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Research Article

The Analysis of Contour Integrals

Tanfer Tanriverdi[1] and John Bryce Mcleod[2]

[1] *Department of Mathematics, Harran University, Osmanbey Campus, Sanlurfa 63100, Turkey*
[2] *Department of Mathematics, University of Pittsburgh, 301 Thackeray Hall, Pittsburgh, PA 15260, USA*

Correspondence should be addressed to Tanfer Tanriverdi, ttanriverdi@harran.edu.tr

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For any n, the contour integral $y = \cosh^{n+1} x \oint_C (\cosh(zs)/(\sinh z - \sinh x)^{n+1}) dz$, $s^2 = -\lambda$, is associated with differential equation $d^2y(x)/dx^2 + (\lambda + n(n+1)/\cosh^2 x)y(x) = 0$. Explicit solutions for $n = 1$ are obtained. For $n = 1$, eigenvalues, eigenfunctions, spectral function, and eigenfunction expansions are explored. This differential equation which does have solution in terms of the trigonometric functions does not seem to have been explored and it is also one of the purposes of this paper to put it on record.

1. Introduction

When one considers eigenfunction expansions associated with second-order ordinary differential equations, as Titchmarsh does in his book [1], one is concerned with solutions of the equation

$$-\frac{d^2y(x)}{dx^2} + q(x)y(x) = \lambda y(x) \tag{1.1}$$

along with certain boundary conditions, and one tends to say that the only case in which one can solve this equation explicitly in elementary terms for all λ is the case $q(x) = 0$, when the solutions are of course trigonometric functions.

Now in fact this is not true, and there is in particular one problem which does not seem to have been explored, and it is the purpose of this paper to put it on record. Here is the problem:

$$\frac{d^2y(x)}{dx^2} + \left(\lambda + n(n+1)\operatorname{sech}^2 x\right)y(x) = 0, \tag{1.2}$$

which can be solved explicitly in elementary terms when n is integral. The explicit solution was known to Kamke [2], but Kamke does not anyway explore the consequences for eigenfunction expansions nor does Titchmarsh discuss this problem, although he does discuss problems close to it, for example,

$$\frac{d^2y(x)}{dx^2} + \left(\lambda + \left(\nu^2 - \frac{1}{4}\right)\sec^2 x\right)y(x) = 0, \tag{1.3}$$

on $(-\pi/2, \pi/2)$, which leads, when $\nu = n$, to an expansion in series involving associated Legendre functions.

It is perhaps worth remarking how our interest in this problem arises. In [3] there is the question of travelling waves and steady solutions for a discrete reaction-diffusion equation of the type

$$u_n' = u_{n+1} - 2u_n + u_{n-1} + f(u_n), \tag{1.4}$$

where the function f is "bistable". That is, there exist three numbers $U_1, U_2, U_3, U_1 < U_2 < U_3$, such that

$$f(U_1) = f(U_2) = f(U_3) = 0 \tag{1.5}$$

with $f < 0$ in (U_1, U_2) and $f > 0$ in (U_2, U_3). A prototypical f would be

$$f(u) = -A\sin(u) + F, \tag{1.6}$$

where A and F are positive constants and $F < A$, so that

$$U_1 = \sin^{-1}\left(\frac{F}{A}\right), \qquad U_2 = \pi + \sin^{-1}\left(\frac{F}{A}\right), \qquad U_3 = 2\pi + \sin^{-1}\left(\frac{F}{A}\right). \tag{1.7}$$

Such equations arise in a number of different applications, for example, in dislocation theory where u_n is the displacement of the nth atom in some material, or in neurobiology where u_n is typically the electric potential of the nth nerve cell, and in both these applications the interest is in monotonic solutions u_n with $u_n \to U_1$ as $n \to -\infty$, $u_n \to U_3$ as $n \to \infty$ [3–6].

The basic question is whether there exist such solutions with the form of a travelling wave, $u(n - ct)$, $c \neq 0$, or of a steady solution or standing wave, where $c = 0$, and there is an important distinction between these two cases. For a travelling wave, $c \neq 0$, u_n is clearly a function of the continuous variable t, and indeed because of (1.4), a differentiable function of t. This leads to the difference-differential equation

$$u_t(x,t) = u(x+1,t) - 2u(x,t) + u(x-1,t) + f\left(u(x,t)\right). \tag{1.8}$$

If, however, $c = 0$, then, as in [3], we have to study the purely difference equation

$$u(x+1,t) - 2u(x,t) + u(x-1,t) + f\left(u(x,t)\right) = 0, \tag{1.9}$$

and the solutions may be discontinuous since there is nothing that now connects values of $u(x)$ with values of $u(x+\delta)$ for $|\delta| < 1$. It is best therefore to think of the solution of (1.9) as a number of (monotonic) sequences u_n^α indexed by α, each satisfying

$$u_{n+1} - 2u_n + u_{n-1} + f(u_n) = 0. \tag{1.10}$$

The simplest case would be that there is just one such sequence (modulo the translation $n \to n + k$, k integral), but it is possible that there may be a finite number, or even a partial or total continuum.

