MULTIPLE POSITIVE SOLUTIONS OF SINGULAR DISCRETE $p$-LAPLACIAN PROBLEMS VIA VARIATIONAL METHODS

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We obtain multiple positive solutions of singular discrete $p$-Laplacian problems using variational methods.

1. Introduction

We consider the boundary value problem

$$
-\Delta(p) (\Delta u(k - 1)) = f(k, u(k)), \quad k \in [1, n],
$$

$$
u(k) > 0, \quad k \in [1, n],
$$

$$
u(0) = 0 = u(n + 1),
$$

where $n$ is an integer greater than or equal to 1, $[1, n]$ is the discrete interval $\{1, \ldots, n\}$, $\Delta u(k) = u(k + 1) - u(k)$ is the forward difference operator, $\varphi_p(s) = |s|^{p-2}s$, $1 < p < \infty$, and we only assume that $f \in C([1, n] \times (0, \infty))$ satisfies

$$
a_0(k) \leq f(k, t) \leq a_1(k) t^{-\gamma}, \quad (k, t) \in [1, n] \times (0, t_0)
$$

for some nontrivial functions $a_0, a_1 \geq 0$ and $\gamma, t_0 > 0$, so that it may be singular at $t = 0$ and may change sign.

Let $\lambda_1, \varphi_1 > 0$ be the first eigenvalue and eigenfunction of

$$
-\Delta(p) (\Delta u(k - 1)) = \lambda \varphi_p(u(k)), \quad k \in [1, n],
$$

$$
u(0) = 0 = u(n + 1).
$$

Theorem 1.1. If (1.2) holds and

$$
\limsup_{t \to \infty} \frac{f(k, t)}{t^{p-1}} < \lambda_1, \quad k \in [1, n],
$$

then (1.1) has a solution.
Theorem 1.2. If (1.2) holds and
\[ f(k,t_1) \leq 0, \quad k \in [1,n], \]
for some \( t_1 > t_0 \), then (1.1) has a solution \( u_1 < t_1 \). If, in addition,
\[ \liminf_{t \to \infty} \frac{f(k,t)}{t^{p-1}} > \lambda_1, \quad k \in [1,n], \]
then there is a second solution \( u_2 > u_1 \).

Example 1.3. Problem (1.1) with \( f(k,t) = t^{-\gamma} + \lambda t^\beta \) has a solution for all \( \gamma > 0 \) and \( \lambda \) (resp., \( \lambda < \lambda_1, \lambda \leq 0 \)) if \( \beta < p-1 \) (resp., \( \beta = p-1, \beta > p-1 \)) by Theorem 1.1.

Example 1.4. Problem (1.1) with \( f(k,t) = t^{-\gamma} + e^t - \lambda \) has two solutions for all \( \gamma > 0 \) and sufficiently large \( \lambda > 0 \) by Theorem 1.2.

Our results seem new even for \( p = 2 \). Other results on discrete \( p \)-Laplacian problems can be found in [1, 2] in the nonsingular case and in [3, 4, 5, 6] in the singular case.

2. Preliminaries

First we recall the weak comparison principle (see, e.g., Jiang et al. [2]).

Lemma 2.1. If
\[ -\Delta(\varphi_p(\Delta u(k-1))) \geq -\Delta(\varphi_p(\Delta v(k-1))), \quad k \in [1,n], \]
\[ u(0) \geq v(0), \quad u(n+1) \geq v(n+1), \]
then \( u \geq v \).

Next we prove a local comparison result.

Lemma 2.2. If
\[ -\Delta(\varphi_p(\Delta u(k-1))) \geq -\Delta(\varphi_p(\Delta v(k-1))), \]
\[ u(k) = v(k), \quad u(k+1) \geq v(k+1), \]
then \( u(k \pm 1) = v(k \pm 1) \).

