We consider a nonlinear problem for the mean curvature equation in the hyperbolic space with a Dirichlet boundary data \( g \). We find solutions in a Sobolev space under appropriate conditions on \( g \).

1. Introduction

Let \( M \) be the open unit ball in \( \mathbb{R}^3 \) of center 0 and let

\[
g_{ij}(x) = \frac{4\delta_{ij}}{(1 - |x|^2)^2}
\]

be the hyperbolic metric on \( M \). Let \( \Omega \subset \mathbb{R}^2 \) be a bounded domain with smooth boundary \( \partial \Omega \in C^{1,1} \), and let \( (u,v) \) be the variables in \( \mathbb{R}^2 \). We consider in this paper the Dirichlet problem for a function \( X : \overline{\Omega} \to M \) which satisfies the equation of prescribed mean curvature

\[
\nabla_{X_u} X_u + \nabla_{X_v} X_v = -2H(X)X_u \wedge X_v \quad \text{in } \Omega,
\]

\[
X = g \quad \text{on } \partial \Omega,
\]

where \( H : M \to \mathbb{R} \) is a given continuous function, and \( g \in W^{2,p}(\Omega, \mathbb{R}^3) \) for \( 1 < p < \infty \), with \( \|g\|_{\infty} < 1 \).

In the above equation \( X_u, X_v, X_u \wedge X_v : \Omega \to TM \) are the vector fields given by

\[
X_u(u,v) = \frac{3}{X_u} \sum_{k=1}^{3} \frac{\partial X_k}{\partial u} \bigg|_{(u,v)} \frac{\partial}{\partial x_k} X(u,v), \quad X_v(u,v) = \frac{3}{X_v} \sum_{k=1}^{3} \frac{\partial X_k}{\partial v} \bigg|_{(u,v)} \frac{\partial}{\partial x_k} X(u,v),
\]

\[
X_u \wedge X_v(u,v) = \frac{3}{X_u \wedge X_v} \sum_{k=1}^{3} \left( X_u \wedge X_v \right)^k (u,v) \frac{\partial}{\partial x_k} X(u,v),
\]
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where

\[
(X_u \wedge X_v)^1(u,v) = \frac{1}{2} \left(\frac{\partial X_2}{\partial u}(u,v) \frac{\partial X_3}{\partial v}(u,v) - \frac{\partial X_3}{\partial u}(u,v) \frac{\partial X_2}{\partial v}(u,v)\right),
\]

\[
(X_u \wedge X_v)^2(u,v) = \frac{1}{2} \left(\frac{\partial X_3}{\partial u}(u,v) \frac{\partial X_1}{\partial v}(u,v) - \frac{\partial X_1}{\partial u}(u,v) \frac{\partial X_3}{\partial v}(u,v)\right),
\]

\[
(X_u \wedge X_v)^3(u,v) = \frac{1}{2} \left(\frac{\partial X_1}{\partial u}(u,v) \frac{\partial X_2}{\partial v}(u,v) - \frac{\partial X_2}{\partial u}(u,v) \frac{\partial X_1}{\partial v}(u,v)\right),
\]

for \( \phi(x) = \frac{4}{(1 - |x|^2)}^2 \).

We remark that if \( X_u \) and \( X_v \) are linearly independent, then \( X(Z\Omega_1) \subset M \) is an imbedded submanifold and \( X_u \wedge X_v(u,v) \) is the only vector orthogonal to \( X(Z\Omega_1) \) at \( X(u,v) \) that satisfies, for any \( z = \sum_{k=1}^3 z_k(\partial/\partial x_k) \big|_{X(u,v)} \)

\[
\langle z, X_u \wedge X_v(u,v) \rangle = \omega(X(u,v))\langle z, X_u(u,v), X_v(u,v) \rangle.
\]

where \( \omega \) is the volume element of \((M, \langle , \rangle)\), namely

\[
\omega = \sqrt{\det(g_{ij})} dx_1 \wedge dx_2 \wedge dx_3 = \phi^{3/2} dx_1 \wedge dx_2 \wedge dx_3.
\]

If \( \nabla \) is the Levi-Civita connection associated to \( \langle , \rangle \) and \( \Gamma^k_{ij} : M \to \mathbb{R} \) are the Christoffel symbols

\[
\Gamma^k_{ij}(x) = \Gamma^i_{kj}(x) = \frac{2x_j}{1-|x|^2}, \quad \Gamma^k_{ii}(x) = \begin{cases} -\frac{2x_k}{1-|x|^2} & \text{if } k \neq i, \\ 0 & \text{otherwise}. \end{cases}
\]

