

ZERO DISTRIBUTION OF SEQUENCES OF CLASSICAL ORTHOGONAL POLYNOMIALS

PLAMEN SIMEONOV

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We obtain the zero distribution of sequences of classical orthogonal polynomials associated with Jacobi, Laguerre, and Hermite weights. We show that the limit measure is the extremal measure associated with the corresponding weight.

1. Introduction

In this paper, we study the zero distribution of sequences of Jacobi, Laguerre, and Hermite polynomials. Our approach is based on identifying these orthogonal polynomials with certain Fekete polynomials defined below, and using monotonicity properties of the zeros of the polynomials.

Let $E \subset \mathbb{R}$ be a closed set that consists of finitely many intervals. Let $w : E \rightarrow [0, \infty)$ be a weight function such that $w(x) > 0$, $x \in \text{Int}(E)$, and $|x|w(x) \rightarrow 0$ as $|x| \rightarrow \infty$, $x \in E$, if E is unbounded. Consider the function

$$V_n(x_1, \dots, x_n) := \prod_{1 \leq i < j \leq n} [w(x_i)w(x_j)|x_j - x_i|], \quad (1.1)$$

$\{x_i\}_{i=1}^n \subset E$. It can be shown that V_n attains its maximum for some set $\mathcal{F}_n = \{x_i\}_{i=1}^n \subset E$ called n th weighted Fekete set or simply Fekete set.

We introduce the following notation: if μ is a measure, its logarithmic potential $U^\mu(z)$ is defined by

$$U^\mu(z) := \int \log \frac{1}{|z - t|} d\mu(t), \quad (1.2)$$

and if w is a weight as defined above, μ_w denotes the corresponding extremal measure [4], which is the unique measure that minimizes the weighted

logarithmic energy

$$I_w(\mu) := \iint \log \frac{1}{w(z)w(t)|z-t|} d\mu(z)d\mu(t) \tag{1.3}$$

over all probability Borel measures supported on E . The support of the measure μ_w will be denoted by S_w .

The asymptotic distribution of Fekete points is known (see [4, Chapter III, Theorem 1.3]).

THEOREM 1.1. *Let $\nu_{\mathcal{F}_n}$ be the discrete measure that has mass $1/n$ at each Fekete point $x_i \in \mathcal{F}_n$. Then,*

$$\nu_{\mathcal{F}_n} \xrightarrow{w^*} \mu_w, \quad n \rightarrow \infty, \tag{1.4}$$

that is, $\lim_{n \rightarrow \infty} \nu_{\mathcal{F}_n} = \mu_w$ in the weak-star topology of measures. Furthermore, if F_n is the n th degree monic polynomial with zero set \mathcal{F}_n ,

$$\lim_{n \rightarrow \infty} |F_n(z)|^{1/n} = \exp(-U^{\mu_w}(z)) \tag{1.5}$$

uniformly on compact subsets of $\mathbb{C} \setminus S_w$.

We will assume that $w(x) = 0$ when $x \in E \setminus \text{Int}(E)$ and x is finite. This condition implies that every Fekete set $\mathcal{F}_n \subset \text{Int}(E)$. Consequently, the partial derivatives of $\log(V_n^2)$ vanish at the Fekete points:

$$\begin{aligned} \frac{d}{dx_i} (\log V_n^2) &= 2(n-1) \frac{w'(x_i)}{w(x_i)} + 2 \sum_{k \neq i} \frac{1}{x_i - x_k} \\ &= 2(n-1) \frac{w'(x_i)}{w(x_i)} + \frac{F_n''(x_i)}{F_n'(x_i)} = 0, \quad i = 1, \dots, n. \end{aligned} \tag{1.6}$$

In **Section 2**, we study the zero distribution of Jacobi polynomials $P_n^{(\alpha_n, \beta_n)}$ with parameters $\alpha_n > 0$ and $\beta_n > 0$ that satisfy $\lim_{n \rightarrow \infty} \alpha_n/n = 2\alpha > 0$ and $\lim_{n \rightarrow \infty} \beta_n/n = 2\beta > 0$.