Proof. We have
\[ -\varphi_p(\Delta u(k)) + \varphi_p(\Delta u(k-1)) \geq -\varphi_p(\Delta v(k)) + \varphi_p(\Delta v(k-1)), \]
\[ \Delta u(k) \geq \Delta v(k), \quad \Delta u(k-1) \leq \Delta v(k-1). \]
Combining with the strict monotonicity of \( \varphi_p \) shows that
\[ 0 \leq \varphi_p(\Delta u(k)) - \varphi_p(\Delta v(k)) \leq \varphi_p(\Delta u(k-1)) - \varphi_p(\Delta v(k-1)) \leq 0, \]
and hence, the equalities hold in (2.4). \( \square \)
The following strong comparison principle is now immediate.

**Lemma 2.3.** If
\[
-\Delta(\varphi_p(\Delta u(k-1))) \geq -\Delta(\varphi_p(\Delta v(k-1))), \quad k \in [1,n],
\]
\[
u(0) \geq \nu(0), \quad u(n+1) \geq v(n+1),
\]
then either \(u > v\) in \([1,n]\), or \(u \equiv v\). In particular, if
\[
-\Delta(\varphi_p(\Delta u(k-1))) \geq 0, \quad k \in [1,n],
\]
\[
u(0) \geq 0, \quad u(n+1) \geq 0,
\]
then either \(u > 0\) in \([1,n]\) or \(u \equiv 0\).

Consider the problem
\[
-\Delta(\varphi_p(\Delta u(k-1))) = g(k,u(k)), \quad k \in [1,n],
\]
\[
u(0) = 0 = u(n+1),
\]
where \(g \in C([1,n] \times \mathbb{R})\). The class \(W\) of functions \(u : [0,n+1] \rightarrow \mathbb{R}\) such that \(u(0) = 0 = u(n+1)\) is an \(n\)-dimensional Banach space under the norm
\[
\|u\| = \left( \sum_{k=1}^{n+1} |\Delta u(k-1)|^p \right)^{1/p}.
\]

Define
\[
\Phi_g(u) = \sum_{k=1}^{n+1} \left[ \frac{1}{p} |\Delta u(k-1)|^p - G(k,u(k)) \right], \quad u \in W,
\]
where \(G(k,t) = \int_0^t g(k,s)ds\). Then the functional \(\Phi_g\) is \(C^1\) with
\[
(\Phi'_g(u),v) = \sum_{k=1}^{n+1} \left[ \varphi_p(\Delta u(k-1)) \Delta v(k-1) - g(k,u(k)) v(k) \right]
\]
\[
= - \sum_{k=1}^n \left[ \Delta(\varphi_p(\Delta u(k-1))) + g(k,u(k)) \right] v(k)
\]
(summing by parts), so solutions of (2.8) are precisely the critical points of \(\Phi_g\).

**Lemma 2.4.** If
\[
\limsup_{|t| \rightarrow \infty} \frac{g(k,t)}{|t|^{p-2}t} < \lambda_1, \quad k \in [1,n],
\]
then \(\Phi_g\) has a global minimizer.
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**Proof.** By (2.12), there is a $\lambda \in [0, \lambda_1)$ such that

$$G(k, t) \leq \frac{\lambda}{p} |t|^p + C,$$

where $C$ denotes a generic positive constant. Since

$$\lambda_1 = \min_{u \in W \setminus \{0\}} \frac{\sum_{k=1}^{n+1} |\Delta u(k - 1)|^p}{\sum_{k=1}^{n} |u(k)|^p},$$

then

$$\Phi_g(u) \geq \frac{1}{p} \left(1 - \frac{\lambda}{\lambda_1}\right)\|u\|^p - C\|u\|,$$

so $\Phi_g$ is bounded from below and coercive. □

**Lemma 2.5.** If

$$\liminf_{t \to +\infty} \frac{g(k, t)}{t^{p-1}} > \lambda_1, \quad \lim_{t \to -\infty} \frac{g(k, t)}{|t|^{p-1}} = 0, \quad k \in [1, n],$$

then $\Phi_g$ satisfies the Palais-Smale compactness condition (PS): every sequence $(u_j)$ in $W$ such that $\Phi_g(u_j)$ is bounded and $\Phi'_g(u_j) \to 0$ has a convergent subsequence.

