EXISTENCE OF ENTROPY SOLUTIONS FOR SOME NONLINEAR PROBLEMS IN ORLICZ SPACES

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We study in the framework of Orlicz Sobolev spaces \( W^{1,0}_{LM}(\Omega) \), the existence of entropic solutions to the nonlinear elliptic problems:

\[
- \text{div} a(x, u, \nabla u) + \text{div} \phi(u) = f \quad \text{in} \ \Omega,
\]

for the case where the second member of the equation \( f \in L^1(\Omega) \), and \( \phi \in (C^0(\mathbb{R}))^N \).

1. Introduction

Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) and let \( A(u) = -\text{div} a(x, u, \nabla u) \) be a Leray-Lions operator defined on \( W^{1,p}_0(\Omega) \), \( 1 < p < \infty \).

We consider the nonlinear elliptic problem

\[
- \text{div} a(x, u, \nabla u) = f - \text{div} \phi(u) \quad \text{in} \ \Omega,
\]

\[
u = 0 \quad \text{on} \ \partial \Omega,
\]

where

\[
f \in L^1(\Omega), \quad \phi \in (C^0(\mathbb{R}))^N.
\]

Note that no growth hypothesis is assumed on the function \( \phi \), which implies that the term \( \text{div} \phi(u) \) may be meaningless, even as a distribution. The notion of entropy solution, used in [8], allows us to give a meaning to a possible solution of (1.1).

In fact Boccardo proved in [8], for \( p \) such that \( 2 - 1/N < p < N \), the existence and regularity of an entropy solution \( u \) of problem (1.1), that is,

\[
u \in W^{1,q}_0(\Omega), \quad q < \bar{p} = \frac{(p-1)N}{N-1},
\]

\[
T_k(u) \in W^{1,p}_0(\Omega), \quad \forall k > 0,
\]
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\[ \int_{\Omega} a(x,u,\nabla u) \nabla T_k[u-\varphi] \, dx \leq \int_{\Omega} f T_k[u-\varphi] \, dx + \int_{\Omega} \phi(u) \nabla T_k[u-\varphi] \, dx \]
\[ \forall \varphi \in W^{1,p}_0(\Omega) \cap L^\infty(\Omega), \] (1.3)

where

\[ T_k(s) = s \quad \text{if } |s| \leq k \]
\[ T_k(s) = k \frac{s}{|s|} \quad \text{if } |s| > k. \] (1.4)

For the case \( \phi = 0 \) and \( f \) is a bounded measure, Bénilan et al. proved in [3] the existence and uniqueness of entropy solutions.

We mention as a parallel development, the work of Lions and Murat [14] who consider similar problems in the context of the renormalized solutions introduced by Diperna and Lions [10] for the study of the Boltzmann equations. They can prove existence and uniqueness of renormalized solution.

The functional setting in these works is that of the usual Sobolev space \( W^{1,p} \). Accordingly, the function \( a \) is supposed to satisfy polynomial growth conditions with respect to \( u \) and its derivatives \( \nabla u \). When trying to generalize the growth condition on \( a \), one is led to replace \( W^{1,p} \) by a Sobolev space \( W^{1,L_M} \) built from an Orlicz space \( L_M \) instead of \( L^p \). Here the \( N \)-function \( M \) which defines \( L_M \) is related to the actual growth of the function \( a \).

It is our purpose, in this paper, to prove the existence of entropy solution for problem (1.1) in the setting of the Orlicz Sobolev space \( W^{1,0}_{0,L_M}(\Omega) \). Our result, Theorem 3.5, generalizes [8, Theorem 2.1] and gives in particular a refinement of his result (see Remark 3.6).

For some existence results for strongly nonlinear elliptic equations in Orlicz spaces [4, 5, 6].

2. Preliminaries

2.1. Let \( M: \mathbb{R}^+ \to \mathbb{R}^+ \) be an \( N \)-function, that is, \( M \) is continuous, convex, with \( M(t) > 0 \) for \( t > 0 \), \( M(t)/t \to 0 \) as \( t \to 0 \) and \( M(t)/t \to \infty \) as \( t \to \infty \).

Equivalently, \( M \) admits the representation \( M(t) = \int_0^t a(\tau) \, d\tau \), where \( a: \mathbb{R}^+ \to \mathbb{R}^+ \) is nondecreasing, right continuous, with \( a(0) = 0 \), \( a(t) > 0 \) for \( t > 0 \) and \( a(t) \to \infty \) as \( t \to \infty \).

In the following, we assume, for convenience, that all \( N \)-functions are twice continuously differentiable, see Benkirane and Gossez [7].

The \( N \)-function \( \tilde{M} \) conjugate to \( M \) is defined by \( \tilde{M}(t) = \int_0^t \tilde{a}(\tau) \, d\tau \), where \( \tilde{a}: \mathbb{R}^+ \to \mathbb{R}^+ \) is given by \( \tilde{a}(t) = \sup\{s: a(s) \leq t\} \), see [1, 13].