In view of applications, where the distance between atoms or nerve cells is small, it is more natural to think of (1.8) in the form

$$u_t(x,t) = u(x+\epsilon,t) - 2u(x,t) + u(x-\epsilon,t) + f\big(u(x,t)\big), \tag{1.11}$$

where ϵ is small and represents the distance between atoms or nerve cells. A tempting approximation is then

$$u_t(x,t) \sim \epsilon^2 u_{xx}(x,t) + f\big(u(x,t)\big), \tag{1.12}$$

and in order to make sense of the scaling, in [3] the authors introduced a factor ϵ^2 in front of f. This therefore leads to a comparison between the solutions of

$$u_n' = u_{n+1} - 2u_n + u_{n-1} + \epsilon^2 f(u_n) = 0, \tag{1.13}$$

$$u_t(x,t) = u_{xx}(x,t) + f(u). \tag{1.14}$$

For the continuous diffusion problem, the answer is both simple and well known [4, 5].

Given a function f that is bistable, there is just one possible wave-speed c, and this value of c is 0, that is, there is a steady solution if and only if

$$\int_{U_1}^{U_3} f(u)\,du = 0. \tag{1.15}$$

(The proof is a simple phase plane argument, and $c = 0$ implies (1.15) follows by multiplying (1.14) by u' and integrating.)

The solution in the discrete case is however different, as discussed in [6]. There may continue to be steady solutions where (1.15) no longer holds. Consider specifically the case (1.6), so that

$$u_n' = u_{n+1} - 2u_n + u_{n-1} - \epsilon^2 \sin u_n + F. \tag{1.16}$$

The case corresponding to (1.15) is $F = 0$, but the authors, in [3], have shown that for F sufficiently small, say $|F| < F_{\text{crit}}$, there exist precisely two steady solutions of (1.16), and F_{crit}, which of course depends on ϵ, can be evaluated for small ϵ. Specifically,

$$F_{\text{crit}} \sim B e^{-\pi^2/\epsilon}, \tag{1.17}$$

where the constant $B = 64\pi \int_0^{\pi} (\sin^2(s)/s)\,ds$ is given. For $|F| > F_{\text{crit}}$, the solutions move and the equation has travelling wave solutions instead of steady solutions.

In order to prove results such as (1.17), one has to regard (1.16) as a singular perturbation of the steady continuous-diffusion equation

$$u_{xx} - \sin u = 0, \tag{1.18}$$

for which the solution (satisfying $u(-\infty) = 0, u(\infty) = 2\pi$) is $U = 4\tan^{-1}e^x$. If we linearize (1.18) about U, we obtain

$$\phi'' - \cos(U)\phi = 0. \tag{1.19}$$

But multiplying (1.18) by U' and integrating lead to $(1/2)(U')^2 = 1 - \cos U$, so that since $U' = 2\text{sech}(x)$, we have $\cos U = 1 - 2\text{sech}^2(x)$. The linearization (1.19) thus becomes $\phi'' + (-1 + 2\text{sech}^2 x)\phi = 0$, which is of course (1.2) with $\lambda = -1$ and $n = 1$. Thus the selfadjoint operator T given (in $L^2(-\infty,\infty)$) by $T\phi = -\phi'' - 2\text{sech}^2(x)\phi$ has an eigenvalue at -1, with eigenfunction U' (differentiation of (1.18) shows that U' satisfies (1.19)). This fact, together with the additional fact that the spectrum of T is continuous above 0 (since $\text{sech}^2(x) \in L(0,\infty)$), is highly relevant to the work in [3] and led to our interest more generally in the spectral problem (1.2).

The explicit solution for any n using contour integrals different from what Kamke did is known to [7]. For more information on this problem one can see [7, 8].