In **Section 3**, we consider Laguerre polynomials $L_n^{(\alpha_n)}$ with parameters $\alpha_n > 0$ that satisfy $\lim_{n \rightarrow \infty} \alpha_n = 2\alpha > 0$.

In **Section 4**, we obtain the zero distribution of the Hermite polynomials H_n .

Asymptotics and zero distribution of classical orthogonal polynomials have been studied in [1, 2, 3, 5]. Here, we extend these results using a simple method that works for all classical orthogonal polynomials.

2. Zero distribution of Jacobi polynomials

The Jacobi weight $w_{\alpha, \beta}(x)$ is defined by

$$w_{\alpha, \beta}(x) = (1-x)^\alpha (1+x)^\beta, \quad x \in [-1, 1], \tag{2.1}$$

with positive α and β . The corresponding extremal measure is given by [4, Chapter IV, Section 5]

$$d\mu_{w_{\alpha,\beta}}(t) = \frac{1}{\pi} \frac{(1 + \alpha + \beta)}{1 - t^2} ((t - a)(b - t))^{1/2} dt, \quad t \in S_{w_{\alpha,\beta}}, \quad (2.2)$$

with support [4, Chapter IV, Section 1]

$$S_{w_{\alpha,\beta}} = [a, b] = [\theta_2^2 - \theta_1^2 - \Delta^{1/2}, \theta_2^2 - \theta_1^2 + \Delta^{1/2}], \quad (2.3)$$

where $\theta_1 = \alpha/(1 + \alpha + \beta)$, $\theta_2 = \beta/(1 + \alpha + \beta)$, and $\Delta = (1 - (\theta_1 + \theta_2)^2)(1 - (\theta_1 - \theta_2)^2)$.

Let $P_n^{(\alpha,\beta)}$ and $q_{n,\alpha,\beta}$ denote the orthonormal polynomial of degree n and the monic orthogonal polynomial of degree n , respectively, with respect to the weight $w_{\alpha,\beta}$. Let

$$\nu_{n,\alpha,\beta} := \frac{1}{n} \sum_{x:P_n^{(\alpha,\beta)}(x)=0} \delta(x) \quad (2.4)$$

denote the discrete probability measure with mass $1/n$ at each zero of $P_n^{(\alpha,\beta)}$. Here, $\delta(x)$ denotes the discrete probability measure with support x (the point mass at x).

We first show that the Fekete polynomials for Jacobi weights $w_{\alpha,\beta}$ with $\alpha > 0$ and $\beta > 0$ are, in fact, Jacobi polynomials.

Let $\alpha > 0$ and $\beta > 0$ be fixed and set $w = w_{\alpha,\beta}^{1/(n-1)}$ in the function V_n defined with (1.1). Since

$$\frac{w'(x)}{w(x)} = \frac{1}{(n-1)} \frac{w'_{\alpha,\beta}(x)}{w_{\alpha,\beta}(x)} = \frac{1}{(n-1)} \frac{\beta - \alpha - (\alpha + \beta)x}{(1 - x^2)}, \quad x \in (-1, 1), \quad (2.5)$$

equations (1.6) yield

$$2(\beta - \alpha - (\alpha + \beta)x_i)F'_n(x_i) + (1 - x_i^2)F''_n(x_i) = 0, \quad i = 1, \dots, n. \quad (2.6)$$

Thus, the polynomial $(1 - x^2)F''_n(x) + 2(\beta - \alpha - (\alpha + \beta)x)F'_n(x)$ of degree n with leading coefficient $-n(n + 2\alpha + 2\beta - 1)$ has zeros at x_1, \dots, x_n , and therefore

$$(1 - x^2)F''_n(x) + 2(\beta - \alpha - (\alpha + \beta)x)F'_n(x) + n(n + 2\alpha + 2\beta - 1)F_n(x) = 0. \quad (2.7)$$