The \( N \)-function \( M \) is said to satisfy the \( \Delta_2 \)-condition (resp., near infinity) if for some \( k \) and for every \( t \geq 0 \),

\[ M(2t) \leq k M(t) \quad \text{(resp., for } t \geq \text{ some } t_0). \] (2.1)
Let $M$ and $P$ be two $N$-functions. The notation $P \ll M$ means that $P$ grows essentially less rapidly than $M$, that is, for each $\epsilon > 0$, $P(t)/M(\epsilon t) \to 0$ as $t \to \infty$. This is the case if and only if $\lim_{t \to \infty} M^{-1}(t)/P^{-1}(t) = 0$. We will extend all $N$-functions into even functions on all $\mathbb{R}$.

### 2.2.

Let $\Omega$ be an open subset of $\mathbb{R}^N$. The Orlicz class $K_M(\Omega)$ (resp., the Orlicz space $L_M(\Omega)$) is defined as the set of (equivalence classes of) real-valued measurable functions $u$ on $\Omega$ such that

$$\int_{\Omega} M(u(x)) \, dx < \infty$$

(resp., $\int_{\Omega} M(u(x)/\lambda) \, dx < \infty$ for some $\lambda > 0$). The space $L_M(\Omega)$ is a Banach space under the norm

$$\|u\|_M = \inf \left\{ \lambda > 0 : \int_{\Omega} M\left(\frac{u(x)}{\lambda}\right) \, dx \leq 1 \right\}$$

and $K_M(\Omega)$ is a convex subset of $L_M(\Omega)$. The closure in $L_M(\Omega)$ of the set of bounded measurable functions with compact support in $\Omega$ is denoted by $E_M(\Omega)$.

The equality $E_M(\Omega) = L_M(\Omega)$ holds if and only if $M$ satisfies the $\Delta_2$ condition, for all $t$ or for $t$ large according to whether $\Omega$ has infinity measure or not.

The dual of $E_M(\Omega)$ can be identified with $\bar{L}_M(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x) \, dx$, and the dual norm on $\bar{L}_M(\Omega)$ is equivalent to $\|\cdot\|_M$. We say that $u_n$ converges to $u$ for the modular convergence in $L_M(\Omega)$ if for some $\lambda > 0$

$$\int_{\Omega} M\left(\frac{|u_n - u|}{\lambda}\right) \, dx \to 0 \quad \text{as} \quad n \to \infty. \quad (2.4)$$

If $M$ satisfies the $\Delta_2$-condition, then the modular convergence coincide with the norm convergence.

### 2.3.

The Orlicz Sobolev space $W^{1}L_M(\Omega)$ (resp., $W^{1}E_M(\Omega)$) is the space of all functions $u$ such that $u$ and its distributional derivatives up to order one lie in $L_M(\Omega)$ (resp., $E_M(\Omega)$). It is a Banach space under the norm

$$\|u\|_{1,M} = \sum_{|\alpha| \leq 1} \|D^\alpha u\|_M. \quad (2.5)$$

Thus, $W^{1}L_M(\Omega)$ and $W^{1}E_M(\Omega)$ can be identified with subspaces of the product of $N + 1$ copies of $L_M(\Omega)$. Denoting this product by $\prod L_M$, we will use the weak topologies $\sigma(\prod L_M, \prod E_M)$ and $\sigma(\prod L_M, \prod \bar{L}_M)$.

The space $W^{1}_{0}E_M(\Omega)$ is defined as the norm closure of $\mathcal{D}(\Omega)$ in $W^{1}E_M(\Omega)$ and the space $W^{1}_{0}L_M(\Omega)$ as the $\sigma(\prod L_M, \prod \bar{L}_M)$ closure of $\mathcal{D}(\Omega)$ in $W^{1}L_M(\Omega)$. 

We say that \( u_n \) converges to \( u \) for the modular convergence in \( W^1 L_M(\Omega) \) if for some \( \lambda > 0 \)
\[
\int_\Omega M\left(\frac{|D^\alpha u_n - D^\alpha u|}{\lambda}\right) dx \to 0 \quad \forall |\alpha| \leq 1.
\]
(2.6)

This implies the convergence \( \sigma(\prod L_M, \prod L_M) \).

2.4. Let \( W^{-1} L_M(\Omega) \) (resp., \( W^{-1} E_M(\Omega) \)) denote the space of distributions on \( \Omega \) which can be written as sums of derivatives of order \( \leq 1 \) of functions in \( L_M(\Omega) \) (resp., \( E_M(\Omega) \)). It is a Banach space under the usual quotient norm.

If the open set \( \Omega \) has the segment property, then the space \( \bar{\mathfrak{D}}(\Omega) \) is dense in \( W^1_0 L_M(\Omega) \) for the modular convergence and thus for the topology \( \sigma(\prod L_M, \prod L_M) \). Consequently, the action of a distribution in \( W^{-1} L_M(\Omega) \) on an element of \( W^1_0 L_M(\Omega) \) is well defined.

2.5. We recall the following lemmas.

Lemma 2.1 (see [5]). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with finite measure. Let \( M, P, \) and \( Q \) be \( N \)-functions such that \( Q \ll P, \) and let \( f : \Omega \times \mathbb{R} \to \mathbb{R}^N \) be a Carathéodory function such that
\[
|f(x,s)| \leq c(x) + k_1 P^{-1} M(k_2 |s|) \quad \text{a.e. } x \in \Omega, \forall s \in \mathbb{R},
\]
(2.7)
where \( k_1, k_2 \in \mathbb{R}_+, c(x) \in E_Q(\Omega) \). Let \( N_f \) be the Nemytskii operator defined from \( P(E_M(\Omega), 1/k_2) = \{ u \in L_M(\Omega) : d(u, E_M(\Omega)) < 1/k_2 \} \) to \( (E_Q(\Omega))^N \) by \( N_f(u)(x) = f(x, u(x)) \). Then \( N_f \) is strongly continuous.

