GENERIC UNIQUENESS OF A MINIMAL SOLUTION FOR VARIATIONAL PROBLEMS ON A TORUS

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We study minimal solutions for one-dimensional variational problems on a torus. We show that, for a generic integrand and any rational number $\alpha$, there exists a unique (up to translations) periodic minimal solution with rotation number $\alpha$.

1. Introduction

In this paper, we consider functionals of the form

$$I^f(a, b, x) = \int_a^b f(t, x(t), x'(t)) \, dt,$$

(1.1)

where $a$ and $b$ are arbitrary real numbers satisfying $a < b$, $x \in W^{1,1}(a, b)$ and $f$ belongs to a space of functions described below. By an appropriate choice of representatives, $W^{1,1}(a, b)$ can be identified with the set of absolutely continuous functions $x : [a, b] \to \mathbb{R}$, and henceforth we will assume that this has been done.

Denote by $\mathcal{M}$ the set of integrands $f = f(t, x, p) : \mathbb{R}^3 \to \mathbb{R}$ which satisfy the following assumptions:

(A1) $f \in C^3$ and $f(t, x, p)$ has period 1 in $t, x$;

(A2) $\delta_f \leq f_{pp}(t, x, p) \leq \delta^{-1}_f$ for every $(t, x, p) \in \mathbb{R}^3$;

(A3) $|f_{xp}| + |f_{tp}| \leq c_f(1 + |p|)$, $|f_{xx}| + |f_{xt}| \leq c_f(1 + p^2)$,

with some constants $\delta_f \in (0, 1)$, $c_f > 0$.

Clearly, these assumptions imply that

$$\delta_f p^2 - \bar{c}_f \leq f(t, x, p) \leq \delta^{-1}_f p^2 + \bar{c}_f$$

(1.2)

for every $(t, x, p) \in \mathbb{R}^3$ for some constants $\bar{c}_f > 0$ and $0 < \bar{\delta}_f < \delta_f$.

In this paper, we analyse extremals of variational problems with integrands $f \in \mathcal{M}$. The following optimality criterion was introduced by Aubry and Le

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Let $f \in \mathcal{M}$. A function $x(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ is called an $(f)$-minimal solution if

$$I^f(a, b, y) \geq I^f(a, b, x)$$

(1.3)

for each pair of numbers $a < b$ and each $y \in W^{1,1}(a, b)$ which satisfies $y(a) = x(a)$ and $y(b) = x(b)$ (see [2, 9, 10, 12]).

Our work follows Moser [9, 10], who studied the existence and structure of minimal solutions in the spirit of Aubry-Mather theory [2, 7].

Consider any $f \in \mathcal{M}$. It was shown in [9, 10] that $(f)$-minimal solutions possess numerous remarkable properties. Thus, for every $(f)$-minimal solution $x(\cdot)$, there is a real number $\alpha$ satisfying

$$\sup \{|x(t) - \alpha t| : t \in \mathbb{R}^1\} < \infty$$

(1.4)

which is called the rotation number of $x(\cdot)$, and given any real $\alpha$ there exists an $(f)$-minimal solution with rotation number $\alpha$. Senn [11] established the existence of a strictly convex function $E_f : \mathbb{R}^1 \to \mathbb{R}^1$, which is called the minimal average action of $f$ such that, for each real $\alpha$ and each $(f)$-minimal solution $x(\cdot)$ with rotation number $\alpha$,

$$(T_2 - T_1)^{-1} I^f(T_1, T_2, x) \to E_f(\alpha) \text{ as } T_2 - T_1 \to \infty. \quad (1.5)$$

This result is an analogue of Mather’s theorem about the average energy function for Aubry-Mather sets generated by a diffeomorphism of the infinite cylinder [8].

In this paper, we show that for a generic integrand $f$ and any rational $\alpha$, there exists a unique (up to translations) $(f)$-minimal periodic solution with rotation number $\alpha$.

Let $k \geq 3$ be an integer. Set $\mathcal{M}_k = \mathcal{M} \cap C^k(\mathbb{R}^3)$. For $f \in \mathcal{M}_k$ and $q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3$ satisfying $q_1 + q_2 + q_3 \leq k$, we set

$$|q| = q_1 + q_2 + q_3, \quad D^q f = \frac{\partial^{|q|} f}{\partial t^{q_1} \partial x^{q_2} \partial p^{q_3}}. \quad (1.6)$$