2. Preliminaries

We want to know expansion of an arbitrary function $f(x)$ in terms of eigenfunctions. So one needs to know the following. Let $\theta(x,\lambda)$ and $\phi(x,\lambda)$ be the solutions of (1.1) such that

$$\phi(0,\lambda) = \sin\alpha, \qquad \phi'(0,\lambda) = -\cos\alpha, \qquad \theta(0,\lambda) = \cos\alpha, \qquad \theta'(0,\lambda) = \sin\alpha, \tag{2.1}$$

where α is real. $W_x(\phi,\theta) = W_0(\phi,\theta) = 1$. The general solution of (1.1) is of the form

$$\psi(x,\lambda) = \theta(x,\lambda) + m(\lambda)\phi(x,\lambda) \in L^2(0,\infty). \tag{2.2}$$

The spectrum is defined by means of the function

$$k(\lambda) = \lim_{\delta\to 0}\int_0^\lambda -\Im\{m(u+i\delta)\}du, \tag{2.3}$$

which exists for all real λ and $k(\lambda)$ is a nondecreasing function. The expansion of a function $f(x)$ in terms of the spectral function depends on the following lemmas taken from [1].

Lemma 2.1. *Without detailing, let the interval be* $(0,\infty)$*:*

$$f(x) = \frac{1}{\pi}\int_0^\infty \phi(x,\lambda)dk(\lambda)\int_0^\infty \phi(t,\lambda)f(t)dt. \tag{2.4}$$

If $m(\lambda)$ *has poles, then*

$$f(x) = \sum_{N=0}^{\infty}\phi_N(t)\int_0^\infty \phi_N(t,\lambda)f(t)dt. \tag{2.5}$$

Lemma 2.2. *Without detailing, let the interval be* $(-\infty,\infty)$*. If* $q(x)$ *is an even function, then* $m_1(\lambda) = -m_2(\lambda)$*. So the expansion formula is*

$$f(x) = \frac{1}{\pi}\int_{-\infty}^\infty \theta(x,\lambda)d\xi\int_{-\infty}^\infty \theta(y,\lambda)f(y)dy + \frac{1}{\pi}\int_{-\infty}^\infty \phi(x,\lambda)d\zeta\int_{-\infty}^\infty \phi(y,\lambda)f(y)dy, \tag{2.6}$$

where

$$\xi(\lambda)=\lim_{\delta\to 0}\int_0^{\lambda}-\Im\left\{\frac{1}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du,\qquad \xi'(\lambda)=\Im\left\{\frac{1}{2m_2(\lambda)}\right\},$$
$$\zeta(\lambda)=\lim_{\delta\to 0}\int_0^{\lambda}-\Im\left\{\frac{m_1(u+i\delta)m_2(u+i\delta)}{m_1(u+i\delta)-m_2(u+i\delta)}\right\}du,\qquad \zeta'(\lambda)=-\frac{1}{2}\Im\{m_2(\lambda)\}. \tag{2.7}$$

3. Main results

We are now dealing with (1.2) in the case where n is integral. Without loss of generality, we may suppose $n\geq 0$, but since $n=0$ reduces (1.2) to the simple trigonometric case, we are in fact interested only in $n>0$. We first prove that a solution is given by

$$y=\cosh^{n+1}x\oint_C\frac{\cosh(zs)}{(\sinh z-\sinh x)^{n+1}}dz,\qquad s^2=-\lambda, \tag{3.1}$$

where the contour C is taken round the point $z=x$ and no other zero of $\sinh z-\sinh x$. This is slight variant of a form which Titchmarsh uses in his discussion of (1.3). The proof below will show (3.1), being continuous at least formally, to be a solution of (1.2) where n is not an integer, but the difficulty then is to choose a suitable contour, since the integrand has a branch point at $z=x$.

Remark 3.1. We also remark that it is obvious that we can express the solution (3.1) equivalently ignoring some multiplicative constants as

$$y(x)=\cosh^{n+1}x\frac{d^n}{(\cosh x dx)^n}\oint_C\frac{\cosh(zs)}{\sinh z-\sinh x}dz. \tag{3.2}$$

Theorem 3.2. *The contour integral* (3.1) *satisfies the differential equation* (1.2).