Lemma 2.2 (see [5]). Let \( f : \mathbb{R} \to \mathbb{R} \) be uniformly Lipschitzian, with \( F(0) = 0 \). Let \( M \) be an \( N \)-function and let \( u \in W^1_0 L_M(\Omega) \) (resp., \( W^1_0 E_M(\Omega) \)). Then \( F(u) \in W^1_0 L_M(\Omega) \) (resp., \( W^1_0 E_M(\Omega) \)). Moreover, if the set \( D \) of discontinuity points of \( F' \) is finite, then
\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e. in } \{ x \in \Omega : u(x) \notin D \}, \\ 0 & \text{a.e. in } \{ x \in \Omega : u(x) \in D \}. \end{cases}
\]
(2.8)

Then \( F : W^1_0 L_M(\Omega) \to W^1_0 L_M(\Omega) \) is sequentially continuous with respect to the weak* topology \( \sigma(\prod L_M, \prod E_M) \).

Lemma 2.3 (see [11]). Let \( \Omega \) have the segment property. Then for each \( \nu \in W^1_0 L_M(\Omega) \), there exists a sequence \( \nu_n \in \mathfrak{D}(\Omega) \) such that \( \nu_n \) converges to \( \nu \) for the modular convergence in \( W^1_0 L_M(\Omega) \). Furthermore, if \( \nu \in W^1_0 L_M(\Omega) \cap L^\infty(\Omega) \) then
\[
\|\nu_n\|_{L^\infty(\Omega)} \leq (N + 1) \|\nu\|_{L^\infty(\Omega)}.
\]
(2.9)
2.6. We introduce the following notation, see [2, 15].

Definition 2.4. Let $M$ be an $N$-function, and define the following set:

$$
\mathcal{A}_M = \left\{ Q : Q \text{ is an } N \text{-function such that } \frac{Q''}{Q} \leq \frac{M''}{M'}, \quad \int_0^1 Q \circ H^{-1} \left( \frac{1}{r^{1-1/N}} \right) dr < \infty \text{ where } H(r) = \frac{M(r)}{r^N} \right\}.
$$

(2.10)

Remark 2.5. Let $M(t) = t^p$ and $Q(t) = t^q$, then the condition $Q \in \mathcal{A}_M$ is equivalent to the following conditions:

(i) $2 - 1/N < p < N$
(ii) $q < \bar{p} = \left( \frac{p-1}{N} \right) N/(N-1)$, see (1).

Remark 2.6. We can give some examples of $N$-functions $M$ for which the set $\mathcal{A}_M$ is not empty. Here, the $N$-functions $M$ are defined only at infinity.

(1) For $M(t) = t^2 \log t$ and $Q(t) = t \log t$, we have $H(t) = t \log t$ and $H^{-1}(t) = t(\log t)^{-1}$ at infinity, see [13]. Then the $N$-function $Q$ belongs to $\mathcal{A}_M$.

(2) For $M(t) = t^2 \log^2 t$ at infinity and $Q(t) = t \log^2 t$, we have $H(t) = t \log^2 t$ and $H^{-1}(t) = t(\log t)^{-2}$ at infinity, see [13]. Then the $N$-function $Q$ belongs to $\mathcal{A}_M$.

3. Definition and existence of entropy solutions

Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ with the segment property. Let $M, P$ be two $N$-functions such that $P \ll M$.

Let $A : D(A) \subset W^{1}_0 L_M(\Omega) \to W^{-1} L_{\bar{M}}(\Omega)$ be a mapping (not defined everywhere) given by $A(u) = -\text{div} a(x, u, \nabla u)$ where $a : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ is a Carathéodory function satisfying for a.e. $x \in \Omega$ and all $t \in \mathbb{R}$, $\xi, \bar{\xi}$ with $\xi \neq \bar{\xi}$,

$$
|a(x, t, \xi)| \leq d(x) + k_1 \bar{p}^{-1} M(k_2 |t|) + k_3 \bar{M}^{-1} M(k_4 |\xi|),
$$

(3.1)

$$
[a(x, t, \xi) - a(x, t, \bar{\xi})] [\xi - \bar{\xi}] > 0,
$$

(3.2)

$$
a(x, t, \xi) \xi \geq a M \left( \frac{|\xi|}{\lambda} \right),
$$

(3.3)

where $d(x) \in E_M(\Omega)$, $d \geq 0$, $\alpha, \lambda \in \mathbb{R}^*_+$, $k_1, k_2, k_3, k_4 \in \mathbb{R}^*_+$.