For $N, \epsilon > 0$ we set

$$E_k(N, \epsilon) = \{ (f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)|$$

$$\leq \epsilon + \epsilon \max \{|D^q f(t, x, p)|, |D^q g(t, x, p)|\}$$

$$\forall q \in \{0, 1, 2\}^3 \text{ satisfying } |q| \leq 2, \forall (t, x, p) \in \mathbb{R}^3\}$$

$$\cap \{ (f, g) \in \mathcal{M}_k \times \mathcal{M}_k : |D^q f(t, x, p) - D^q g(t, x, p)| \leq \epsilon$$

$$\forall q \in \{0, \ldots, k\}^3 \text{ satisfying } |q| \leq k, \forall (t, x, p) \in \mathbb{R}^3$$

$$\text{such that } |p| \leq N\}. \quad (1.7)$$
It is easy to verify that, for the set $\mathcal{M}_k$ there exists a uniformity which is determined by the base $E_k(N,\epsilon), N,\epsilon > 0$, and that the uniform space $\mathcal{M}_k$ is metrizable and complete [3]. We establish the existence of a set $\mathcal{F}_k \subset \mathcal{M}_k$ which is a countable intersection of open everywhere dense subsets of $\mathcal{M}_k$ such that, for each $f \in \mathcal{F}_k$ and each rational $\alpha \in \mathbb{R}$, there exists a unique (up to translations) $(f)$-minimal periodic solution with rotation number $\alpha$.

2. Properties of minimal solutions

Consider any $f \in \mathcal{M}$. We note that, for each pair of integers $j$ and $k$ the translations $(t,x) \rightarrow (t+j,x+k)$ leave the variational problem invariant. Therefore, if $x(\cdot)$ is an $(f)$-minimal solution, so is $x(\cdot)+j+k$. Of course, on the torus, this represents the same curve as does $x(\cdot)$. This motivates the following terminology [9, 10].

We say that a function $x(\cdot) \in W_{\text{loc}}^{1,1}(\mathbb{R}^1)$ has no self-intersections if for all pairs of integers $j,k$ the function $t \rightarrow x(t+j)+k-x(t)$ is either always positive, or always negative, or identically zero.

Denote by $\mathbb{Z}$ the set of all integers. We have the following result (see [6, Proposition 3.2] and [9, 10]).

**Proposition 2.1.** (i) Let $f \in \mathcal{M}$. Given any real $\alpha$ there exists a nonself-intersecting $(f)$-minimal solution with rotation number $\alpha$.

(ii) For any $f \in \mathcal{M}$ and any $(f)$-minimal solution $x$, there is the rotation number of $x$.

For each $f \in \mathcal{M}$, each rational number $\alpha$, and each natural number $q$ satisfying $qa \in \mathbb{Z}$, we define

$$\mathcal{N}(\alpha, q) = \{ x(\cdot) \in W_{\text{loc}}^{1,1}(\mathbb{R}^1) : x(t+q) = x(t) + aq, \ t \in \mathbb{R}^1 \},$$

$$\mathcal{M}_f(\alpha, q) = \{ x(\cdot) \in \mathcal{N}(\alpha, q) : I^f(0,q,x) \leq I^f(0,q,y) \ \forall y \in \mathcal{N}(\alpha, q) \}. \quad (2.1)$$

We have the following result [9, Theorems 5.1, 5.2, 5.4, and Corollaries 5.3 and 5.5].

**Proposition 2.2.** Let $f \in \mathcal{M}$, let $\alpha$ be a rational number, and let $p,q \geq 1$ be integers satisfying $pa, qa \in \mathbb{Z}$. Then $\mathcal{M}_f(\alpha, q) = \mathcal{M}_f(\alpha, p) \neq \emptyset$, each $x \in \mathcal{M}_f(\alpha, q)$ is a nonself-intersecting $(f)$-minimal solution with rotation number $\alpha$ and the set $\mathcal{M}_f(\alpha, q)$ is totally ordered, that is, if $x, y \in \mathcal{M}_f(\alpha, q)$, then either $x(t) < y(t)$ for all $t$, or $x(t) > y(t)$ for all $t$, or $x(t) = y(t)$ identically.

For any $f \in \mathcal{M}$ and any rational number $\alpha$ we set $\mathcal{M}_f^{\text{per}}(\alpha) = \mathcal{M}_f(\alpha, q)$, where $q$ is a natural number satisfying $qa \in \mathbb{Z}$.

We have the following result (see [6, Theorem 1.1]).

