}\left(\lambda_i t_{ij}^{(k)} + \lambda_j t_{ji}^{(k)}\right), \quad i \neq j,\ i,j = 1,2,\dots,n.$$

Hence, the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1) is unique.

Theorem 3.1. *If $t_{ij}^{(k)} \neq 0$, for some $i,j = 1,2,\dots,n$, $k = 1,2,\dots,K$, and (H_1)–(H_5) are satisfied, then the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1) is globally asymptotically stable.*

Proof. Let

$$\begin{aligned} x_j(t) &= u_j(t) - u_j^*, \quad j = 1,2,\dots,n, \\ s_j(x_j(t)) &= f_j(x_j(t) + u_j^*) - f_j(u_j^*), \quad j = 1,2,\dots,n, \\ h_j(x_j(t)) &= b_j(x_j(t) + u_j^*) - b_j(u_j^*), \quad j = 1,2,\dots,n. \end{aligned} \tag{3.4}$$

Then, the stability properties of the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1) are equivalent to that of the trivial solution of the following system:

$$\dot{x}_i(t) = -a_i(x_i(t) + u_i^*)\left[h_i(x_i(t)) - \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} s_j(x_j(t - \tau_k(t)))\right]. \tag{3.5}$$

We construct the following Lyapunov function:

$$V_1 = \sum_{i=1}^{n} \lambda_i \int_0^{x_i(t)} \frac{s_i(\xi)}{a_i(\xi + u_i^*)} d\xi. \tag{3.6}$$

Then its upper right Dini derivative is

$$\begin{aligned} D^+V_1|_{(3.5)} &= \sum_{i=1}^{n} \lambda_i s_i(x_i(t)) \left[-h_i(x_i(t)) + \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} s_j(x_j(t-\tau_k(t))) \right] \\ &= \sum_{i=1}^{n} \lambda_i s_i(x_i(t)) \left[-h_i(x_i(t)) + \sum_{k=0}^{K} t_{ii}^{(k)} s_j(x_j(t)) \right] \\ &\quad + \sum_{i=1}^{n} \lambda_i s_i(x_i(t))\alpha_i + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_i t_{ij}^{(k)} s_i(x_i(t)) s_j(x_j(t)), \end{aligned} \tag{3.7}$$

where

$$\alpha_i = \sum_{k=0}^{K}\sum_{j=1}^{n} t_{ij}^{(k)} \int_t^{t-\tau_k(t)} s_j'(x_j(\xi)) x_j'(\xi) d\xi. \tag{3.8}$$

Note that

$$p_{i,\varepsilon} \equiv \min_{-(N_i+\varepsilon)\leq w \leq N_i+\varepsilon} s_i'(w) \leq \max_{-(N_i+\varepsilon)\leq w \leq N_i+\varepsilon} s_i'(w) \equiv q_{i,\varepsilon}. \tag{3.9}$$

We also note that for sufficiently large t,

$$|x_i(t)| \leq \frac{|s_i(x_i(t))|}{p_{i\varepsilon}}. \tag{3.10}$$

Then

$$\begin{aligned} |s_i(x_i(t))\alpha_i| &\leq |s_i(x_i(t))| \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \int_{t-\tau_k(t)}^{t} |s_j'(x_j(\xi))| |x_j'(\xi)| d\xi \\ &\leq |s_i(x_i(t))| \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \int_{t-\tau_k(t)}^{t} q_{j\varepsilon} \\ &\quad \times \left(|h_j(x_j(\xi))| + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| s_l(x_l(\xi - \tau_k(\xi))) \right) d\xi \end{aligned}$$