By [6, Theorem 4.2.1], the polynomial $q_{n,2\alpha-1,2\beta-1}$ satisfies (2.7) as well. However, (2.7) has a unique monic polynomial solution of degree n . Indeed, the polynomial

$$r := F_n - q_{n,2\alpha-1,2\beta-1} = \sum_{j=0}^{n-1} c_j q_{j,2\alpha-1,2\beta-1} \quad (2.8)$$

satisfies (2.7). Substituting r in (2.7), we obtain

$$\begin{aligned}
 0 &= (1 - x^2)r''(x) + 2(\beta - \alpha - (\alpha + \beta)x)r'(x) + n(n + 2\alpha + 2\beta - 1)r(x) \\
 &= \sum_{j=0}^{n-1} c_j \left[(1 - x^2)q''_{j,2\alpha-1,2\beta-1}(x) + 2(\beta - \alpha - (\alpha + \beta)x)q'_{j,2\alpha-1,2\beta-1}(x) \right. \\
 &\quad \left. + n(n + 2\alpha + 2\beta - 1)q_{j,2\alpha-1,2\beta-1}(x) \right] \\
 &= \sum_{j=0}^{n-1} c_j (-j^2 + n^2 + (n - j)(2\alpha + 2\beta - 1))q_{j,2\alpha-1,2\beta-1}(x),
 \end{aligned} \tag{2.9}$$

where (2.7) was applied to $q_{j,2\alpha-1,2\beta-1}$, $j = 0, \dots, n - 1$. Since $(n - j)(n + j + 2\alpha + 2\beta - 1) > 0$ for $j = 0, \dots, n - 1$, (2.9) implies $c_j = 0$, $j = 0, \dots, n - 1$, and the uniqueness of the polynomial solution of (2.7) follows.

We have shown that for positive α and β , the n th Fekete polynomial $F_{n,\alpha,\beta}$ associated with the Jacobi weight $w_{\alpha,\beta}$ is the Jacobi polynomial $q_{n,2(n-1)\alpha-1,2(n-1)\beta-1}$.

THEOREM 2.1. *Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences of positive numbers satisfying*

$$\frac{\alpha_n}{n} \longrightarrow 2\alpha > 0, \quad \frac{\beta_n}{n} \longrightarrow 2\beta > 0, \quad n \longrightarrow \infty. \tag{2.10}$$

If α and β are finite, then

$$\nu_{n,\alpha_n,\beta_n} \xrightarrow{w^*} \mu_{\alpha,\beta}, \quad n \longrightarrow \infty. \tag{2.11}$$

If $\alpha = \infty$ and β is finite, the limit of the measures ν_{n,α_n,β_n} is the point mass at -1 .

If α is finite and $\beta = \infty$, the limit of the measures ν_{n,α_n,β_n} is the point mass at 1 .

If $\alpha = \beta = \infty$ and $\alpha_n/\beta_n \rightarrow \lambda > 0$ as $n \rightarrow \infty$, the limit measure is the point mass at $(1 - \lambda)/(1 + \lambda)$.

Proof. For fixed $\alpha > 0$ and $\beta > 0$, let $\{x_{i,n}^{(\alpha,\beta)}\}_{i=1}^n$ be the n th Fekete set, and let $\nu_{n,\alpha,\beta}$ denote the discrete probability measure having mass $1/n$ at each Fekete point $x_{i,n}^{(\alpha,\beta)}$. By Theorem 1.1,

$$\nu_{n,\alpha,\beta} \xrightarrow{w^*} \mu_{w_{\alpha,\beta}}, \quad n \longrightarrow \infty. \tag{2.12}$$

From (2.10), it follows that

$$\tilde{\alpha}_n := \frac{\alpha_n + 1}{2(n - 1)} \longrightarrow \alpha, \quad \tilde{\beta}_n := \frac{\beta_n + 1}{2(n - 1)} \longrightarrow \beta, \quad n \longrightarrow \infty. \tag{2.13}$$