**Proposition 2.3.** Let $f \in \mathcal{M}$. Then there exist a strictly convex function $E_f : \mathbb{R}^1 \rightarrow \mathbb{R}^1$ satisfying $E_f(\alpha) \rightarrow \infty$ as $|\alpha| \rightarrow \infty$ and a monotonically increasing function $\Gamma_f : (0,\infty) \rightarrow [0,\infty)$ such that for each real $\alpha$, each $(f)$-minimal solution $x$ with
By Proposition 2.3 for each \( f \in \mathcal{M} \) there exists a unique number \( \alpha(f) \) such that
\[
E_f(\alpha(f)) = \min \{ E_f(\beta) : \beta \in \mathbb{R}^1 \}. 
\] (2.3)

Note that assumptions (A1), (A2), and (A3) play an important role in the proofs of Propositions 2.1, 2.2, and 2.3 (see [9, 10]).

3. The main results

Theorem 3.1. Let \( k \geq 3 \) be an integer and \( \alpha \) be a rational number. Then there exists a set \( \mathcal{F}_k \subset \mathcal{M}_k \) which is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \) such that for each \( f \in \mathcal{M}_k \) the following assertions hold:

1. If \( x, y \in \mathcal{M}_f^{(\text{per})}(\alpha) \), then there are integers \( p, q \) such that \( y(t) = x(t + p) - q \) for all \( t \in \mathbb{R}^1 \).

2. Let \( x \in \mathcal{M}_f^{(\text{per})}(\alpha) \) and \( \varepsilon > 0 \). Then there exists a neighborhood \( \mathcal{U} \) of \( f \) in \( \mathcal{M}_k \) such that for each \( g \in \mathcal{U} \) and each \( y \in \mathcal{M}_g^{(\text{per})}(\alpha) \) there are integers \( p, q \) such that \( |y(t) - x(t + p) + q| \leq \varepsilon \) for all \( t \in \mathbb{R}^1 \).

It is not difficult to see that Theorem 3.1 implies the following result.

Theorem 3.2. Let \( k \geq 3 \) be an integer. Then there exists a set \( \mathcal{F}_k \subset \mathcal{M}_k \) which is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \) such that for each \( f \in \mathcal{M}_k \) and each rational number \( \alpha \) the assertions (1) and (2) of Theorem 3.1 hold.

Note that minimal solutions with irrational rotation numbers were studied in [2, 7, 9, 10, 12].

4. An auxiliary result

Let \( k \geq 3 \) be an integer and \( \beta \in \mathbb{R}^1 \). For each \( f \in \mathcal{M}_k \), define \( \mathcal{A}f \in C^3(\mathbb{R}^3) \) by
\[
(\mathcal{A}f)(t, x, u) = f(t, x, u) - \beta u, \quad (t, x, u) \in \mathbb{R}^3. 
\] (4.1)

Clearly \( \mathcal{A}f \in \mathcal{M}_k \) for each \( f \in \mathcal{M}_k \).

Proposition 4.1. The mapping \( \mathcal{A} : \mathcal{M}_k \rightarrow \mathcal{M}_k \) is continuous.

Proof. Let \( f \in \mathcal{M}_k \) and let \( N, \varepsilon > 0 \). In order to prove the proposition, it is sufficient to show that there exists \( \varepsilon_0 \in (0, \varepsilon) \) such that
\[
\mathcal{A}\left( \{ g \in \mathcal{M}_k : (f, g) \in E_k(N, \varepsilon_0) \} \right) \subset \{ h \in \mathcal{M}_k : (h, \mathcal{A}f) \in E_k(N, \varepsilon) \}. 
\] (4.2)

Set
\[
\Delta_0 = 2(|\beta| + 1). 
\] (4.3)
Equation (1.2) implies that there exists $c_0 > 0$ such that

$$\Delta_0|u| - c_0 \leq f(t, x, u) \quad \forall (t, x, u) \in \mathbb{R}^3. \quad (4.4)$$

Choose a number $\varepsilon_0$ such that

$$0 < \varepsilon_0 < \min\{1, \varepsilon\}, \quad 4\varepsilon_0 + 4\varepsilon_0(1 - \varepsilon_0)^{-1}(4 + c_0) < \varepsilon. \quad (4.5)$$

It follows from (4.3) and (4.4) that for each $(t, x, u) \in \mathbb{R}^3$,

$$\begin{align*}
|f(t, x, u) - \beta u| &\geq |f(t, x, u)| - |\beta u| \geq |f(t, x, u)| - |\beta|\Delta_0^{-1}(f(t, x, u) + c_0) \\
&\geq |f(t, x, u)|(1 - |\beta|\Delta_0^{-1}) - |\beta|\Delta_0^{-1}c_0 \\
&\geq 2^{-1}|f(t, x, u)| - 2^{-1}c_0.
\end{align*} \quad (4.6)$$