Let \( E, F, G : \Omega \to \mathbb{R} \) be the coefficients of the first fundamental form, and the unit normal \( N : \Omega \to TM \) be given by

\[
N = \frac{1}{\sqrt{EG-F^2}} X_u \wedge X_v
\]

which is orthogonal to the tangent space \( \{X(\Omega)\}_x \) for any \( x = X(u,v) \). Then, if \( H : \Omega \to \mathbb{R} \) is the mean curvature of \( X(\Omega) \) we obtain

\[
\left( N, \frac{G}{EG-F^2} \nabla_{X_u} X_u + \frac{E}{EG-F^2} \nabla_{X_v} X_v - 2\frac{F}{EG-F^2} \nabla_{X_u} X_v \right) = -2H.
\]

In particular, if \( X \) is isothermal, that is, \( E = G, F = 0 \), then \( \langle \nabla_{X_u} X_u + \nabla_{X_v} X_v, X_u \rangle = 0 = \langle \nabla_{X_u} X_u + \nabla_{X_v} X_v, X_v \rangle \) and consequently

\[
\nabla_{X_u} X_u + \nabla_{X_v} X_v = -2H X_u \wedge X_v.
\]

Thus, (1.11) is the equation of prescribed mean curvature for an imbedded submanifold of \( M \).
2. A Dirichlet problem for (1.11)

With the notations of the previous section, we consider the Dirichlet problem (1.2). The equation of prescribed mean curvature for a surface in $\mathbb{R}^3$ has been studied for constant $H$ in [3, 5], and for $H$ nonconstant in [1, 2].

Without loss of generality, we may assume that $g$ is harmonic in $\Omega$. Our existence result reads as follows.

**Theorem 2.1.** Let $c_0$ and $c_1$ be some positive constants to be specified. Then (1.2) is solvable for any $g \in W^{2,p}(\Omega, \mathbb{R}^3)$ harmonic such that

$$\|g\|_\infty + 2 \left( c_1 + \sqrt{c_1(c_1 + c_0)} \right) \|\text{grad}(g)\|_p \leq 1. \quad (2.1)$$

In the proof of Theorem 2.1, we ignore the canonical isomorphism $\partial/\partial x_k |_{X(u,v)} \rightarrow e_k$ (with $\{e_k\}$ the usual basis of $\mathbb{R}^3$), and considering $X_u, X_v \in \mathbb{R}^3$ we may write (1.2) as a system

$$-\Delta X_k = \psi_k(X, X_u, X_v) \quad \text{in } \Omega,$$

$$X_k = g_k \quad \text{on } \partial \Omega \quad (2.2)$$

with $\psi_k(X, X_u, X_v) = 2H(X)(X_u \wedge X_v)^k + \sum_{i,j} \Gamma_{ij}^k(X) \text{grad}(X_i) \text{grad}(X_j), 1 \leq k \leq 3$.

For fixed $\overline{X} \in W^{1,2}_0(\Omega, \mathbb{R}^3)$ such that $\|g + \overline{X}\|_\infty < 1$, we define $X = T\overline{X}$ as the unique solution in $W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow W^{1,2}_0(\Omega, \mathbb{R}^3)$ of the linear problem

$$-\Delta X_k = \psi_k \left( \overline{X} + g, (\overline{X} + g)_u, (\overline{X} + g)_v \right) \quad \text{in } \Omega,$$

$$X_k = 0 \quad \text{on } \partial \Omega \quad (2.3)$$

Then, for $B = \{X \in W^{1,2}_0(\Omega, \mathbb{R}^3) \mid \|g + X\|_\infty < 1\}$ the operator $T : B \rightarrow W^{1,2}_0(\Omega, \mathbb{R}^3)$ is well defined and a strong solution of (1.2) in $W^{2,p}$ can be regarded as $Y = g + X$, where $X$ is a fixed point of $T$. By the usual a priori bounds for the Laplacian and the compactness of the imbedding $W^{2,p}(\Omega, \mathbb{R}^3) \hookrightarrow W^{1,2}_0(\Omega, \mathbb{R}^3)$ we get the following lemma.

**Lemma 2.2.** $T : B \rightarrow W^{1,2}_0(\Omega, \mathbb{R}^3)$ is continuous. Furthermore, if

$$C_{R_1, R_2} = \{X \in W^{1,2}_0(\Omega, \mathbb{R}^3) \mid \|g + X\|_\infty \leq R_1, \|\text{grad}(X)\|_p \leq R_2\} \quad (2.4)$$

with $R_1 < 1$, then $T(C_{R_1, R_2})$ is precompact.