Furthermore,

$$F_{n,\tilde{\alpha}_n,\tilde{\beta}_n} = q_{n,\alpha_n,\beta_n}. \tag{2.14}$$

Assume first that α and β are both finite. Let $\epsilon > 0$ be fixed and let $N(\epsilon)$ be such that $\alpha - \epsilon \leq \tilde{\alpha}_n \leq \alpha + \epsilon$ and $\beta - \epsilon \leq \tilde{\beta}_n \leq \beta + \epsilon$ for $n \geq N(\epsilon)$. We will use

a certain monotonicity property of the zeros of the Jacobi polynomials. For $0 < \alpha_1 < \alpha_2$ and $0 < \beta_1 < \beta_2$,

$$\frac{w_{\alpha_1, \beta}(x)}{w_{\alpha_2, \beta}(x)} = (1-x)^{\alpha_1 - \alpha_2}, \quad \frac{w_{\alpha, \beta_2}(x)}{w_{\alpha, \beta_1}(x)} = (1+x)^{\beta_2 - \beta_1} \tag{2.15}$$

are increasing functions on $(-1, 1)$. By [6, Theorem 6.12.2],

$$x_{j,n}^{(\alpha_2, \beta)} < x_{j,n}^{(\alpha_1, \beta)}, \quad x_{j,n}^{(\alpha, \beta_1)} < x_{j,n}^{(\alpha, \beta_2)}, \quad j = 1, \dots, n. \tag{2.16}$$

Therefore,

$$x_{j,n}^{(\alpha+\epsilon, \beta-\epsilon)} < x_{j,n}^{(\tilde{\alpha}_n, \tilde{\beta}_n)} < x_{j,n}^{(\alpha-\epsilon, \beta+\epsilon)}, \quad j = 1, \dots, n. \tag{2.17}$$

Let $A \subset S_{w_{\alpha, \beta}}$ be an interval. We have

$$\left| \nu_{n, \tilde{\alpha}_n, \tilde{\beta}_n}(A) - \mu_{w_{\alpha, \beta}}(A) \right| \leq \left| \nu_{n, \tilde{\alpha}_n, \tilde{\beta}_n}(A) - \nu_{n, \alpha, \beta}(A) \right| + \left| \nu_{n, \alpha, \beta}(A) - \mu_{w_{\alpha, \beta}}(A) \right|. \tag{2.18}$$

In view of (2.12), it is enough to estimate the first term in (2.18). For any measurable set B and fixed $\alpha_0 > 0$ and $\beta_0 > 0$, from (2.2) and (2.12), it follows that

$$\begin{aligned} & \left| \nu_{n, \alpha', \beta'}(B) - \nu_{n, \alpha_0, \beta_0}(B) \right| \\ & \leq \left| \nu_{n, \alpha', \beta'}(B) - \mu_{w_{\alpha', \beta'}}(B) \right| + \left| \mu_{w_{\alpha', \beta'}}(B) - \mu_{w_{\alpha_0, \beta_0}}(B) \right| \\ & \quad + \left| \nu_{n, \alpha_0, \beta_0}(B) - \mu_{w_{\alpha_0, \beta_0}}(B) \right| \rightarrow 0 \end{aligned} \tag{2.19}$$

if we let $n \rightarrow \infty$ first, and then $\alpha' \rightarrow \alpha_0$ and $\beta' \rightarrow \beta_0$.

Next, define

$$J_{n, \alpha, \beta}^L(a) := \max \{ j : x_{j,n}^{(\alpha, \beta)} < a \}, \quad J_{n, \alpha, \beta}^R(a) := \min \{ j : x_{j,n}^{(\alpha, \beta)} > a \}. \tag{2.20}$$

Let $A = [c, d]$. By (2.19),

$$n^{-1} \left| J_{n, \alpha \pm \epsilon, \beta \pm \epsilon}^L(c) - J_{n, \alpha, \beta}^L(c) \right| = \left| (\nu_{n, \alpha \pm \epsilon, \beta \pm \epsilon} - \nu_{n, \alpha, \beta})((-\infty, c)) \right| \rightarrow 0 \tag{2.21}$$

as $n \rightarrow \infty$ first, and then $\epsilon \rightarrow 0$. Similarly,

$$n^{-1} \left| J_{n, \alpha \pm \epsilon, \beta \pm \epsilon}^R(d) - J_{n, \alpha, \beta}^R(d) \right| \rightarrow 0, \quad n \rightarrow \infty, \epsilon \rightarrow 0. \tag{2.22}$$