Assume that

$$g \in \mathcal{M}_k, \quad (f, g) \in E_k(N, \varepsilon_0). \quad (4.7)$$

By (1.7) and (4.7) for each $(t, x, u) \in \mathbb{R}^3$,

$$\begin{align*}
|f(t, x, u) - g(t, x, u)| &\leq \varepsilon_0 + \varepsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\}, \\
\max\{|f(t, x, u)|, |g(t, x, u)|\} - \min\{|f(t, x, u)|, |g(t, x, u)|\} \\
&\leq \varepsilon_0 + \varepsilon_0 \max\{|f(t, x, u)|, |g(t, x, u)|\}, \\
(1 - \varepsilon_0) \max\{|f(t, x, u)|, |g(t, x, u)|\} &\leq \min\{|f(t, x, u)|, |g(t, x, u)|\} + \varepsilon_0, \\
g(t, x, u) &\leq (1 - \varepsilon_0)^{-1}|f(t, x, u)| + (1 - \varepsilon_0)^{-1}\varepsilon_0.
\end{align*} \quad (4.8)$$

We show that $(\mathcal{A}f, \mathcal{A}g) \in E_k(N, \varepsilon)$. It follows from (1.7), (4.1), (4.5), and (4.7) that, for each $q = (q_1, q_2, q_3) \in \{0, \ldots, k\}^3$ satisfying $|q| \leq k$ and each $(t, x, p) \in \mathbb{R}^3$ satisfying $|p| \leq N$,

$$|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)| \leq \varepsilon_0 < \varepsilon. \quad (4.9)$$

Let $q \in \{0, 1, 2\}^3, |q| \in \{0, 2\}$, and $(t, x, p) \in \mathbb{R}^3$. Equation (4.1) implies that

$$|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| = |D^q f(t, x, p) - D^q g(t, x, p)|. \quad (4.10)$$

If $|q| = 2$, then by (1.7), (4.1), (4.5), (4.7), and (4.10),

$$\begin{align*}
|D^q(\mathcal{A}f)(t, x, p) - D^q(\mathcal{A}g)(t, x, p)| \\
&\leq \varepsilon_0 + \varepsilon_0 \max\{|D^q f(t, x, p)|, |D^q g(t, x, p)|\} \\
&< \varepsilon + \varepsilon \max\{|D^q(\mathcal{A}f)(t, x, p)|, |D^q(\mathcal{A}g)(t, x, p)|\}.
\end{align*} \quad (4.11)$$
Assume that $q = 0$. By (1.7), (4.1), (4.5), (4.6), (4.7), and (4.8),

\[
\left| D^q(\varphi f)(t,x,p) - D^q(\varphi g)(t,x,p) \right| \\
= \left| f(t,x,p) - g(t,x,p) \right| \leq \varepsilon_0 + \varepsilon_0 \max \left\{ \left| f(t,x,p) \right|, \left| g(t,x,p) \right| \right\} \\
\leq \varepsilon_0 + \varepsilon_0 \max \left\{ \left| f(t,x,p) \right|, (1 - \varepsilon_0)^{-1} \left| f(t,x,p) \right| + (1 - \varepsilon_0)^{-1} \varepsilon_0 \right\} \\
= \varepsilon_0 + \varepsilon_0 (1 - \varepsilon_0)^{-1} \left| f(t,x,p) \right| + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} \\
\leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + \varepsilon_0 (1 - \varepsilon_0)^{-1} \varepsilon_0 \\
\leq \varepsilon_0 + \varepsilon_0^2 (1 - \varepsilon_0)^{-1} + 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} \varepsilon_0 + 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} \left| f(t,x,p) - \beta p \right| \\
\leq 2 \varepsilon_0 (1 - \varepsilon_0)^{-1} \left| (\varphi f)(t,x,p) \right| + \varepsilon \leq \varepsilon + \varepsilon \left| (\varphi f)(t,x,p) \right|.
\]

Equations (4.9), (4.11), and (4.12) imply that $(\varphi f, \varphi g) \in E_h(N, \varepsilon)$. Proposition 4.1 is proved.

Let $-\infty < T_1 < T_2 < \infty$ and $x \in W^{1,1}(T_1, T_2)$. By (4.1) we have

\[
I^{\varphi f}(T_1, T_2, x) = \int_{T_1}^{T_2} \left( f(t,x(t),x'(t)) - \beta x'(t) \right) dt \\
= I^f(T_1, T_2, x) - \beta x(T_2) + \beta x(T_1).
\]

Therefore, each $x \in W^{1,1}_{loc}(\mathbb{R}^1)$ is an $(\varphi f)$-minimal solution if and only if $x(\cdot)$ is an $(f)$-minimal solution.