$$
\begin{aligned}
&\le |s_i(x_i(t))| \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \int_{t-\tau_k(t)}^{t} q_{j\varepsilon} \\
&\quad \times \left(\frac{\overline{B_j}}{p_{j\varepsilon}} |s_j(x_j(\xi))| + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| s_l(x_l(\xi - \tau_k(\xi))) \right) d\xi \\
&\le \frac{1}{2} \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| q_{j\varepsilon} \int_{t-\tau_k(t)}^{t} \\
&\quad \times \Bigg\{ \frac{\overline{B_j}}{p_{j\varepsilon}} [s_i^2(x_i(t)) + s_j^2(x_j(t))] \\
&\qquad + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| [s_i^2(x_i(t)) + s_l^2(x_l(\xi - \tau_k(\xi)))] \Bigg\} d\xi \\
&= \frac{1}{2} \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| q_{j\varepsilon}\tau_k(t) \left[\frac{\overline{B_j}}{p_{j\varepsilon}} + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| \right] s_i^2(x_l(t)) \\
&\quad + \frac{1}{2} \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \int_{t-\tau_k(t)}^{t} q_{j\varepsilon} \\
&\quad \times \left[\frac{\overline{B_j}}{p_{j\varepsilon}} s_j^2(x_j(\xi)) + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| s_l^2(x_l(\xi - \tau_k(\xi))) \right] d\xi \\
&= \sum_{k=0}^{K}\sum_{j=1}^{n} |t_{ij}^{(k)}| \left(A_{jk\varepsilon} s_i^2(x_i(t)) + \int_{t-\tau_k(t)}^{t} \mu_{j\varepsilon}(\xi)\, d\xi \right),
\end{aligned}
\tag{3.11}
$$

where

$$
\begin{aligned}
A_{jk\varepsilon} &= \frac{1}{2} q_{j\varepsilon}\tau_k(t) \left[\frac{\overline{B_j}}{p_{j\varepsilon}} + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| \right], \\
\mu_{j\varepsilon}(\xi) &= \frac{1}{2} q_{j\varepsilon} \left[\frac{\overline{B_j}}{p_{j\varepsilon}} s_j^2(x_j(\xi)) + \sum_{k=0}^{K}\sum_{l=1}^{n} |t_{jl}^{(k)}| s_l^2(x_l(\xi - \tau_k(\xi))) \right].
\end{aligned}
\tag{3.12}
$$

Furthermore, by (H_1), we have for $t \ge T + \Delta$ that

$$
\begin{aligned}
& s_i(x_i(t)) \left[-h_i(x_i(t)) + \sum_{k=0}^{K} t_{ii}^{(k)} s_i(x_i(t)) \right] \\
&\quad \le -\underline{B_i} x_i(t) f_i(x_i(t)) + \left(\sum_{k=0}^{K} t_{ii}^{(k)} \right) s_i^2(x_i(t)) \le \left[-\frac{\underline{B_i}}{q_{i\varepsilon}} + \left(\sum_{k=0}^{K} t_{ii}^{(k)} \right) \right] s_i^2(x_i(t)).
\end{aligned}
\tag{3.13}
$$

Let

$$V_2 = \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i \left| t_{ij}^{(k)} \right| \left(\int_{t-\tau_k(t)}^{t}\int_{\theta}^{t}\mu_{j\varepsilon}(\xi)d\xi d\theta + \frac{\tau_k(t)q_{j\varepsilon}}{2}\sum_{k=0}^{K}\sum_{l=1}^{n}\left| t_{jl}^{(k)} \right| \int_{t-\tau_k(t)}^{t} s_l^2(x_l(\xi))d\xi \right). \tag{3.14}$$

Its derivative is

$$\begin{aligned} D^+V_2|_{(3.5)} = {} & \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i q_{j\varepsilon}}{2}\left| t_{ij}^{(k)} \right| \tau_k(t)\left(\frac{\overline{B_j}}{p_{j\varepsilon}} s_j^2(x_j(t)) + \sum_{k=0}^{K}\sum_{l=1}^{n}\left| t_{jl}^{(k)} \right| s_l^2(x_l(t)) \right) \\ & - \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i \left| t_{ij}^{(k)} \right| \int_{t-\tau_k(t)}^{t}\mu_{j\varepsilon}(\xi)d\xi. \end{aligned} \tag{3.15}$$