Consider the nonlinear elliptic problem (1.1) where

$$
f \in L^1(\Omega)
$$

(3.4)

and $\phi = (\phi_1, \ldots, \phi_N)$ satisfies

$$
\phi \in (C^0(\mathbb{R}))^N.
$$

(3.5)

As in [8], we define the following notion of an entropy solution, which gives a meaning to a possible solution of (1.1).
Definition 3.1. Assume that (3.1), (3.2), (3.3), (3.4), and (3.5) hold true and \( \mathcal{M} \neq \emptyset \). A function \( u \) is an entropy solution of problem (1.1) if

\[
\begin{align*}
    u & \in W^1_0 L_Q(\Omega) \quad \forall Q \in \mathcal{M}, \\
    T_k(u) & \in W^1_0 L_M(\Omega) \quad \forall k > 0,
\end{align*}
\]

\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi] \, dx \leq \int_{\Omega} f T_k[u - \varphi] \, dx + \int_{\Omega} \phi(u) \nabla T_k[u - \varphi] \, dx
\]

\[\forall \varphi \in W^1_0 L_M(\Omega) \cap L^{\infty}(\Omega).\]

We cannot use the solution \( u \) as a test function in (1.1), because \( u \) does not belong to \( W^1_0 L_M(\Omega) \). An important role is played by \( T_k(u) \) and the test functions\n
\[
T_k[u - \varphi], \quad \varphi \in W^1_0 L_M(\Omega) \cap L^{\infty}(\Omega)
\]

because both belong to \( W^1_0 L_M(\Omega) \).

In Theorem 3.5, we prove the existence of solution of problem (1.1), in the framework of entropy solutions.

Lemma 3.2. Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with the segment property. If \( u \in (W^1_0 L_M(\Omega))^N \) then \( \int_{\Omega} \text{div} u \, dx = 0 \).

Proof of Lemma 3.2. It is sufficient to use an approximation of \( u \). \( \square \)

We recall the following lemma (see [15, Lemma 2]).

Lemma 3.3. Let \( M \) be an \( N \)-function, \( u \in W^1 L_M(\Omega) \) such that \( \int_{\Omega} M(|\nabla u|) \, dx < \infty \), then

\[
-\mu'(t) \geq NC_N^{1/N} \mu^{1-1/N}(t)
\]

\[
\times C \left( \frac{1}{NC_N^{1/N} \mu^{1-1/N}(t)} \frac{d}{dt} \int_{|u| > r} M(|\nabla u|) \, dx \right) \quad \forall t > 0,
\]

where \( C \) is the function defined as

\[
C(t) = \frac{1}{\sup \{ r \geq 0, H(r) \leq t \}}, \quad H(r) = \frac{M(r)}{r}.
\]

The function \( C_N \) is the measure of the unit ball of \( \mathbb{R}^N \), and \( \mu(t) = \text{meas}\{ |u| > t \} \).

Lemma 3.4. Let \( (X, \tau, \mu) \) be a measurable set such that \( \mu(X) < \infty \). Let \( \gamma \) be a measurable function \( \gamma : X \to [0, \infty) \) such that

\[
\mu(\{ x \in X : \gamma(x) = 0 \}) = 0,
\]

then for each \( \epsilon > 0 \), there exists \( \delta > 0 \) such that \( \int_{A} \gamma(x) \, dx < \delta \) implies

\[
\mu(A) \leq \epsilon.
\]
Theorem 3.5. Under assumptions (3.1), (3.2), (3.3), (3.4), and (3.5), with $\mathcal{A}_M \neq \emptyset$, there exists an entropy solution $u$ of problem (1.1) (in the sense of Definition 3.1).

Remark 3.6. In the case $M(t) = t^p$, Theorem 3.5 gives a refinement of the regularity result (1) (i.e., $u \in W_0^{1,q}(\Omega)$, $q < \tilde{p} = ((p-1)N/N-1)$). In fact, by Theorem 3.5, we have $u \in W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$ (for example for $Q(t) = t^{\tilde{p}}/\log(\alpha (e+t))$, $\alpha > 1$).

Proof of Theorem 3.5

Step 1. Define, for each $n > 0$, the approximations

$$\phi_n(s) = \phi(T_n(s)), \quad f_n(s) = T_n[f(s)].$$

Consider the nonlinear elliptic problem

$$u_n \in W_0^1 L_M(\Omega), \quad -\text{div} a(x, u_n, \nabla u_n) = f_n - \text{div} \phi_n(u_n) \quad \text{in } \Omega. \quad (3.13)$$

From Gossez and Mustonen [12, Proposition 1, Remark 2], problem (3.13) has at least one solution.

Step 2. We will prove that $(u_n)$ is bounded in $W_0^1 L_Q(\Omega)$ for each $Q \in \mathcal{A}_M$. Let $\varphi$ be the truncation defined, for each $t, h > 0$, by

$$\varphi(\xi) = \begin{cases} 0 & \text{if } 0 \leq \xi \leq t, \\ \frac{1}{h} (\xi - t) & \text{if } t < \xi < t + h, \\ 1 & \text{if } \xi \geq t + h, \\ -\varphi(-\xi) & \text{if } \xi < 0. \end{cases} \quad (3.14)$$

Using the test function $v = \varphi(u_n)$ in (3.13) ($v \in W_0^1 L_M(\Omega)$ by Lemma 2.2), we have

$$\int_{\Omega} a(x, u_n, \nabla u_n) \varphi'(u_n) \nabla u_n \, dx = \int_{\Omega} f_n \varphi(u_n) \, dx + \int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) \, dx. \quad (3.15)$$