**Proof.** It suffices to show that $(u_j)$ is bounded since $W$ is finite dimensional, so suppose that $\rho_j := \|u_j\| \to \infty$ for some subsequence. We have

$$o(1)\|u_j\| = \langle \Phi'_g(u_j), u_j \rangle \leq -\|u_j\|^p - \sum_{k=1}^{n+1} g(k, -u_j^-(k))u_j^-(k),$$

where $u_j^- = \max\{-u_j, 0\}$ is the negative part of $u_j$, so it follows from (2.16) that $(u_j^-)$ is bounded. So, for a further subsequence, $\tilde{u}_j := u_j/\rho_j$ converges to some $\tilde{u} \geq 0$ in $W$ with $\|\tilde{u}\| = 1$.

We may assume that for each $k$, either $(u_j(k))$ is bounded or $u_j(k) \to \infty$. In the former case, $\tilde{u}(k) = 0$ and $g(k, u_j(k))/\rho_j^{p-1} \to 0$, and in the latter case, $g(k, u_j(k)) \geq 0$ for large $j$ by (2.16). So it follows from

$$o(1) = \frac{\langle \Phi'_g(u_j), v \rangle}{\rho_j^{p-1}} = \sum_{k=1}^{n+1} \left[ \varphi_p(\Delta \tilde{u}_j(k - 1)) \Delta v(k - 1) - \frac{g(k, u_j(k))}{\rho_j^{p-1}} v(k) \right]$$

that

$$\sum_{k=1}^{n+1} \varphi_p(\Delta \tilde{u}(k - 1)) \Delta v(k - 1) \geq 0 \quad \forall v \geq 0,$$
and hence, $\tilde{u} > 0$ in $[1, n]$ by Lemma 2.3. Then $u_j(k) \to \infty$ for each $k$, and hence, (2.18) can be written as

$$\sum_{k=1}^{n+1} \left[ \varphi_p(\Delta \tilde{u}_j(k-1)) \Delta v(k-1) - \alpha_j(k) \tilde{u}_j(k)^{p-1} v(k) \right] = o(1),$$

(2.20)

where

$$\alpha_j(k) = \frac{g(k, u_j(k))}{u_j(k)^{p-1}} \geq \lambda, \quad j \text{ large},$$

(2.21)

for some $\lambda > \lambda_1$ by (2.16).

Choosing $v$ appropriately and passing to the limit shows that each $\alpha_j(k)$ converges to some $\alpha(k) \geq \lambda$ and

$$-\Delta(\varphi_p(\Delta \tilde{u}(k-1))) = \alpha(k) \tilde{u}(k)^{p-1}, \quad k \in [1, n],$$

$$\tilde{u}(0) = 0 = \tilde{u}(n+1).$$

(2.22)

This implies that the first eigenvalue of the corresponding weighted eigenvalue problem is given by

$$\min_{u \in W \setminus \{0\}} \frac{\sum_{k=1}^{n+1} |\Delta u(k-1)|^p}{\sum_{k=1}^n \alpha(k) |u(k)|^p} = 1.$$  

(2.23)

Then

$$1 \leq \frac{\sum_{k=1}^{n+1} |\Delta \varphi_1(k-1)|^p}{\sum_{k=1}^n \alpha(k) \varphi_1(k)^p} \leq \frac{\lambda_1}{\lambda} < 1,$$