Hence,

$$\begin{aligned} D^+V = {} & D^+V_1 + D^+V_2 \\ \le {} & \sum_{i=1}^{n}\lambda_i\left[-\frac{\underline{\underline{B_i}}}{q_{i\varepsilon}} + \left(\sum_{k=0}^{K} t_{ii}^{(k)} \right) \right] s_i^2(x_i(t)) + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i \left| t_{ij}^{(k)} \right| A_{jk\varepsilon} s_i^2(x_i(t)) \\ & + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i t_{ij}^{(k)} s_i(x_i(t)) s_j(x_j(t)) \\ & + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i q_{j\varepsilon}}{2}\left| t_{ij}^{(k)} \right| \tau_k(t)\left(\frac{\overline{B_j}}{p_{j\varepsilon}} s_j^2(x_j(t)) + \sum_{k=0}^{K}\sum_{l=1}^{n}\left| t_{jl}^{(k)} \right| s_l^2(x_l(t)) \right) \\ = {} & \sum_{i=1}^{n}\lambda_i\left[-\frac{\underline{\underline{B_i}}}{q_{i\varepsilon}} + \left(\sum_{k=0}^{K} t_{ii}^{(k)} \right) + \sum_{k=0}^{K}\sum_{j=1}^{n}\left| t_{ij}^{(k)} \right| A_{jk\varepsilon} \right] s_i^2(x_i(t)) \\ & + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i t_{ij}^{(k)} s_i(x_i(t)) s_j(x_j(t)) \\ & + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\lambda_i q_{j\varepsilon}}{2}\left| t_{ji}^{(k)} \right| \tau_k(t)\frac{\overline{B_j}}{p_{i\varepsilon}} s_i^2(x_i(t)) \\ & + \sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sum_{k=0}^{K}\sum_{l=1}^{n}\frac{\lambda_l q_{j\varepsilon}}{2}\left| t_{lj}^{(k)} \right| \tau_k(t)\left| t_{ji}^{(k)} \right| s_i^2(x_i(t)) \right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n}\Bigg\{\lambda_i\Bigg[-\frac{\underline{\underline{B_i}}}{q_{i\varepsilon}}+\Bigg(\sum_{k=0}^{K}t_{ii}^{(k)}\Bigg)+\sum_{k=0}^{K}\sum_{j=1}^{n}\,|\,t_{ij}^{(k)}\,|\,\frac{q_{j\varepsilon}\tau_k(t)}{2}\Bigg(\frac{\overline{B_j}}{p_{j\varepsilon}}+\sum_{k=0}^{K}\sum_{l=1}^{n}\,|\,t_{jl}^{(k)}\,|\Bigg)\Bigg]\\
&\qquad\qquad +\frac{q_{i\varepsilon}\underline{\underline{B_i}}}{2p_{i\varepsilon}}\sum_{k=0}^{K}\sum_{j=1}^{n}\lambda_j\,|\,t_{ji}^{(k)}\,|\,\tau_k(t)\\
&\qquad\qquad +\sum_{k=0}^{K}\sum_{j=1}^{n}\frac{q_{j\varepsilon}}{2}\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l\tau_k(t)\,|\,t_{lj}^{(k)}\,|\,|\,t_{ji}^{(k)}\,|\Bigg)\Bigg\}s_i^2(x_i(t))\\
&\quad +\sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i t_{ij}^{(k)}s_i(x_i(t))s_j(x_j(t))\\
&= \sum_{i=1}^{n}\Bigg\{\lambda_i\Bigg(-\frac{\underline{\underline{B_i}}}{q_{i\varepsilon}}+\Bigg(\sum_{k=0}^{K}t_{ii}^{(k)}\Bigg)\Bigg)+\frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\Bigg[\frac{\lambda_i q_{j\varepsilon}\overline{B_j}}{p_{j\varepsilon}}\,|\,t_{ij}^{(k)}\,|\,\tau_k(t)+\frac{\lambda_j q_{i\varepsilon}\underline{\underline{B_i}}}{p_{i\varepsilon}}\,|\,t_{ji}^{(k)}\,|\,\tau_k(t)\Bigg]\\
&\qquad +\frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\Bigg[\lambda_i q_{j\varepsilon}\tau_k(t)\,|\,t_{ij}^{(k)}\,|\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{n}\,|\,t_{jl}^{(k)}\,|\Bigg)\\
&\qquad\qquad\qquad +q_{j\varepsilon}\tau_k(t)\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l\,|\,t_{lj}^{(k)}\,|\,|\,t_{ji}^{(k)}\,|\Bigg)\Bigg]\Bigg\}s_i^2(x_i(t))\\
&\quad +\sum_{k=0}^{K}\sum_{i=1}^{n}\sum_{j=1}^{n}\lambda_i t_{ij}^{(k)}s_i(x_i(t))s_j(x_j(t))\\
&= \frac{1}{2}\big(s_1(x_1(t)),s_2(x_2(t)),\dots,s_n(x_n(t))\big)R(\varepsilon)\big(s_1(x_1(t)),s_2(x_2(t)),\dots,s_n(x_n(t))\big)^T,
\end{aligned}
\tag{3.16}
$$