Furthermore, (2.17) implies

$$\begin{aligned} J_{n, \alpha - \epsilon, \beta + \epsilon}^L(c) & \leq J_{n, \tilde{\alpha}_n, \tilde{\beta}_n}^L(c) \leq J_{n, \alpha + \epsilon, \beta - \epsilon}^L(c), \\ J_{n, \alpha - \epsilon, \beta + \epsilon}^R(d) & \leq J_{n, \tilde{\alpha}_n, \tilde{\beta}_n}^R(d) \leq J_{n, \alpha + \epsilon, \beta - \epsilon}^R(d). \end{aligned} \tag{2.23}$$

From (2.21), (2.22), and (2.23) it follows that

$$\left| \nu_{n, \tilde{\alpha}_n, \tilde{\beta}_n}(A) - \nu_{n, \alpha, \beta}(A) \right| \rightarrow 0, \quad n \rightarrow \infty, \tag{2.24}$$

and this completes the proof for finite α and β .

If α is finite and $\beta = \infty$, β is finite and $\alpha = \infty$, or α and β are both infinite, and $\alpha_n/\beta_n \rightarrow \lambda \geq 0$ as $n \rightarrow \infty$, it immediately follows from (2.3) that the supports of the extremal measures $S_{w_{\alpha_n, \beta_n}}$ shrink to the single point 1, -1 , or $(1 - \lambda)/(1 + \lambda)$, respectively, which establishes the proof in these cases. \square

3. Zero distribution of Laguerre polynomials

Let $L_n^{(\alpha)}(x)$ denote the monic Laguerre polynomials that are orthogonal with respect to the Laguerre weight $w_\alpha(x) = x^\alpha e^{-x}$ on $[0, \infty)$, when $\alpha > -1$. Furthermore, $y = L_n^{(\alpha)}$ is the unique polynomial solution of degree n of the differential equation

$$xy'' + (\alpha + 1 - x)y' + ny = 0. \tag{3.1}$$

When $\alpha > 0$, the extremal measure μ_{w_α} is given by (see [4, Chapter IV, Section 5])

$$d\mu_{w_\alpha}(t) = \frac{1}{\pi t} ((t - a_\alpha)(b_\alpha - t))^{1/2} dt, \quad t \in S_{w_\alpha}, \tag{3.2}$$

where (see [4, Chapter IV, Section 1])

$$S_{w_\alpha} = [a_\alpha, b_\alpha] = [\alpha + 1 - (2\alpha + 1)^{1/2}, \alpha + 1 + (2\alpha + 1)^{1/2}]. \tag{3.3}$$

To show that the Fekete polynomials for Laguerre weights w_α with $\alpha > 0$ are Laguerre polynomials, we set $w = w_\alpha$ in (1.1). Since $w'(x)/w(x) = (\alpha/x - 1)$, (1.6) takes the form

$$x_i F_n''(x_i) + 2(n - 1)(\alpha - x_i)F_n'(x_i) = 0, \quad i = 1, \dots, n, \tag{3.4}$$

where $F_n = F_{n, \alpha}$ is the n th Fekete polynomial for the weight w_α . Since $2(n - 1)(\alpha - x)F_n'(x) + xF_n''(x)$ is a polynomial of degree n with leading coefficient $-2n(n - 1)$, the above equations imply that $z = F_n$ satisfies the differential equation

$$tz'' + 2(n - 1)(\alpha - t)z' + 2n(n - 1)z = 0. \tag{3.5}$$

Setting $z(t) = y(x)$ with $x = \lambda t$, we get $d^k z/dt^k = \lambda^k d^k y/dx^k$ for every $k \geq 0$, and (3.5) becomes