(2.24)

a contradiction. $\square$

3. Proofs

The problem

$$-\Delta(\varphi_p(\Delta u(k-1))) = a_0(k), \quad k \in [1, n],$$

$$u(0) = 0 = u(n+1),$$

(3.1)

has a unique solution $u_0 > 0$ by Lemmas 2.3 and 2.4. Fix $\varepsilon \in (0, 1]$ so small that $u := \varepsilon^{1/(p-1)} u_0 < t_0$. Then

$$-\Delta(\varphi_p(\Delta u(k-1))) - f(k, u(k)) \leq -(1 - \varepsilon)a_0(k) \leq 0$$

(3.2)

by (1.2), so $u$ is a subsolution of (1.1). Let

$$f_u(k, t) = \begin{cases} f(k, t), & t \geq u(k), \\ f(k, u(k)), & t < u(k). \end{cases}$$

(3.3)
Proof of Theorem 1.1. By (1.4), there are \( \lambda \in [0, \lambda_1) \) and \( T > t_0 \) such that
\[
f(k, t) \leq \lambda t^{p-1}, \quad (k, t) \in [1, n] \times (T, \infty). \tag{3.4}
\]
Then
\[
f_u(k, t) \begin{cases} 
\leq a_1(k)u(k)^{-r} + \max f([1, n] \times [t_0, T]) + \lambda t^{p-1}, & t \geq 0, \\
\geq a_0(k), & t < 0,
\end{cases} \tag{3.5}
\]
by (1.2), so the modified problem
\[
-\Delta (\varphi_p(\Delta u(k-1))) = f_u(k, u(k)), \quad k \in [1, n],
\]
\[
u(0) = 0 = u(n + 1), \tag{3.6}
\]
has a solution \( u \) by Lemma 2.4. By Lemma 2.1, \( u \geq u_1 \), and hence, also a solution of (1.1). \( \square \)

Proof of Theorem 1.2. Noting that \( t_1 \) is a supersolution of (3.6), let
\[
\tilde{f}_u(k, t) = \begin{cases} 
 f_u(k, t_1), & t > t_1, \\
 f_u(k, t), & t \leq t_1.
\end{cases} \tag{3.7}
\]
By (1.2),
\[
\tilde{f}_u(k, t) \begin{cases} 
\leq a_1(k)u(k)^{-r} + \max f([1, n] \times [t_0, t_1]), & t \geq 0, \\
\geq a_0(k), & t < 0,
\end{cases} \tag{3.8}
\]
so \( \Phi_{\tilde{f}_u} \) has a global minimizer \( u_1 \) by Lemma 2.4. By Lemmas 2.1 and 2.2, \( u \leq u_1 < t_1 \), so \( \Phi_{\tilde{f}_u} = \Phi_{f_u} \) near \( u_1 \) and hence, \( u_1 \) is a local minimizer of \( \Phi_{f_u} \). Let
\[
f_{u_1}(k, t) = \begin{cases} 
f(k, t), & t \geq u_1(k), \\
f(k, u_1(k)), & t < u_1(k),
\end{cases} \tag{3.9}
\]
Since \( u_1 \) is also a subsolution of (1.1), repeating the above argument with \( u_1 \) in place of \( u \), we see that \( \Phi_{f_{u_1}} \) also has a local minimizer, which we assume is \( u_1 \) itself, for otherwise we are done. By (1.6), there are \( \lambda > \lambda_1 \) and \( T > t_1 \) such that
\[
f(k, t) \geq \lambda t^{p-1}, \quad (k, t) \in [1, n] \times (T, \infty), \tag{3.10}
\]
so
\[
\Phi_{f_{u_1}}(t\varphi_1) \leq -\frac{t^p}{p} \left( \frac{\lambda}{\lambda_1} - 1 \right) + Ct < \Phi_{f_{u_1}}(u_1), \quad t > 0 \text{ large.} \tag{3.11}
\]
Since \( \Phi_{f_{u_1}} \) satisfies (PS) by Lemma 2.5, the mountain-pass lemma now gives a second critical point \( u_2 \), which is greater than \( u_1 \) by Lemmas 2.1 and 2.2. \( \square \)
References


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