where

$$
\begin{aligned}
R(\varepsilon) &= \begin{pmatrix} \eta_1(\varepsilon) & r_{12} & r_{13} & \cdots & r_{1n}\\ r_{21} & \eta_2(\varepsilon) & r_{23} & \cdots & r_{2n}\\ \vdots & \vdots & \vdots & \vdots & \vdots\\ r_{n1} & r_{n2} & r_{n3} & \cdots & \eta_n(\varepsilon)\end{pmatrix},\\
\eta_i &= \lambda_i\Bigg(-\frac{\underline{\underline{B_i}}}{q_{i\varepsilon}}+\Bigg(\sum_{k=0}^{K}t_{kk}^{(k)}\Bigg)\Bigg)+\frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\Bigg[\frac{\lambda_i q_{j\varepsilon}\overline{B_j}}{p_{j\varepsilon}}\,|\,t_{ij}^{(k)}\,|+\frac{\lambda_j q_{i\varepsilon}\underline{\underline{B_i}}}{p_{i\varepsilon}}\,|\,t_{ji}^{(k)}\,|\Bigg]\\
&\quad +\frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\Bigg[\lambda_i q_{j\varepsilon}\,|\,t_{ij}^{(k)}\,|\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{n}\,|\,t_{il}^{(k)}\,|\Bigg)+q_{j\varepsilon}\Bigg(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l\,|\,t_{lj}^{(k)}\,|\,|\,t_{ji}^{(k)}\,|\Bigg)\Bigg]\\
r_{ij} &= \frac{1}{2}\sum_{k=0}^{K}\big(\lambda_i t_{ij}^{(k)}+\lambda_j t_{ji}^{(k)}\big),\quad i\neq j,\ i,j=1,2,\dots,n.
\end{aligned}
\tag{3.17}
$$

Moreover, the upper right derivative D^+V of V along solution (3.5) satisfies

$$D^+V|_{(3.5)} \le -\delta(\varepsilon)\sum_{i=1}^{n} s_i^2(x_i(t)) \tag{3.18}$$

for $t \ge T+\overline{\tau}$. Here $\delta(\varepsilon) > 0$ is some constant. $x_t = x(t+s)$ for $-\overline{\tau} \le s \le 0$.