Proof. We see that

$$y'=(n+1)\tanh(x)y+(n+1)\cosh^{n+2}x\oint_C\frac{\cosh(zs)}{(\sinh z-\sinh x)^{n+2}}dz, \tag{3.3}$$

$$y''+n(n+1)\operatorname{sech}^2(x)y=(n+1)\cosh^{n+1}x\oint_C\frac{\cosh(zs)\{(n+1)\sinh^2 z+\sinh z\sinh x+n+2\}}{(\sinh z-\sinh x)^{n+3}}dz. \tag{3.4}$$

Integrating (3.1) by parts,

$$y(x)=\frac{n+1}{s^2}\oint_C\frac{\cosh(zs)\{(n+1)\sinh^2 z+\sinh z\sinh x+n+2\}}{(\sinh z-\sinh x)^{n+3}}dz, \tag{3.5}$$

so that

$$\lambda y=-(n+1)\cosh^{n+1}x\oint_C\frac{\cosh(zs)\{(n+1)\sinh^2 z+\sinh z\sinh x+n+2\}}{(\sinh z-\sinh x)^{n+3}}dz. \tag{3.6}$$

Comparing (3.4) and (3.6), we see that $y''+n(n+1)\operatorname{sech}^2(x)y=-\lambda y$, so that $y(x)$ satisfies (1.2), as required. □

Remark 3.3. We now point out that the factor $\cosh(zs)$ played little part in the argument. Certainly, the argument would have washed equally well if we had replaced $\cosh(zs)$ by $\sinh(zs)$:

$$y_2(x) = \cosh^{n+1} x \oint_C \frac{\sinh(zs)}{(\sinh z - \sinh x)^{n+1}} dz. \tag{3.7}$$

Theorem 3.4. *The contour integral* (3.7) *satisfies the differential equation* (1.2).

Proof. Proof is the same as the above theorem. So we omit it. □

Remark 3.5. Furthermore, once the integrands have poles at $z = x$, the solution can be evaluated by calculating the relevant residues. For example, in the trivial case $n = 0$, when we should recover the trigonometric functions, the residues of

$$\frac{\cosh(zs)}{\sinh z - \sinh x} \tag{3.8}$$

are

$$\frac{\cosh(xs)}{\cosh x}, \tag{3.9}$$

so that the solution (3.7) becomes multiples of $\cos(x\sqrt{\lambda})$ (similarly $\sin(x\sqrt{\lambda})$), as we expect.

We can generalize Theorems (3.2) and (3.4) by defining the following operator:

$$Tf(x) = \cosh^{n+1} x \oint_C \frac{f(z)}{(\sinh z - \sinh x)^{n+1}} dz, \tag{3.10}$$

where f is a differentiable function as long as one can pick up residue.

Corollary 3.6. *If $f(z) = \cosh(zs)(\sinh(zs))$, then we obtain Theorems (3.2) and (3.4). The operator T is also linear.*

4. The explicit solution given by residues for $n = 1$

We now require the residues of

$$\frac{\cosh(zs)}{(\sinh z - \sinh x)^2}, \qquad \frac{\sinh(zs)}{(\sinh z - \sinh x)^2}. \tag{4.1}$$

Since

$$\frac{\cosh(zs)}{(\sinh z - \sinh x)^2} = \frac{\{\cosh(xs) + (z-x)s\sinh(xs) + \cdots\}\{1 - (z-x)\tanh(x) + \cdots\}}{(z-x)^2\cosh^2(x)}, \tag{4.2}$$

we see that the residue at $z = x$ is

$$\frac{s\sinh(xs) - \tanh(x)\cosh(xs)}{\cosh^2(x)}, \tag{4.3}$$

so that one solution is

$$y_1 = \sqrt{\lambda}\sin\left(x\sqrt{\lambda}\right) + \tanh(x)\cos\left(x\sqrt{\lambda}\right). \tag{4.4}$$

By examining the residue of the second equation of (4.1), we see that a second solution is

$$y_2 = \sqrt{\lambda}\cos\left(x\sqrt{\lambda}\right) - \tanh(x)\sin\left(x\sqrt{\lambda}\right). \tag{4.5}$$

Remark 4.1. The solution can also be obtained from (3.2). For we have already seen, from our brief discussion of the case $n = 0$, that the integral in (3.2) is just a multiple of $\cos(x\sqrt{\lambda})/\cosh x$ (or of $\sin(x\sqrt{\lambda})/\cosh x$ if we replace $\cosh(zs)$ by $\sinh(zs)$) hence in the first case, (3.2) gives a multiple of

$$\cosh(x)\frac{d}{dx}\left(\frac{\cos(x\sqrt{\lambda})}{\cosh(x)}\right) = -\sqrt{\lambda}\sin\left(x\sqrt{\lambda}\right) - \tanh(x)\cos(x\lambda), \tag{4.6}$$

in accordance with (4.4).

Remark 4.2. Wronskian $W(y_1(x), y_2(x)) = -\sqrt{\lambda}(\lambda + 1)$. We now have two linearly independent solutions.

Remark 4.3. The general solution is $y(x) = c_1 y_1(x) + c_2 y_2(x)$.