Let $x \in W^{1,1}_{loc}(\mathbb{R}^1)$ be an $(f)$-minimal solution with rotation number $r$. By Proposition 2.1 there exists $c_1 > 0$ such that for all $s, t \in \mathbb{R}^1$,

\[
|x(t+s) - x(t) - rs| \leq c_1.
\]

Proposition 2.3 implies that there exists a constant $c_2 > 0$ such that for each $s \in \mathbb{R}^1$ and each $t > 0$,

\[
\left| I^f(s,s+t,x) - E_f(r) t \right| \leq c_2, \\
\left| I^{\varphi f}(s,s+t,x) - E_{\varphi f}(r) t \right| \leq c_2.
\]

It follows from (4.13), (4.14), (4.15), and (4.16) that, for each $s \in \mathbb{R}^1$ and each $t > 0$,

\[
\left| E_{\varphi f}(r) t + \beta t r - E_f(r) t \right| \\
\leq \left| E_{\varphi f}(r) t - I^{\varphi f}(s,s+t,x) \right| + \left| I^{\varphi f}(s,s+t,x) + \beta t r - I^f(s,s+t,x) \right| \\
+ \left| I^f(s,s+t,x) - E_f(r) t \right| \\
\leq c_2 + \left| \beta t r - \beta(x(t+s) - x(s)) \right| + c_2 \leq 2 c_2 + \beta |c_1|.
\]

These inequalities imply that

\[
E_{\varphi f}(r) = E_f(r) - \beta r \quad \forall r \in \mathbb{R}^1.
\]
5. Proof of Theorem 3.1

Let \( g \in \mathcal{M} \). We define

\[
\mu(g) = \inf \left\{ \liminf_{T \to \infty} T^{-1} I^g(0, T, x) : x(\cdot) \in W^{1,1}_{\text{loc}}([0, \infty)) \right\}.
\]  

(5.1)

In [13, Section 5] we showed that the number \( \mu(g) \) is well defined and proved the following result [13, Theorem 5.1].

**Proposition 5.1.** Let \( f \in \mathcal{M} \). Then there exists a constant \( M_0 > 0 \) such that:

(i) \( I^f(0, T, x) - \mu(f) T \geq -M_0 \) for each \( x \in W^{1,1}_{\text{loc}}([0, \infty)) \) and each \( T > 0 \).

(ii) For each \( a \in \mathbb{R}^1 \) there exists \( x \in W^{1,1}_{\text{loc}}([0, \infty)) \) such that \( x(0) = a \) and

\[
|I^f(0, T, x) - \mu(f) T| \leq 4M_0 \quad \forall T > 0.
\]

(5.2)

Note that assertion (ii) of Proposition 5.1 holds by the periodicity of \( f \) in \( x \).

Let \( f \in \mathcal{M} \). A function \( x \in W^{1,1}_{\text{loc}}([0, \infty)) \) is called \((f)\)-good (see [5]) if

\[
\sup \{|I^f(0, T, x) - \mu(f) T| : T \in (0, \infty)\} < \infty.
\]

(5.3)

By [6, Theorem 4.1],

\[
E_f(\alpha(f)) = \mu(f) \quad \forall f \in \mathcal{M}.
\]

(5.4)

For \( f \in \mathcal{M}, x, y, T_1 \in \mathbb{R}^1 \), and \( T_2 > T_1 \) we set

\[
U^f(T_1, T_2, x, y) = \inf \left\{ I^f(T_1, T_2, \nu) : \nu \in W^{1,1}(T_1, T_2), \nu(T_1) = x, \nu(T_2) = y \right\}.
\]

(5.5)

It is not difficult to see that for each \( x, y, T_1 \in \mathbb{R}^1, T_2 > T_1 \),

\[
U^f(T_1, T_2, x+1, y+1) = U^f(T_1, T_2, x, y),
\]

\[
U^f(T_1 + 1, T_2 + 1, x, y) = U^f(T_1, T_2, x, y), \quad -\infty < U^f(T_1, T_2, x, y) < \infty,
\]

\[
\inf \{ U^f(T_1, T_2, a, b) : a, b \in \mathbb{R}^1 \} > -\infty.
\]

(5.6)

Denote by \( \mathcal{M}_{\text{per}} \) the set of all \( f \in \mathcal{M} \) such that \( \alpha(f) \) is rational and denote by \( \mathcal{M}_{\text{per}}^0 \) the set of all \( g \in \mathcal{M}_{\text{per}} \) for which there exist an \((g)\)-minimal solution \( w \in C^2(\mathbb{R}^1) \), a continuous function \( \pi : \mathbb{R}^1 \to \mathbb{R}^1 \), and integers \( m, n \) such that the following properties hold:

(P1) \( \pi(x+1) = \pi(x), x \in \mathbb{R}^1 \);