Integrating (3.18) over $[T+\overline{\tau}, t]$ yields

$$V(x_t) + \delta(\varepsilon)\int_{T+\overline{\tau}}^{t}\sum_{i=1}^{n} s_i^2(x_i(\xi))\,d\xi \le V(x_{T+\overline{\tau}}), \tag{3.19}$$

which implies that

$$\sum_{i=1}^{n}\int_0^{+\infty} s_i^2(x_i(\xi))\,d\xi < +\infty. \tag{3.20}$$

Moreover, from Lemma 2.2 and (H_2)–(H_3), we see that $s_i^2(x_i(t))$, $i = 1,2,\dots,n$, are also uniformly continuous on $\mathbb{R}_+$. Hence, Lemma 2.4 implies that

$$\lim_{t\to+\infty} |s_i(x_i(t))| = 0, \quad i = 1,2,\dots,n. \tag{3.21}$$

Again from Lemma 2.2 and (H_2), we have

$$\lim_{t\to+\infty} x_i(t) = 0, \quad i = 1,2,\dots,n, \tag{3.22}$$

that is,

$$\lim_{t\to+\infty} u_i(t) = u_i^*, \quad i = 1,2,\dots,n, \tag{3.23}$$

which show that the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1) is globally attractive.

Furthermore, note (H_2), (H_3), and the following inequalities:

$$p_i \le f_i'(u_i^*) \le q_i, \quad i = 1,2,\dots,n. \tag{3.24}$$

We see that (H_5) implies (H_6) of Theorem 3.4. Thus, the equilibrium $(u_1^*, u_2^*, \dots, u_n^*)$ of system (2.1) is also locally asymptotically stable. This proves Theorem 3.1. □

(H_5') There exist positive constants λ_i, $i = 1,2,\dots,n$, such that

$$\begin{aligned}\gamma_i = {} & \lambda_i\left(-\frac{B_i}{q_{i\varepsilon}} + \left(\sum_{k=0}^{K} t_{ii}^{(k)}\right)\right) + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\frac{\lambda_i q_j \overline{B_j}}{p_j}|t_{ij}^{(k)}| + \frac{\lambda_j q_i \underline{B_i}}{p_i}|t_{ji}^{(k)}|\right] \\ & + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\lambda_i q_j |t_{ij}^{(k)}|\left(\sum_{k=0}^{K}\sum_{l=1}^{n}|t_{jl}^{(k)}|\right) + q_j\left(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l |t_{lj}^{(k)}||t_{ji}^{(k)}|\right)\right] \\ & + \frac{1}{2}\sum_{k=0}^{K}\left[\lambda_i |t_{ij}^{(k)}| + \lambda_j |t_{ji}^{(k)}|\right] < 0.\end{aligned} \tag{3.25}$$

By the process of the proof of Theorem 3.1, we can easily obtain the following.

Corollary 3.2. *If $t_{ij}^{(k)} \neq 0$, for some $i, j = 1,2,\dots,n$, and (H_1)–(H_4) and (H_5') are satisfied, then the equilibrium $(u_1^*, u_2^*,\dots,u_n^*)$ of system (2.1) is also globally asymptotically stable.*

Remark 3.3. In [11, 14, 18, 24], the authors require that the time-varying delays satisfy $\tau_k'(t) \le R < 1$. However, in our theorem, these delays are not necessarily continuous and differentiable. They only need to satisfy the condition $0 \le \tau_k(t) \le \overline{\tau}$. Hence, our results are less restrictive and conservative than the known result [7, 19].

(H_6) There exist positive constants λ_i, $i = 1,2,\dots,n$, such that the matrix

$$R^* = \begin{pmatrix} \eta_1^* & r_{12} & r_{13} & \cdots & r_{1n} \\ r_{21} & \eta_2^* & r_{23} & \cdots & r_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ r_{n1} & r_{n2} & r_{n3} & \cdots & \eta_n^* \end{pmatrix} \tag{3.26}$$