Lemma 4.4. *$y_1(x)$ is an odd function but $y_2(x)$ is an even function.*

Proof.

$$y_1(-x) = -y_1(x), \qquad y_2(-x) = y_2(x). \tag{4.7}$$

□

5. Eigenvalues and eigenfunctions for $n = 1$ when $y(0) = y(b) = 0$

Theorem 5.1. *Eigenvalues with associated boundary conditions $y(0) = y(b) = 0$ are the zeros of*

$$\sqrt{\lambda}\tan\left(b\sqrt{\lambda}\right) + \tanh(b) = 0; \tag{5.1}$$

furthermore, one and only one eigenvalue lies in the interval

$$\left(k - \frac{1}{2}\right)\pi < b\sqrt{\lambda} < \left(k + \frac{1}{2}\right)\pi \tag{5.2}$$

for every integral $k \neq 0$.

Proof. One can see from Remark 4.3 that if $y(0) = 0$, then $c_2 = 0$. So that $y(b) = 0$ implies

$$\sqrt{\lambda}\sin\left(b\sqrt{\lambda}\right) + \tanh(b)\cos\left(b\sqrt{\lambda}\right) = 0, \qquad \sqrt{\lambda}\tan\left(b\sqrt{\lambda}\right) = -\tanh(b). \tag{5.3}$$

One can see immediately that the eigenvalues belong to the interval (5.2).

To prove the second part we use the following strategy. Multiply (5.1) by b and set $x = b\sqrt{\lambda}$, then denote $h(x) = x\tan(x) + b\tanh(b)$. So $h'(x) = \sec^2(x)\{(1/2)\sin(2x) + x\}$. Notice that $h(x)$ is an even and does not intersect x-axis, where $-1/2 < x < 1/2$. If $x < -1/2$, then $h'(x) < 0$ and if $x > 1/2$, then $h'(x) > 0$. So the monotonicity of $h(x)$ implies that $h(x)$ has only one zero belonging to the interval (5.2) for every integral $k \neq 0$. □

Remark 5.2. So y_k is the eigenfunction in the form of (4.4):

$$y_k = \sqrt{\lambda_k}\sin\left(x\sqrt{\lambda_k}\right) + \tanh(x)\cos\left(x\sqrt{\lambda_k}\right). \tag{5.4}$$

Corollary 5.3. *One can orthonormalize the eigenfunctions.*

Proof.

$$\int_0^b y_k^2 dx = \frac{-4\sqrt{\lambda_k}\tanh(b)\cos^2(b\sqrt{\lambda_k}) + 2\sqrt{\lambda_k}(\lambda_k+1)b - (\lambda_k-1)\sin(2b\sqrt{\lambda_k})}{4\sqrt{\lambda_k}}. \tag{5.5}$$

The orthonormalized eigenfunctions denoted by $\Phi(x,\lambda_k)$, $\Phi(x,\lambda_k) = (\sqrt{\lambda_k}\sin(x\sqrt{\lambda_k}) + \tanh(x)\cos(x\sqrt{\lambda_k}))/\sqrt{\int_0^b y_k^2 dx}$. □

Remark 5.4. An arbitrary function $f(x)$ in terms of eigenfunctions follows:

$$f(x) = \sum_{k=0}^{\infty} C_k \Phi(x,\lambda_k), \quad \text{where } C_k = \int_0^b f(x)\Phi(x,\lambda_k)dx. \tag{5.6}$$

6. Eigenvalues and eigenfunctions for $n = 1$ when $y'(0) = y'(b) = 0$

Theorem 6.1. *The eigenvalues with $y'(0) = y'(b) = 0$ are the zeros of* (6.1)*; furthermore, there exists one and only one eigenvalue lying in the interval* (5.2) *for every integral $k \neq 0$.*

Proof. One can see from Remark 4.3 that if $y'(0) = 0$, then either $c_1 = 0$ or $\lambda = -1$. If $\lambda = -1$, then the associated eigenfunction is zero. So this is useless. Hence, $c_1 = 0$. $y'(b) = 0$ implies

$$\lambda \tan\left(b\sqrt{\lambda}\right) + \operatorname{sech}^2(b)\tan\left(b\sqrt{\lambda}\right) + \sqrt{\lambda}\tanh(b) = 0. \tag{6.1}$$

So it is obvious that the zeros (eigenvalues) belong to the interval (5.2).