(P2) \( n \geq 1 \) and \( \alpha(g) = mn^{-1} \) is an irreducible fraction;

(P3) \( w(t+n) = w(t) + m \) for all \( t \in \mathbb{R}^1 \);

(P4) \( U^g(0, 1, x, y) - \mu(g) - \pi(x) + \pi(y) \geq 0 \) for each \( x, y \in \mathbb{R}^1 \);

(P5) for any \( u \in W^{1,1}(0, n) \), the equality

\[
I^g(0, n, u) = n\mu(g) + \pi(u(0)) - \pi(u(n))
\]

holds if and only if there are integers \( i, j \) such that \( u(t) = w(t+i) - j \) for all \( t \in [0, n] \).
Consider the manifold \((\mathbb{R}^1/\mathbb{Z})^2\) and the canonical mapping \(P : \mathbb{R}^2 \rightarrow (\mathbb{R}^1/\mathbb{Z})^2\).
We have the following result \([13, \text{Proposition 6.2}]\).

**Proposition 5.2.** Let \(\Omega\) be a closed subset of \((\mathbb{R}^1/\mathbb{Z})^2\). Then there exists a bounded nonnegative function \(\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)\) such that
\[
\Omega = \{ x \in (\mathbb{R}^1/\mathbb{Z})^2 : \phi(x) = 0 \}.
\]

\textbf{Proposition 5.2} is proved by using \([1, \text{Chapter 2, Section 3, Theorem 1}]\) and the partition of unity (see \([4, \text{Appendix 1}]\)).

We also have the following result (see \([13, \text{Proposition 6.3}]\)).

**Proposition 5.3.** Suppose that \(f \in M_{\text{per}}, \alpha(f) = mn^{-1}\) is an irreducible fraction (\(m, n\) are integers, \(n \geq 1\)) and \(w \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)\) is an \((f)\)-minimal solution satisfying \(w(t + n) = w(t) + m\) for all \(t \in \mathbb{R}^1\). Let \(\phi \in C^\infty((\mathbb{R}^1/\mathbb{Z})^2)\) be as guaranteed in \textbf{Proposition 5.2} with
\[
\Omega = \{ P(t, w(t)) : t \in [0, n] \},
\]
and let
\[
g(t, x, p) = f(t, x, p) + \phi(P(t, x)), \quad (t, x, p) \in \mathbb{R}^3. \quad (5.10)
\]

Then \(g \in M^0_{\text{per}}\) and there is a continuous function \(\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1\) such that the properties (P1), (P2), (P3), (P4), and (P5) hold with \(g, w, \pi, m, n\) and \(\alpha(g) = \alpha(f)\).

In the sequel we need the following two lemmas proved in \([13]\).

**Lemma 5.4** \([13, \text{Lemma 6.6}]\). Assume that \(k \geq 3\) is an integer, \(g \in M^0_{\text{per}} \cap M_k\), and properties (P1), (P2), (P3), (P4), and (P5) hold with a \(g\)-minimal solution \(w(\cdot) \in C^2(\mathbb{R}^1)\), a continuous function \(\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1\) and integers \(m, n\). Then for each \(e \in (0, 1)\), there exists a neighborhood \(\mathcal{U}\) of \(g\) in \(M_k\) such that for each \(h \in \mathcal{U}\) and each \((h)\)-good function \(v \in W^{1,1}_{\text{loc}}([0, \infty))\) there are integers \(p, q\) such that
\[
|v(t) - w(t + p) - q| \leq e \quad \text{for all large enough } t. \quad (5.11)
\]

**Lemma 5.5** \([13, \text{Corollary 6.1}]\). Assume that \(k \geq 3\) is an integer, \(g \in M^0_{\text{per}} \cap M_k\), and properties (P1), (P2), (P3), (P4), and (P5) hold with a \(g\)-minimal solution \(w(\cdot) \in C^2(\mathbb{R}^1)\), a continuous function \(\pi : \mathbb{R}^1 \rightarrow \mathbb{R}^1\) and integers \(m, n\). Then there exist a neighborhood \(\mathcal{U}\) of \(g\) in \(M_k\) and a number \(L > 0\) such that for each \(h \in \mathcal{U}\) and each \((h)\)-good function \(v \in W^{1,1}_{\text{loc}}([0, \infty))\), the following property holds.