We claim now that

$$\int_{\Omega} \phi_n(u_n) \nabla \varphi(u_n) \, dx = 0. \quad (3.16)$$

Indeed,

$$\nabla \varphi(u_n) = \varphi'(u_n) \nabla u_n, \quad (3.17)$$

where

$$\varphi'(\xi) = \begin{cases} \frac{1}{h} & \text{if } t < |\xi| < t + h, \\ 0 & \text{otherwise}, \end{cases} \quad (3.18)$$
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define \( \theta(s) = \phi_n(s)(1/h)\chi_{|t|<|u_n|<t+h} \), and \( \hat{\theta}(s) = \int_0^s \theta(t) \, dt \), we have by Lemma 2.2, \( \hat{\theta}(u_n) \in (W_0^{1/L}(\Omega))^N \), which implies

\[
\int \phi_n(u_n) \nabla \varphi(u_n) \, dx = \int \phi_n(u_n) \frac{1}{h} \chi_{|t|<|u_n|<t+h} \nabla u_n \, dx = \int \hat{\theta}(u_n) \nabla u_n \, dx \]

\[
= \int \operatorname{div} (\hat{\theta}(u_n)) \, dx = 0 \quad \text{(see Lemma 3.2).} \tag{3.19}
\]

This proves (3.16). By (3.3) and (3.15), we have (where we can suppose without loss of generality that \( \lambda = 1 \), since one can take \( u'_n = u_n/\lambda \))

\[
\frac{\alpha}{h} \int_{t<|u_n|<t+h} M(|\nabla u_n|) \, dx \leq \| f \|_{1,\Omega}. \tag{3.20}
\]

Let \( h \to 0 \), then

\[
- \frac{d}{dt} \int_{|u_n|>t} Q(|\nabla u_n|) \, dx \leq C \quad \text{with} \quad C = \frac{\| f \|_{1,\Omega}}{\alpha}. \tag{3.21}
\]

We prove the following inequality, which allows us to obtain the boundedness of \( (u_n) \) in \( W_0^{1/LQ}(\Omega) \),

\[
- \frac{d}{dt} \int_{|u_n|>t} Q(|\nabla u_n|) \, dx \leq -\mu'_n(t) Q \circ H^{-1} \left( - \frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right). \tag{3.22}
\]

Indeed, let \( C(s) = 1/H^{-1}(s) \), where \( H(r) = M(r)/r \) and \( H^{-1}(s) = \sup\{ r \geq 0, \, H(r) \leq s \} \). Then

\[
C(s) = \frac{s}{M \circ H^{-1}(s)}. \tag{3.23}
\]

By Lemma 3.3 we have, with \( \mu_n(t) = \text{meas}\{|u_n| > t\}, \)

\[
-\mu'_n(t) \geq NC_N^{1/N} \mu_n(t)^{1-1/N} \times C \left( - \frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right), \tag{3.24}
\]

then

\[
-\mu'_n(t) M \circ H^{-1} \left( - \frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right) \geq NC_N^{1/N} \mu_n(t)^{1-1/N} \left( - \frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right), \tag{3.25}
\]
and also
\[
\frac{1}{\mu_n(t)} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx
\]
\[
\leq M \circ H^{-1} \left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right)
\]

(3.26)

which gives
\[
M^{-1} \left( -\frac{1}{\mu_n(t)} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right)
\]
\[
\leq H^{-1} \left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right).
\]

(3.27)

Let \( Q \in \mathcal{M} \) and let \( D(s) = M(Q^{-1}(s)) \), \( D \) is then convex, and the Jensen's inequality gives
\[
D\left( \int_{\{t<|u_n|<t+h\}} \frac{Q(|\nabla u_n|)}{-\mu_n(t+h)+\mu_n(t)} \, dx \right) \leq \int_{\{t<|u_n|<t+h\}} \frac{M(|\nabla u_n|)}{-\mu_n(t+h)+\mu_n(t)} \, dx,
\]

(3.28)

then
\[
Q^{-1} \left( -\frac{1}{\mu_n(t)} \frac{d}{dt} \int_{|u_n|>t} Q(|\nabla u_n|) \, dx \right)
\]
\[
\leq M^{-1} \left( -\frac{1}{\mu_n(t)} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right)
\]
\[
\leq H^{-1} \left( -\frac{1}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \frac{d}{dt} \int_{|u_n|>t} M(|\nabla u_n|) \, dx \right)
\]

(3.29)

which gives (3.22). By (3.21) and (3.22) and since the function
\[
t \mapsto \int_{|u_n|>t} Q(|\nabla u_n|) \, dx
\]

(3.30)
is absolutely continuous (see [15]), we have
\[
\int_{\Omega} Q(|\nabla u_n|) \, dx = \int_{0}^{\infty} \left( -\frac{d}{dt} \int_{|u_n|>t} Q(|\nabla u_n|) \right) \, dt
\]
\[
\leq \int_{0}^{\infty} -\mu_n(t) Q \circ H^{-1} \left( \frac{C}{NC_N^{1/N} \mu_n(t)^{1-1/N}} \right) \, dt
\]
\[
\leq \frac{1}{C} \int_{0}^{\text{C-meas}(\Omega)} Q \circ H^{-1} \left( \frac{1}{r^{1-1/N}} \right) \, dr < \infty
\]