**Proof.** For $X = T(\overline{X})$, $Y = T(\overline{Y})$, as $X = Y$ on $\partial \Omega$ we obtain that

$$\|\text{grad}(X_k - Y_k)\|_p \leq c \|\Delta(X_k - Y_k)\|_p$$

$$= c \|\psi_k(\overline{X} + g, (\overline{X} + g)_u, (\overline{X} + g)_v) - \psi_k(\overline{Y} + g, (\overline{Y} + g)_u, (\overline{Y} + g)_v)\|_p \quad (2.5)$$
A boundary value problem in the hyperbolic space and the continuity of $T$ follows. On the other hand, if $X \in C_{R_1, R_2}$, then
\[
\|\nabla (X_k)\|_2^2 \leq c \left\| \nabla (X + g, (X + g)_u, (X + g)_v) \right\|_p \leq \overline{c} (R_2 + \|\nabla (g)\|_2^2)^2
\] (2.6)
for some constant $\overline{c}$ and the result follows. □

**Remark 2.3.** By definition of $\psi_k$, it is clear that $c \leq c_1/(1 - R_1)$ for some constant $c_1$.

**Proof of Theorem 2.1.** With the notation of the previous lemma, by Schauder fixed point theorem, it suffices to see that $C_{R_1, R_2}$ is $T$-invariant for some $R_1, R_2$. From the previous computations, we have
\[
\|\nabla (X)\|_2^2 \leq \frac{c_1}{1 - R_1} (R_2 + \|\nabla (g)\|_2^2)^2.
\] (2.7)
Moreover, by Poincaré’s inequality
\[
\|g + X\|_\infty \leq \|g\|_\infty + c_0 \|\nabla (X)\|_2^2.
\] (2.8)
Thus, a sufficient condition for obtaining $T(C_{R_1, R_2}) \subset C_{R_1, R_2}$ is that
\[
\frac{c_1}{1 - R_1} (R_2 + \|\nabla (g)\|_2^2)^2 \leq R_2, \quad \|g\|_\infty + c_0 R_2 \leq R_1.
\] (2.9)
For $R$ small enough we may fix $R_1 = \|g\|_\infty + c_0 R < 1$, and then the theorem is proved if
\[
c_1 (R + \|\nabla (g)\|_2^2)^2 \leq R (1 - \|g\|_\infty - c_0 R)
\] (2.10)
for some $R > 0$. As last condition is equivalent to our hypothesis, the result holds. □

### 3. Regularity of the solutions of problem (1.2)

In this section, we state the following regularity result.

**Theorem 3.1.** Let $X \in W^{1,2p}(\Omega, \mathbb{R}^3)$ be a solution of (1.2). Then
(a) if $g \in W^{2,q}(\Omega, \mathbb{R}^3)$ for some $q > 1$, then $X \in W^{2,q}(\Omega, \mathbb{R}^3)$,
(b) if $\partial \Omega \in C^{k+2,\alpha}$, $H \in C^{k,\alpha}(\mathbb{R}^3, \mathbb{R})$, $g \in C^{k+2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$ for some $0 < \alpha < 1$, $\overline{\Omega}$, then $X \in C^{k+2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$.

**Proof.** (a) Let $\Delta X = f \in L^p$. If $p \geq q$, let $Z$ be the unique solution in $W^{2,q}$ of the problem $\Delta Z = f$, $Z|_{\partial \Omega} = g$. As $\Delta (X - Z) = 0$ and $X = Z$ on $\partial \Omega$ the result follows.
On the other hand, if $p < q$, we obtain in the same way that $X \in W^{2,p}$. For $2 \leq p < q$ this implies that $X \in W^{1,2q}$ and the result follows.

Now we consider the case $p < 2, q$. Let $p_0 = p$ and define
\[
p_{n+1} = \begin{cases} 
p_n^*/2 & \text{if } p_n < 2, q, \\
p & \text{otherwise},
\end{cases}
\] (3.1)
where $p_n^*$ is the critical Sobolev exponent $2p_n/(2 - p_n)$. Then $\{p_n\}$ is bounded, and $X \in W^{1,2p_n}$ for every $n$. If $p_n < 2$, $q$ for every $n$, then $p_n$ is increasing and taking $r = \lim_{n \to \infty} p_n$, we obtain that $r/(2 - r) = r$, a contradiction. Hence, $p_n \geq q$ or $q > p_n \geq 2$ for some $n$, and the proof is complete.

(b) Case $k = 0$: by part (a), choosing $q > 2/(1 - \alpha)$ we obtain that $X \in W^{2,q} \hookrightarrow C^{1,\alpha}(\overline{\Omega}, \mathbb{R}^3)$. Then $\Delta X = f \in C^\alpha(\overline{\Omega}, \mathbb{R}^3)$. By [4, Theorem 6.14] the equation $\Delta Z = f$ in $\Omega$, $Z = g$ in $\partial \Omega$ is uniquely solvable in $C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^3)$, and the result follows from the uniqueness in [4, Theorem 9.15].

The general case is now immediate, from [4, Theorem 6.19].

\[ \square \]

Acknowledgement

The authors thank Professor Jean-Pierre Gossez and the referee for their fruitful remarks.

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