There is a number \(T_0 > 0\) such that
\[
|v(t_2) - v(t_1) - \alpha(g)(t_2 - t_1)| \leq L \quad (5.12)
\]
for each \(t_1 \geq T_0\) and each \(t_2 > t_1\).
Completion of the proof of Theorem 3.1. Let $k \geq 3$ be an integer and let $\alpha = \frac{mn}{n} - 1$ be an irreducible fraction ($n \geq 1$ and $m$ are integers). Let $f \in \mathcal{M}_k$. By Proposition 2.2 there exists an $(f)$-minimal solution $w_f(\cdot) \in W^{1,1}_{\text{loc}}(\mathbb{R}^1)$ such that

$$w_f(t + n) = w_f(t) + m \quad \forall t \in \mathbb{R}^1. \quad (5.13)$$

Choose

$$\beta \in \partial E_f(\alpha). \quad (5.14)$$

Consider a mapping $\mathcal{A} : \mathcal{M}_k \rightarrow \mathcal{M}_k$ defined by (4.1). By Proposition 4.1 the mapping $\mathcal{A}$ is continuous. Clearly there exists a continuous $\mathcal{A}^{-1} : \mathcal{M}_k \rightarrow \mathcal{M}_k$. Equations (5.14) and (4.18) imply that

$$0 \in \partial E_{\mathcal{A}f}(\alpha), \quad E_{\mathcal{A}f}(\alpha) = \min \{ E_{\mathcal{A}f}(r) : r \in \mathbb{R}^1 \} = \mu(\mathcal{A}f) \quad (5.15)$$

and that $\mathcal{A}f \in \mathcal{M}_{\text{per}}$. It follows from Proposition 5.2 that there exists a bounded nonnegative function $\phi \in C^\infty((\mathbb{R}^1/Z)^2)$ such that

$$\{ x \in (\mathbb{R}^1/Z)^2 : \phi(x) = 0 \} = \{ P(t, w_f(t)) : t \in [0, n] \}. \quad (5.16)$$

Set $f^{(\beta)} = \mathcal{A}f$ and for each $\gamma \in (0, 1)$ define

$$f_\gamma(t, x, u) = f(t, x, u) + \gamma \phi(P(t, x)), \quad (t, x, u) \in \mathbb{R}^3, \quad f_\gamma^{(\beta)} = \mathcal{A}(f_\gamma). \quad (5.17)$$

Proposition 5.3 implies that for each $\gamma \in (0, 1)$,

$$f_\gamma^{(\beta)} \in \mathcal{M}_{\text{per}}^0 \cap \mathcal{M}_k, \quad f_\gamma \rightarrow f \quad \text{as} \quad \gamma \rightarrow 0^+, \quad f_\gamma^{(\beta)} \rightarrow f^{(\beta)} \quad \text{as} \quad \gamma \rightarrow 0^+ \quad \text{in} \quad \mathcal{M}_k. \quad (5.18)$$

Fix $\gamma \in (0, 1)$ and an integer $n \geq 1$. By Proposition 5.3 the properties (P1), (P2), (P3), (P4), and (P5) hold with $g = f_\gamma^{(\beta)}$, $\alpha(g) = \alpha$ and $w(\cdot) = w_f$.

By Lemmas 5.4 and 5.5, there exists an open neighborhood $V(f, \gamma, n)$ of $f_\gamma^{(\beta)}$ in $\mathcal{M}_\gamma$ and a number $L(f, \gamma, n) > 0$ such that the following properties hold:

(i) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $\nu \in W^{1,1}_{\text{loc}}([0, \infty))$, there are integers $p, q$ such that

$$| \nu(t) - w_f(t + p) - q | \leq \frac{1}{n} \quad (5.19)$$

for all large enough $t$;

(ii) for each $h \in V(f, \gamma, n)$ and each $(h)$-good function $\nu \in W^{1,1}_{\text{loc}}([0, \infty))$, there is a number $T_0$ such that

$$| \nu(t_2) - \nu(t_1) - \alpha(f_\gamma^{(\beta)})(t_2 - t_1) | \leq L \quad (5.20)$$

for each $t_1 \geq T_0$ and each $t_2 > t_1$. 
Let \( h \in V(f, \gamma, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) be an \((h)\)-minimal solution with rotation number \( \alpha(h) \). Then by Proposition 2.3, (2.3), (5.4), and property (ii), \( v|_{[0, \infty)} \) is an \((h)\)-good function and there is \( T_0 \) such that (5.20) holds for each \( t_1 \geq T_0 \) and each \( t_2 > t_1 \). Since \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) has rotation number \( \alpha(h) \), it follows from Proposition 2.1 that there exists \( c_1 > 0 \) such that

\[
|v(t+s) - v(t) - \alpha(h)s| \leq c_1 \quad \forall s, t \in \mathbb{R}. \tag{5.21}
\]