(3.31)
Indeed, we claim that bounded in \( W_0^1 L_Q(\Omega) \) for each \( Q \in \mathcal{A}_M \). Then \( u_n \) is bounded in \( W_0^1 L_Q(\Omega) \) for each \( Q \in \mathcal{A}_M \). Passing to a subsequence if necessary, we can assume that

\[
  u_n \rightharpoonup u \quad \text{weakly in } W_0^1 L_Q(\Omega) \text{ for } \sigma\left(\prod L_Q, \prod E_Q\right), \text{ a.e. in } \Omega. \tag{3.32}
\]

**Step 3.** We prove that \( T_k(u_n) \rightharpoonup T_k(u) \) weakly in \( W_0^1 L_M(\Omega) \) for all \( k > 0 \). Using the test function \( T_k(u_n) \) in (3.13), we obtain

\[
  \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n) \, dx = \int_{\Omega} f_n T_k(u_n) \, dx + \int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) \, dx. \tag{3.33}
\]

We claim that

\[
  \int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) \, dx = 0. \tag{3.34}
\]

Indeed, \( \nabla T_k(u_n) = \nabla u_n \chi_{|u_n| \leq k} \), define \( \theta(t) = \phi_n(t) \chi_{|t| \leq k} \), and \( \hat{\theta}(t) = \int_0^t \theta(\tau) \, d\tau \), we have by Lemma 2.2, \( \hat{\theta}(u_n) \in (W_0^1 L_M(\Omega))^N \),

\[
  \int_{\Omega} \phi_n(u_n) \nabla T_k(u_n) \, dx = \int_{\Omega} \phi_n(u_n) \chi_{|u_n| \leq k} \nabla u_n \, dx \\
  = \int_{\Omega} \theta(u_n) \nabla u_n \, dx \\
  = \int_{\Omega} \text{div}(\hat{\theta}(u_n)) \, dx = 0 \quad \text{(by Lemma 3.2)}
\]

which proves the claim.

On the other hand, (3.33) can be written as

\[
  \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n) \, dx = \int_{\Omega} a(x, u_n, \nabla T_k(u_n)) \nabla T_k(u_n) \, dx \\
  = \int_{\Omega} f_n T_k(u_n) \, dx, \tag{3.36}
\]

which implies, with (3.3), that \( \nabla T_k(u_n) \) is bounded in \( (L_M(\Omega))^N \), and \( T_k(u_n) \) is bounded in \( (W_0^1 L_M(\Omega))^N \). Since \( u_n \rightharpoonup u \) a.e. in \( \Omega \) then \( T_k(u_n) \rightharpoonup T_k(u) \) a.e. in \( \Omega \). Then

\[
  T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W_0^1 L_M(\Omega) \text{ for } \sigma\left(\prod L_M, \prod E_M\right). \tag{3.37}
\]

**Step 4.** We will prove that \( \nabla u_n \rightharpoonup \nabla u \) a.e. in \( \Omega \). Let \( \lambda > 0 \), \( \epsilon > 0 \). For \( B > 1 \), \( k > 0 \), we consider as in [9] for \( n, m \in \mathbb{N} \),

\[
  E_1 = \{|\nabla u_n| > B\} \cup \{|\nabla u_m| > B\} \cup \{|u_n| > B\} \cup \{|u_m| > B\}, \\
  E_2 = \{|u_n - u_m| > k\}, \\
  E_3 = \{|u_n - u_m| \leq k, |u_n| \leq B, |u_m| \leq B, |\nabla u_n| \leq B, |\nabla u_m| \leq B\}.
\]

Then

\[
  |\nabla u_n - \nabla u_m| \geq \lambda \}
\]

we have \(|\nabla u_n - \nabla u_m| \geq \lambda \} \subset E_1 \cup E_2 \cup E_3 \).
Since \((u_n)\) and \((\nabla u_n)\) are bounded in \(L^1(\Omega)\) (since \(u_n\) is bounded in \(W^1_0 L^\infty(\Omega)\)), we have

\[
2B\mu(E_1) < \int_{E_1} |\nabla u_n| + |u_n| \, dx < \int_{\Omega} |\nabla u_n| + |u_n| \, dx < C. \tag{3.39}
\]

Then \(\text{meas} E_1 \leq \epsilon\) for \(B\) sufficiently large enough, independently of \(n, m\). Thus we fix \(B\) in order to have

\[
\text{meas} E_1 \leq \epsilon. \tag{3.40}
\]

Now we claim that \(\text{meas} E_3 \leq \epsilon\) for \(n\) and \(m\) large. Let \(C_1\) be such that \(\|u_n\|_1 \leq C_1\) and \(\|\nabla u_n\|_1 \leq C_1\). As in [9], the assumption (3.2) gives the existence of a measurable function \(γ(x)\) such that

\[
\text{meas} \left( \{ x \in Ω : γ(x) = 0 \} \right) = 0, \quad [a(x, t, ξ) - a(x, t, \tilde{ξ})] [ξ - \tilde{ξ}] \geq γ(x) > 0, \tag{3.41}
\]

for all \(t \in \mathbb{R}, ξ, \tilde{ξ} \in \mathbb{R}^N, |t|, |ξ|, |\tilde{ξ}| \leq B, |ξ - \tilde{ξ}| \geq λ\) a.e. in \(Ω\). We have

\[
\int_{E_3} γ(x) \, dx \leq \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] [\nabla u_n - \nabla u_m] \, dx \leq \int_{E_3} [a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_n)] [\nabla u_n - \nabla u_m] \, dx \tag{3.42}
\]

\[
+ \int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] [\nabla u_n - \nabla u_m] \, dx.
\]