$$\lambda x y'' + 2(n - 1)(\lambda \alpha - x)y' + 2n(n - 1)y = 0. \tag{3.6}$$

Choosing $\lambda = 2(n - 1)$, we obtain

$$xy'' + (2(n - 1)\alpha - x)y' + ny = 0. \tag{3.7}$$

From (3.1) and (3.7) it follows that $y(x) = L_n^{(2(n-1)\alpha-1)}(x)$. Since $F_n(t) = z(t) = y(2(n - 1)t)$ we obtain

$$F_{n,\alpha}(x) = L_n^{(2(n-1)\alpha-1)}(2(n - 1)x). \tag{3.8}$$

Equation (3.8) shows that for every $n \geq 1$ there is a unique n th Fekete set $\{x_{i,n}^{(\alpha)}\}_{i=1}^n$, and if $\{z_{i,n}^{(\gamma)}\}_{i=1}^n$ denotes the zero set of the Laguerre polynomial $L_n^{(\gamma)}$ with $\gamma > 0$, then

$$x_{i,n}^{(\alpha)} = \frac{z_{i,n}^{(2(n-1)\alpha-1)}}{2(n - 1)}, \quad i = 1, \dots, n, \tag{3.9}$$

where both the zeros of the Laguerre polynomial and the Fekete points are arranged in increasing order.

Next, we show that the Fekete sets for a weight w_γ with $\gamma > 0$ are contained in a compact set. By [4, Chapter I, Theorem 1.3],

$$U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x) = F_{w_\gamma}, \quad x \in S_{w_\gamma}, \tag{3.10}$$

where F_{w_γ} is a constant. Furthermore, by [7, Theorem A], $U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x) \geq F_{w_\gamma}$, $x \notin S_{w_\gamma}$. This function is then continuously differentiable on $(0, \infty) \setminus \{a_\gamma, b_\gamma\}$, its first derivative vanishes on (a_γ, b_γ) , and

$$\frac{d^2}{dx^2}(U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x)) = \int_{a_\gamma}^{b_\gamma} \frac{1}{(x - t)^2} d\mu_{w_\gamma}(t) + \frac{\gamma}{x^2} > 0, \quad x > b_\gamma. \tag{3.11}$$

Thus, the first derivative of $U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x)$ is positive for $x > b_\gamma$, and so $U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x) > F_{w_\gamma}$ for $x > b_\gamma$. Therefore,

$$S_{w_\gamma}^* := \{x : U^{\mu_{w_\gamma}}(x) - \log w_\gamma(x) \leq F_{w_\gamma}\} \subset [0, b_\gamma]. \tag{3.12}$$

By [4, Chapter III, Theorem 1.2], $\{x_{i,n}^{(\gamma)}\}_{i=1}^n \subset S_{w_\gamma}^*$. Thus, we conclude that $\{x_{i,n}^{(\gamma)}\}_{i=1}^n \subset [0, b_\gamma]$ for every n .

THEOREM 3.1. *Let $\{\alpha_n\} \subset (0, \infty)$ be a sequence satisfying $\alpha_n/n \rightarrow 2\alpha > 0$ as $n \rightarrow \infty$. Then,*

$$\nu_{n,\alpha_n} := \frac{1}{n} \sum_{i=1}^n \delta\left(\frac{z_{i,n}^{(\alpha_n)}}{2(n - 1)}\right) \xrightarrow{w_\alpha^*} \mu_{w_\alpha}, \quad n \rightarrow \infty. \tag{3.13}$$

Proof. We have $\tilde{\alpha}_n := (\alpha_n + 1)/(2(n - 1)) \rightarrow \alpha > 0$ as $n \rightarrow \infty$. By (3.9), $z_{i,n}^{(\alpha_n)}/(2(n - 1)) = x_{i,n}^{(\tilde{\alpha}_n)}$, $i = 1, \dots, n$, and by [4, Chapter III, Theorem 1.3],

$$\frac{1}{n} \sum_{i=1}^n \delta(x_{i,n}^{(\alpha)}) \xrightarrow{w^*} \mu_{w_\alpha}, \quad n \rightarrow \infty. \tag{3.14}$$