Set $x = b\sqrt{\lambda}$. ($b^2\lambda = x^2\lambda = x^2/b^2$). Equation (6.1) is denoted by $h(x)$:

$$h(x) = \left(\frac{x^2}{b^2} + \operatorname{sech}^2(b)\right)\tan(x) + \frac{x}{b}\tanh(b). \tag{6.2}$$

It is enough to show that $h(x)$ is monotonic:

$$h'(x) = \frac{x^2 + b^2\operatorname{sech}^2(b) + x\sin(2x)}{b^2\cos^2(x)} + \frac{1}{b}\tanh(b). \tag{6.3}$$

We see that $h'(x) > 0$ everywhere. We therefore conclude that $h(x)$ is monotonic. □

Remark 6.2. The associated eigenfunctions are

$$y_k = \sqrt{\lambda_k}\cos\left(x\sqrt{\lambda_k}\right) - \tanh(x)\sin\left(x\sqrt{\lambda_k}\right). \tag{6.4}$$

Corollary 6.3. *One can orthonormalize the eigenfunctions.*

Proof.

$$\int_0^b y_k^2 dx = \frac{-4\sqrt{\lambda_k}\tanh(b)\sin^2\left(b\sqrt{\lambda_k}\right) + 2\sqrt{\lambda_k}(\lambda_k+1)b + (\lambda_k-1)\sin(2b\sqrt{\lambda_k})}{4\sqrt{\lambda_k}}. \tag{6.5}$$

The orthonormalized eigenfunctions denoted by $\Psi(x,\lambda_k)$, $\Psi(x,\lambda_k) = (\sqrt{\lambda_k}\cos(x\sqrt{\lambda_k}) - \tanh(x)\sin(x\sqrt{\lambda_k}))/\sqrt{\int_0^b y_k^2 dx}$. □

Remark 6.4. Therefore, an arbitrary $f(x)$ in terms of orthonormalized eigenfunctions is

$$f(x) = \sum_{k=0}^{\infty} C_k \Psi(x,\lambda_k), \qquad C_k = \int_0^b f(x)\Psi(x,\lambda_k)dx. \tag{6.6}$$

7. Spectral function $m(\lambda)$ over $(0,\infty)$ and expansion

Now let $\theta(x,\lambda)$ and $\phi(x,\lambda)$ be the solutions of

$$\frac{d^2y}{dx^2}+\{\lambda+2\mathrm{sech}^2(x)\}y=0, \tag{7.1}$$

which satisfy (2.1); so that

$$\phi(x,\lambda)=-\frac{\cos\alpha\{\sqrt{\lambda}\sin(x\sqrt{\lambda})+\tanh\,x\cos(x\sqrt{\lambda})\}}{\lambda+1}+\frac{\sin\alpha\{\sqrt{\lambda}\cos(x\sqrt{\lambda})-\tanh\,x\sin(x\sqrt{\lambda})\}}{\sqrt{\lambda}}, \tag{7.2}$$

$$\theta(x,\lambda)=\frac{\sin\alpha\{\sqrt{\lambda}\sin(x\sqrt{\lambda})+\tanh\,x\cos(x\sqrt{\lambda})\}}{\lambda+1}+\frac{\cos\alpha\{\sqrt{\lambda}\cos(x\sqrt{\lambda})-\tanh\,x\sin(x\sqrt{\lambda})\}}{\sqrt{\lambda}}. \tag{7.3}$$

Now we need to find $m(\lambda)$. This suggests that

$$\psi(x,\lambda)=\theta(x,\lambda)+m(\lambda)\phi(x,\lambda)\in L^2(0,\infty). \tag{7.4}$$

To get demanded result, one needs to find the asymptotics of $\theta(x,\lambda)$ and $\phi(x,\lambda)$ as $x\to\infty$ and $\Im\sqrt{\lambda}>0$:

$$\begin{aligned}\phi(x,\lambda)&\sim\frac{\{-\sqrt{\lambda}\cos\alpha+\sqrt{\lambda}(\lambda+1)\sin\alpha+i(-\lambda\cos\alpha-(\lambda+1)\sin\alpha)\}e^{-ix\sqrt{\lambda}}}{2\sqrt{\lambda}(\lambda+1)}=M_1(\lambda)e^{-ix\sqrt{\lambda}},\\ \theta(x,\lambda)&\sim\frac{\{\sqrt{\lambda}\sin\alpha+\sqrt{\lambda}(\lambda+1)\cos\alpha+i(\lambda\sin\alpha-(\lambda+1)\cos\alpha)\}e^{-ix\sqrt{\lambda}}}{2\sqrt{\lambda}(\lambda+1)}=M(\lambda)e^{-ix\sqrt{\lambda}}.\end{aligned} \tag{7.5}$$