Using the test function \(T_k(u_n - u_m)\) in (3.13) and integrating on \(E_3\), we obtain

\[
\int_{E_3} [a(x, u_n, \nabla u_n) - a(x, u_m, \nabla u_m)] \nabla T_k(u_n - u_m) \, dx = \int_{E_3} (f_n - f_m) T_k(u_n - u_m) \, dx \tag{3.43}
\]

\[
+ \int_{E_3} [\phi_n(u_n) - \phi_m(u_m)] \nabla T_k(u_n - u_m) \, dx,
\]

with

\[
\int_{E_3} [\phi_n(u_n) - \phi_m(u_m)] \nabla T_k(u_n - u_m) \, dx \leq 2B \int_{E_3} \phi_n(u_n) - \phi_m(u_m) \, dx \tag{3.44}
\]

\[
\leq 2B \int_{E_3} \left| \phi_n(u_n) - \phi_m(u_m) \right| \, dx \leq 2B \int_{E_3} \left[ |\phi(T_n(u_n)) - \phi(u_n)| + |\phi(u_n) - \phi(u_m)| + |\phi(u_m) - \phi(T_m(u_m))| \right] \, dx.
\]
Let \( n_0 \geq B \), then for \( n, m \geq n_0 \) we have \( T_n(u_n) = u_n \) and \( T_m(u_m) = u_m \) on \( E_3 \), which implies that the first and the third integral of the last inequality vanish. The second integral of (3.42) is bounded for \( n, m \geq n_0 \) by

\[
2k\|f\|_{1, \Omega} + 2B \int_{E_3} |\phi(u_n) - \phi(u_m)| \, dx. \tag{3.45}
\]

For a.e. \( x \in \Omega \) and \( \epsilon_1 > 0 \) there exist \( \eta(x) \geq 0 \) (meas\{\( x \in \Omega : \eta(x) = 0 \} = 0 \) such that \(|s - s'| \leq \eta(x), |s|, |s'|, |\xi| \leq B \) implies

\[
|a(x, s, \xi) - a(x, s', \xi)| \leq \epsilon_1. \tag{3.46}
\]

We use now the continuity of \( \phi \), to obtain for a.e. \( x \in \Omega \) and \( \epsilon_2 > 0 \), \( \eta_1(x) \geq 0 \) (meas\{\( x \in \Omega : \eta_1(x) = 0 \} = 0 \) such that \(|s - s'| \leq \eta_1(x), |s|, |s'| \leq B \) implies

\[
|\phi(s) - \phi(s')| \leq \epsilon_2. \tag{3.47}
\]

Then

\[
\int_{E_3} y(x) \, dx \leq \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} \left[ a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m) \right] \times \left[ \nabla u_n - \nabla u_m \right] \, dx \\
+ \int_{E_3 \cap \{x \in \Omega : \eta(x) \geq k\}} \left[ a(x, u_m, \nabla u_m) - a(x, u_n, \nabla u_m) \right] \times \left[ \nabla u_n - \nabla u_m \right] \, dx \tag{3.48} \\
+ 2k\|f\|_{1, \Omega} + 2B \int_{E_3 \cap \{x \in \Omega : \eta_1(x) < k\}} |\phi(u_n) - \phi(u_m)| \, dx \\
+ 2k\|f\|_{1, \Omega} + 2B \int_{E_3 \cap \{x \in \Omega : \eta_1(x) \geq k\}} |\phi(u_n) - \phi(u_m)| \, dx.
\]

By using for the first integral the definition of \( E_3 \) and condition (3.1), for the second integral the definition of \( E_3 \) and (3.46), for the fourth integral the definition of \( E_3 \) and \(|\phi(u_n)| \leq C(B)\) (since \(|u_n| \leq B \) and \( \phi \) continuous), and for the last integral the definition of \( E_3 \) and (3.47), we obtain

\[
\int_{E_3} y(x) \, dx \leq C(B) \int_{E_3 \cap \{x \in \Omega : \eta(x) < k\}} [1 + d(x)] \, dx + 2C_1(B)\epsilon_1 + 2k\|f\|_{1, \Omega} + 2C(B) \text{meas}\{x \in \Omega : \eta_1(x) < k\} + C_2\epsilon_2. \tag{3.49}
\]

We have \( \text{meas}\{x \in \Omega : \eta(x) < k\} \to 0 \) when \( k \to 0 \), and \( \text{meas}\{x \in \Omega : \eta_1(x) < k\} \to 0 \) when \( k \to 0 \). Let \( \epsilon > 0 \) and let \( \delta \) be the real, in Lemma 3.4, corresponding to \( \epsilon \), we choose \( \epsilon_1, \epsilon_2 \) such that

\[
2C_1(B)\epsilon_1 \leq \frac{\delta}{5}, \quad C_2\epsilon_2 \leq \frac{\delta}{5}. \tag{3.50}
\]
and $k$ such that
\[ C'(B) \int_{E_{3} \cap \{ x \in \Omega : \eta_{1}(x) < k \}} \left[ 1 + d(x) \right] \, dx < \frac{\delta}{5}, \quad 2k \| f \|_{L^{1}(\Omega)} \leq \frac{\delta}{5}. \] (3.51)
\[ 2C(B) \operatorname{meas} \{ x \in \Omega : \eta_{1}(x) < k \} < \frac{\delta}{5}. \]
Then $\int_{E_{3\gamma}} y(x) \, dx < \delta$ and Lemma 3.4 implies that
\[ \operatorname{meas} E_{3} < \epsilon \quad \forall n, m \geq n_{0}. \] (3.52)
This completes the proof of the claim.