Equations (5.15), (5.17), (5.20), and (5.21) imply that

\[
\alpha(h) = \alpha(f^y_\beta) = \alpha(f^i_\beta) = \alpha. \tag{5.22}
\]

Thus we have shown that

\[
\alpha(h) = \alpha \quad \forall h \in V(f, \gamma, n). \tag{5.23}
\]

Let \( h \in V(f, \gamma, n) \) and let \( v \in W^{1,1}_{\text{loc}}(\mathbb{R}^1) \) be an \((h)\)-minimal solution with rotation number \( \alpha \). It follows from Proposition 2.3, (2.3), and (5.4) that \( v|_{[0, \infty)} \) is an \((h)\)-good function. By property (i) there exist integers \( p, q \) such that

\[
|v(t) - w_{f}(t + p) - q| \leq \frac{1}{n} \quad \text{for all large enough } t. \tag{5.24}
\]

Therefore we proved the following property:

(iii) for each \( h \in V(f, \gamma, n) \) and each \((h)\)-minimal solution \( v \in M^\per_{h}(\alpha) \), there exist integers \( p, q \) such that

\[
|v(t) - w_{f}(t + p) - q| \leq \frac{1}{n} \quad \forall t \in \mathbb{R}. \tag{5.25}
\]

Define

\[
\mathcal{U}(f, \gamma, n) = \mathcal{A}^{-1}(V(f, \gamma, n)). \tag{5.26}
\]

Clearly \( \mathcal{U}(f, \gamma, n) \) is an open neighborhood of \( f_\gamma \) in \( \mathcal{M}_k \). By property (iii) the following property holds:

(iv) for each \( \xi \in \mathcal{U}(f, \gamma, n) \) and each \((\xi)\)-minimal solution \( v \in M^\per_{\xi}(\alpha) \), there exist integers \( p, q \) such that (5.25) holds.

Define

\[
\mathcal{F}_{\kappa \alpha} = \cap_{n=1}^{\infty} \{ \mathcal{U}(f, \gamma, i) : f \in \mathcal{M}_k, \gamma \in (0, 1), i \geq n \}. \tag{5.27}
\]

It is not difficult to see that \( \mathcal{F}_{\kappa \alpha} \) is a countable intersection of open everywhere dense subsets of \( \mathcal{M}_k \).
Let \( g \in \mathcal{F}_{k_3}, \epsilon \in (0,1) \) and \( x, y \in \mathcal{M}_k^{(\text{per})}(\alpha) \). Choose a natural number \( n > 8\epsilon^{-1} \).

By (5.27) there exist \( f \in \mathcal{M}_k, \gamma \in (0,1) \) and an integer \( i \geq n \) such that

\[
g \in \mathcal{U}(f, \gamma, i). \tag{5.28}
\]

It follows from (5.28) and property (iv) that there exist integers \( p_1, q_1, p_2, q_2 \) such that

\[
|x(t) - w_f(t + p_1) - q_1| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1, \tag{5.29}
\]

\[
|y(t) - w_f(t + p_2) - q_2| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1, \tag{5.30}
\]

where \( w_f \in \mathcal{M}_f^{(\text{per})}(\alpha) \).

It follows from (5.29) and (5.30) that for all \( t \in \mathbb{R}^1 \),

\[
|x(t - p_1) - w_f(t) - q_1| \leq \frac{1}{i}, \\
|y(t - p_2) - w_f(t) - q_2| \leq \frac{1}{i}, \\
|x(t - p_1 - q_1) - (y(t - p_2) - q_2)| \leq \frac{2}{i}, \\
|x(t + p_2 - p_1) - y(t) - q_1 + q_2| \leq \frac{2}{i} \leq \frac{2}{n} < \epsilon. \tag{5.31}
\]

Since \( \epsilon \) is any number in \((0,1)\), we conclude that there exist integers \( p, q \) such that

\[
x(t + p) - q = y(t) \quad \forall t \in \mathbb{R}^1. \tag{5.32}
\]

Assume that \( h \in \mathcal{U}(f, \gamma, i) \) and \( z \in \mathcal{M}_h^{(\text{per})}(\alpha) \). By the property (iv) there exist integers \( p_3, q_3 \) such that

\[
|z(t) - w_f(t + p_3) - q_3| \leq \frac{1}{i} \quad \forall t \in \mathbb{R}^1. \tag{5.33}
\]

Combined with (5.29) this inequality implies that

\[
|z(t - p_3) - q_3 - x(t - p_1)| \leq \frac{2}{i} \leq \frac{2}{n} < \epsilon \tag{5.34}
\]

for all \( t \in \mathbb{R}^1 \). This completes the proof of Theorem 3.1.

References


Uniqueness of a minimal solution


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