Finally, we must arrange the linear combination so that the terms $e^{-ix\sqrt{\lambda}}$ cancel. That is, $M(\lambda)+m(\lambda)M_1(\lambda)=0$. Hence,

$$\begin{aligned}m(\lambda)&=\frac{-i\sqrt{\lambda}(\lambda+1)-(\lambda^2+\lambda+1)\sin\alpha\cos\alpha}{\lambda\cos^2\alpha+(\lambda+1)^2\sin^2\alpha},\\ \Im m(\lambda)&=\begin{cases}\dfrac{-\sqrt{\lambda}\,(\lambda+1)}{\lambda\cos^2\alpha+(\lambda+1)^2\sin^2\alpha} & \text{when }\lambda>0,\\ 0 & \text{when }\lambda<0.\end{cases}\end{aligned} \tag{7.6}$$

We see that the spectrum is continuous for $\lambda>0$. But we have a point spectrum for $\lambda<0$. So that an arbitrary $f(x)$ is a linear combination of integrand and series. The spectral function is calculated from (2.3). Hence,

$$dk(\lambda)=\begin{cases}\dfrac{\sqrt{\lambda}(\lambda+1)}{\lambda\cos^2\alpha+(\lambda+1)^2\sin^2\alpha} & \text{when }\lambda>0,\\ 0 & \text{when }\lambda<0.\end{cases} \tag{7.7}$$

In particular, if $\alpha = 0$, then

$$dk(\lambda) = \begin{cases} \dfrac{\lambda+1}{\sqrt{\lambda}} & \text{when } \lambda > 0, \\ 0 & \text{when } \lambda < 0. \end{cases} \tag{7.8}$$

So that in this case there is no eigenvalue for $\lambda < 0$. From Lemma 2.1, one can see the expansion of

$$f(x) = \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}\sin(x\sqrt{\lambda}) + \tanh\, x\cos(x\sqrt{\lambda})}{\sqrt{\lambda}} d\lambda \int_0^\infty \frac{\sqrt{\lambda}\sin(y\sqrt{\lambda}) + \tanh\, y\cos(y\sqrt{\lambda})}{\lambda+1} f(y)dy. \tag{7.9}$$

Similarly, if $\alpha = \pi/2$, then

$$\Im m(\lambda) = \begin{cases} -\dfrac{\sqrt{\lambda}}{\lambda+1} & \text{when } \lambda > 0, \\ 0 & \text{when } \lambda < 0.. \end{cases} \tag{7.10}$$

$$\tag{7.11}$$

Hence, there exits only one eigenvalue at $\lambda = -1$ and the corresponding eigenfunction $\phi(x, -1)$ where $\phi(x, \lambda)$ is (7.2). So the spectrum calculated by (2.3) is

$$dk(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}}{\lambda+1} & \text{when } \lambda > 0, \\ 0 & \text{when } \lambda < 0.. \end{cases} \tag{7.12}$$

Therefore, from Lemma 2.1, an expansion of function $f(x)$ in terms of eigenfunctions and spectral function follows:

$$\begin{aligned} f(x) = {} & \frac{1}{\pi}\int_0^\infty \frac{\sqrt{\lambda}\cos(x\sqrt{\lambda}) - \tanh\, x\sin(x\sqrt{\lambda})}{\lambda+1} d\lambda \\ & \times \int_0^\infty \frac{\sqrt{\lambda}\cos(y\sqrt{\lambda}) - \tanh\, y\sin(y\sqrt{\lambda})}{\sqrt{\lambda}} f(y)dy + c_1 \operatorname{sech}(x), \end{aligned} \tag{7.13}$$

where c_1 is a constant. So we have proved the following theorem regarding the nature of $m(\lambda)$.

Theorem 7.1. *If $\alpha = 0$, there is no eigenvalue where $\lambda < 0$. If $\alpha = \pi/2$, then there exists only one eigenvalue at $\lambda = -1$ and the corresponding eigenfunction is $\phi(x, -1) = \operatorname{sech}(x)$.*

Finally, one can ask what is the range of α in the case of $\lambda < 0$ and $n = 1$.