is negative definite, where

$$\begin{aligned} \eta_i^* &= \lambda_i\left(-\frac{\underline{B_i}}{f_i'(u_i^*)} + \left(\sum_{k=0}^{K} t_{ii}^{(k)}\right)\right) + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\lambda_i\overline{B_j}\,|\,t_{ij}^{(k)}\,| + \lambda_j\underline{B_j}\,|\,t_{ij}^{(k)}\,|\right] \\ &\quad + \frac{1}{2}\sum_{k=0}^{K}\sum_{j=1}^{n}\tau_k(t)\left[\lambda_i f_j'(u_j^*)\,|\,t_{ij}^{(k)}\,|\left(\sum_{k=0}^{K}\sum_{l=1}^{n}|\,t_{jl}^{(k)}\,|\right) + f_j'(u_j^*)\left(\sum_{k=0}^{K}\sum_{l=1}^{n}\lambda_l\,|\,t_{lj}^{(k)}\,|\,|\,t_{ji}^{(k)}\,|\right)\right] \\ r_{ij} &= \frac{1}{2}\sum_{k=0}^{K}\left(\lambda_i t_{ij}^{(k)} + \lambda_j t_{ji}^{(k)}\right), \quad i \neq j,\ i,j = 1,2,\dots,n. \end{aligned} \tag{3.27}$$

Hence, we can easily have the following.

Theorem 3.4. *If $t_{ij}^{(k)} \neq 0$, for some $i, j = 1,2,\dots,n$, $k = 1,2,\dots,K$, and (H_1)–(H_4) and (H_6) are satisfied, then the equilibrium $(u_1^*, u_2^*,\dots,u_n^*)$ of system (2.1) is also locally asymptotically stable.*

4. Numerical example and conclusions

Generally, the delay-independent criteria are particularly restrictive and conservative for networks parameters. Moreover, it is reasonable to consider and apply these criteria first. If they are found inappropriate, the delay-dependent criteria will then be applied. To illustrate the results presented in Theorem 3.1 and Corollary 3.2, a simple example is given and a comparison of the results is given based on the results of literature [7] in the following.

We consider the following model system:

$$\begin{aligned} \dot{x}_1(t) &= -(4+\sin(x_1(t)))\left[2x_1(t) - \tanh(x_1(t)) - 0.5\tanh(2x_2(t-\tau)) + 0.5\right], \\ \dot{x}_2(t) &= -(2+\cos(x_2(t)))\left[2x_2(t) - \tanh(x_1(t)) - 0.5\tanh(2x_2(t-\tau)) - 0.5\right]. \end{aligned} \tag{4.1}$$

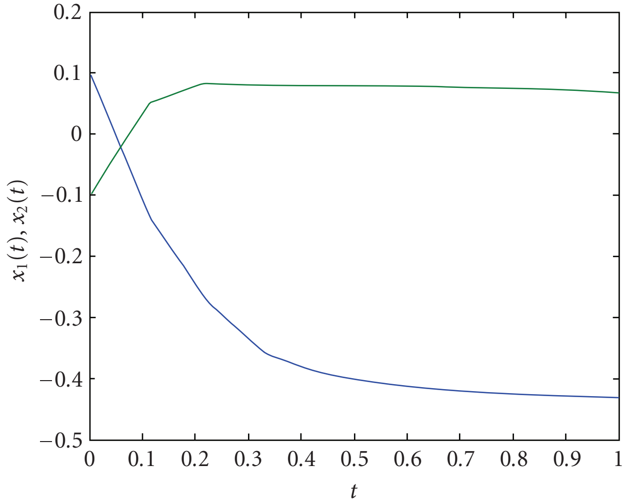

Figure 4.1. Wave form plot for system (4.1) when $\tau = 0.282$.