The rest of the proof follows the argument used in the proof of Theorem 2.1. In this case, the zeros of the Laguerre polynomials $L_n^{(\alpha)}$ are monotone in the sense that if $\alpha_1 > \alpha_2 > -1$, then $z_{i,n}^{(\alpha_2)} < z_{i,n}^{(\alpha_1)}$, $i = 1, \dots, n$. This follows from the fact that $w_{\alpha_1}(x)/w_{\alpha_2}(x) = x^{\alpha_1 - \alpha_2}$ is an increasing function on $[0, \infty)$, and a variation of [6, Theorem 6.12.2] for unbounded intervals. \square

4. Zero distribution of the Hermite polynomials

The monic Hermite polynomials H_n are orthogonal with respect to the weight $w(x) = e^{-x^2}$, $x \in \mathbb{R}$. Furthermore, $y = H_n$ satisfies the differential equation

$$y'' - 2xy' + 2ny = 0, \quad n \geq 0. \tag{4.1}$$

The corresponding extremal measure μ_w is given by (see [4, Chapter IV, Theorem 5.1]),

$$d\mu_w(t) = \frac{2}{\pi} \sqrt{1 - t^2} dt, \quad t \in [-1, 1]. \tag{4.2}$$

To determine the relationship between the zeros of the Hermite polynomials and the Fekete sets for the weight $w(x) = e^{-x^2}$, we set $w(x) = e^{-x^2}$ in (1.1). Since $w'(x)/w(x) = -2x$, (1.6) yields

$$4(n - 1)x_i F'_n(x_i) - F''_n(x_i) = 0, \quad i = 1, \dots, n. \tag{4.3}$$

These equations imply that the n th degree polynomial $4(n - 1)x F'_n(x) - F''_n(x)$ with leading coefficient $4n(n - 1)$ has the same zero set as $F_n(x)$. Therefore, $F_n(x)$ is the polynomial solution of the differential equation

$$z'' - 4(n - 1)xz' + 4n(n - 1)z = 0. \tag{4.4}$$

For $\lambda > 0$, we set $y(x) = z(\lambda x)$. From (4.4), it follows that

$$y'' - 4(n - 1)\lambda^2 xy' + 4\lambda^2 n(n - 1)y = 0, \tag{4.5}$$

and in particular $y(x) = F_n(x/\sqrt{2(n - 1)})$ satisfies (4.1). Since (4.1) has a unique polynomial solution of degree n , we obtain $F_n(x/\sqrt{2(n - 1)}) = H_n(x)$. Then, if $\{x_{i,n}\}_{i=1}^n$ and $\{z_{i,n}\}_{i=1}^n$ denote the zeros of F_n and H_n , respectively, we have

$$x_{i,n} = \frac{z_{i,n}}{\sqrt{2(n - 1)}}, \quad i = 1, \dots, n. \tag{4.6}$$

From [4, Chapter IV, equation (5.5)], it follows that $S_w^* = [-1, 1]$, and then by [4, Chapter III, Theorem 1.2], $\{x_{i,n}\}_{i=1}^n \subset [-1, 1]$ for every $n \geq 1$. Using the argument employed in the previous sections, we establish the following theorem.

THEOREM 4.1. *For every $n \geq 1$, let ν_n denote the discrete probability measure having mass $1/n$ at each zero $z_{i,n}$ of the Hermite polynomial H_n . Then,*

$$\nu_n := \frac{1}{n} \sum_{i=1}^n \delta \left(\frac{z_{i,n}}{\sqrt{2(n-1)}} \right) \xrightarrow{w^*} \mu_w, \quad n \rightarrow \infty. \quad (4.7)$$

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Plamen Simeonov: Department of Computer and Mathematical Sciences, University of Houston-Downtown, Houston, TX 77002, USA

E-mail address: simeonov@uhd.edu