Theorem 7.2. *If $0 < \alpha < \pi/2$, then there are precisely two eigenvalues except at $\alpha = \pi/4$. Hence there are two eigenfunctions, namely, $\phi(x, \lambda_1)$ and $\phi(x, \lambda_2)$.*

Proof. We check the zeros of both the numerator and denominator of (7.6). After working out the algebra, we see that the zeros of the numerator and the denominator are

$$\mu_1 = \frac{-(2+\tan^2\alpha)+\tan\alpha\sqrt{4+\tan^2\alpha+1}}{2}, \qquad \mu_2 = \frac{-(2+\tan^2\alpha)-\tan\alpha\sqrt{4+\tan^2\alpha+1}}{2},$$
$$\lambda_1 = \frac{-(2\tan^2\alpha+1)+\sqrt{4\tan^2\alpha+1}}{2\tan^2\alpha}, \qquad \lambda_2 = \frac{-(2\tan^2\alpha+1)-\sqrt{4\tan^2\alpha+1}}{2\tan^2\alpha}. \tag{7.14}$$

If $m(\lambda)$ has poles, then they are the eigenvalues. If so, the eigenfunctions are $\phi(x,\lambda_1)$ and $\phi(x,\lambda_2)$. Finally, there is one thing to be proved in this case $\alpha=\pi/4$. To do this, we expand the zeros around $\alpha=\pi/4$ and at the end $\alpha\to\pi/4$:

$$\lambda_1 = \frac{-3+\sqrt{5}}{2} + \frac{20-12\sqrt{5}}{10}\left(\alpha-\frac{\pi}{4}\right) + \frac{-400+352\sqrt{5}}{100}\left(\alpha-\frac{\pi}{4}\right)^2 + O\left\{\left(\alpha-\frac{\pi}{4}\right)^3\right\},$$
$$\lambda_2 = \frac{-3-\sqrt{5}}{2} + \frac{20+12\sqrt{5}}{10}\left(\alpha-\frac{\pi}{4}\right) + \frac{-400-352\sqrt{5}}{100}\left(\alpha-\frac{\pi}{4}\right)^2 + O\left\{\left(\alpha-\frac{\pi}{4}\right)^3\right\}. \tag{7.15}$$

Similarly,

$$\mu_1 = \frac{-3+\sqrt{6}}{2} + \frac{7\sqrt{6}-12}{6}\left(\alpha-\frac{\pi}{4}\right) + \frac{-288+118\sqrt{6}}{36}\left(\alpha-\frac{\pi}{4}\right)^2 + O\left\{\left(\alpha-\frac{\pi}{4}\right)^3\right\},$$
$$\mu_2 = \frac{-3-\sqrt{6}}{2} + \frac{-7\sqrt{6}-12}{6}\left(\alpha-\frac{\pi}{4}\right) + \frac{-288-118\sqrt{6}}{36}\left(\alpha-\frac{\pi}{4}\right)^2 + O\left\{\left(\alpha-\frac{\pi}{4}\right)^3\right\}. \tag{7.16}$$

That completes the proof. □

8. Expansion over $(-\infty,\infty)$

Now consider the interval $(-\infty,\infty)$ instead of $(0,\infty)$. From (2.7) the following can be calculated:

$$g\xi'(\lambda) = \begin{cases} \dfrac{\sqrt{\lambda}(\lambda+1)}{2\lambda\sin^2\alpha+2(\lambda+1)^2\cos^2\alpha} & \text{when } \lambda>0, \\ 0 & \text{when } \lambda<0, \end{cases}$$
$$\zeta'(\lambda) = \begin{cases} \dfrac{(\lambda+1)\sqrt{\lambda}}{2\lambda\cos^2\alpha+2(\lambda+1)^2\sin^2\alpha} & \text{when } \lambda>0, \\ 0 & \text{when } \lambda<0. \end{cases} \tag{8.1}$$

Now one can use (7.2), (7.3), and (8.1) gets the following expansion from Lemma (2.2):

$$\begin{aligned} f(x) &= \frac{1}{\pi}\left\{\int_0^{\infty}\frac{\sqrt{\lambda}\cos(x\sqrt{\lambda})-\tanh x\sin(x\sqrt{\lambda})}{2(\lambda+1)}d\lambda\int_{-\infty}^{\infty}\frac{\sqrt{\lambda}\cos(y\sqrt{\lambda})-\tanh y\sin(y\sqrt{\lambda})}{\sqrt{\lambda}}f(y)dy\right. \\ &\quad\left.+\int_0^{\infty}\frac{\sqrt{\lambda}\sin(x\sqrt{\lambda})+\tanh x\cos(x\sqrt{\lambda})}{2\sqrt{\lambda}}d\lambda\int_{-\infty}^{\infty}\frac{\sqrt{\lambda}\sin(y\sqrt{\lambda})+\tanh y\cos(y\sqrt{\lambda})}{\lambda+1}f(y)dy\right\}+c_1\operatorname{sech}(x), \end{aligned} \tag{8.2}$$

where c_1 is a constant.

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