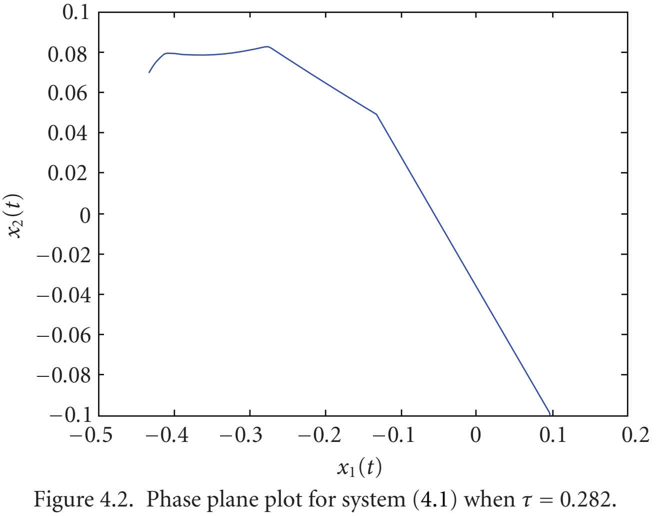

Figure 4.2. Phase plane plot for system (4.1) when $\tau = 0.282$.

We can easily find that the delay-independent conditions given in [7] are not applied and satisfied. This demonstrates that the delay-independent criteria are more conservative and restrictive than the delay-dependent criteria.

For system (4.1), we can obtain $\tau < 0.2828$ from [7, Theorem 3.1]. However, we can also obtain $\tau \leq 0.8246$ based on our results of Theorem 3.1. Numerical simulations have also been performed (see Figures 4.1, 4.2, 4.3 and 4.4). However, the problem of whether the delay superbound is optimal will be studied in a forthcoming paper.

In this paper, we have analyzed Cohen-Grossberg model with time delays in detail. The global asymptotic stability criteria for the equilibrium are derived based on the approach of Lyapunov functional. The obtained results are delay-dependent. Then, the

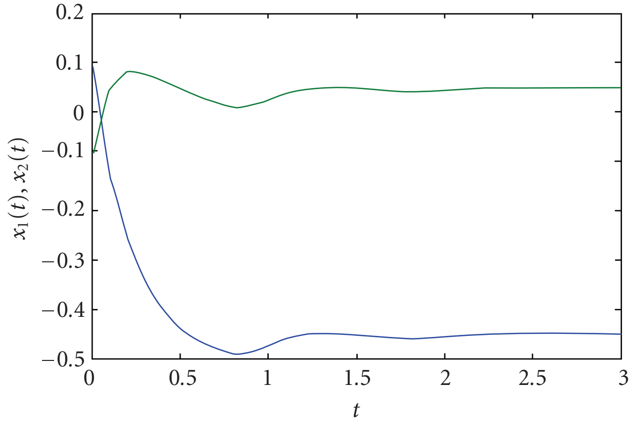

Figure 4.3. Wave form plot for system (4.1) when $\tau = 0.8246$.

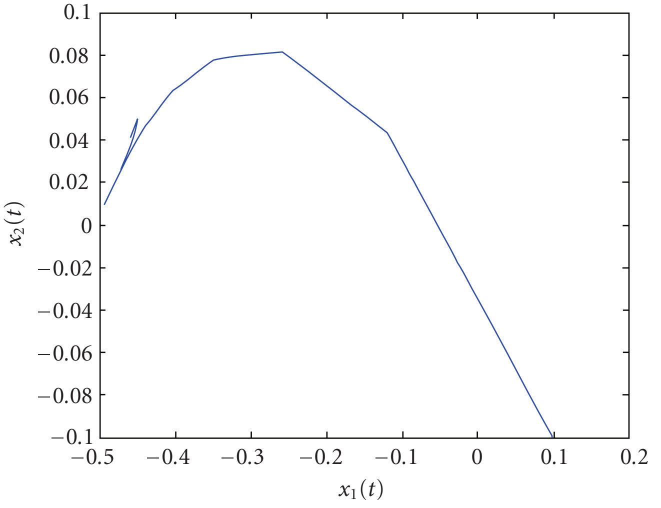

Figure 4.4. Phase plane plot for system (4.1) when $\tau = 0.8246$.

delay-dependent criteria for local asymptotic stability criteria have also been obtained. Hence, our work has complemented and generalized that reported in [7].

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