Let the last $k$ be fixed, $u_{n}$ a Cauchy sequence in measure, we choose $n_{1}$ such that
\[ \operatorname{meas} E_{2} \leq \epsilon \quad \forall n, m \geq n_{1}. \] (3.53)
Then
\[ \operatorname{meas} \{ x \in \Omega : |\nabla u_{n} - \nabla u_{m}| \geq \lambda \} \leq \epsilon \quad \forall n, m \geq \max(n_{1}, n_{0}) \] (3.54)
and $\nabla u_{n} \rightarrow \nabla u$ in measure, consequently
\[ \nabla u_{n} \rightharpoonup \nabla u \quad \text{a.e. in } \Omega. \] (3.55)

Step 5. Let $\varphi \in W_{0}^{1}L_{M}(\Omega) \cap L^{\infty}(\Omega)$. From Lemma 2.3, there exists a sequence $(\varphi_{j}) \in \mathcal{D}(\Omega)$ such that $\varphi_{j}$ converges to $\varphi$ for the modular convergence in $W_{0}^{1}L_{M}(\Omega)$ with
\[ \| \varphi_{j} \|_{L^{\infty}(\Omega)} \leq (N + 1) \| \varphi \|_{L^{\infty}(\Omega)}. \] (3.56)
Using $T_{k}[u_{n} - \varphi_{j}]$ as a test function in (3.13) we obtain
\[ \int_{\Omega} a(x, u_{n}, \nabla u_{n}) \nabla T_{k}[u_{n} - \varphi_{j}] \, dx = \int_{\Omega} f_{n} T_{k}[u_{n} - \varphi_{j}] \, dx + \int_{\Omega} \phi_{n}(u_{n}) \nabla T_{k}[u_{n} - \varphi_{j}] \, dx \] (3.57)
which gives, if $n \rightarrow \infty$,
\[ \liminf_{n \rightarrow \infty} \int_{\Omega} a(x, u_{n}, \nabla u_{n}) \nabla T_{k}[u_{n} - \varphi_{j}] \, dx \geq \liminf_{n \rightarrow \infty} \int_{\Omega} \left[ a(x, u_{n}, \nabla u_{n}) - a(x, u_{n}, \nabla \varphi_{j}) \right] \nabla T_{k}[u_{n} - \varphi_{j}] \, dx \\
+ \lim_{n \rightarrow \infty} \int_{\Omega} a(x, T_{k+i\|\varphi_{j}\|_{L^{\infty}(\Omega)}(u_{n}), \nabla \varphi_{j}) \nabla T_{k}[u_{n} - \varphi_{j}] \, dx \] (3.58)
\[ \geq \int_{\Omega} \left[ a(x, u, \nabla u) - a(x, u, \nabla \varphi_{j}) \right] \nabla T_{k}[u - \varphi_{j}] \, dx \\
+ \int_{\Omega} a(x, u, \nabla \varphi_{j}) \nabla T_{k}[u - \varphi_{j}] \, dx. \]
where we have used Fatou lemma for the first integral, and for the second the convergences \( \nabla T_k[u_n - \varphi_j] \rightarrow \nabla T_k[u - \varphi_j] \) by (3.37) in \( (L_M(\Omega))^N \) for \( \sigma(\prod L_M, \prod E_M) \) and \( a(x, T_k+\|\varphi\|_{L^\infty(\Omega)}(u_n), \nabla \varphi_j) \rightarrow a(x, T_k+\|\varphi\|_{L^\infty(\Omega)}(u), \nabla \varphi_j) \) strongly in \( (E_M(\Omega))^N \) by (3.1), which implies that

\[
\liminf_{n \to \infty} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k[u_n - \varphi_j] \, dx \geq \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx. \tag{3.59}
\]

For \( n \geq k + (N+1)\|\varphi\|_{L^\infty(\Omega)} \),

\[
\int_{\Omega} \phi_n(u_n) \nabla T_k[u_n - \varphi_j] \, dx = \int_{\Omega} \phi(T_k+(N+1)\|\varphi\|_{L^\infty(\Omega)}(u_n)) \nabla T_k[u_n - \varphi_j] \, dx \quad \longrightarrow_{n \to \infty} \int_{\Omega} \phi(T_k+(N+1)\|\varphi\|_{L^\infty(\Omega)}(u)) \nabla T_k[u - \varphi_j] \, dx, \tag{3.60}
\]

we have used the convergences \( \nabla T_k[u_n - \varphi_j] \rightarrow \nabla T_k[u - \varphi_j] \) by (3.37) in \( (L_M(\Omega))^N \) and \( \phi(T_k+(N+1)\|\varphi\|_{L^\infty(\Omega)}(u_n)) \rightarrow \phi(T_k+(N+1)\|\varphi\|_{L^\infty(\Omega)}(u)) \) strongly in \( (E_M(\Omega))^N \) since \( \phi \) is continuous. On the other hand, since \( f_n \rightarrow f \) strongly in \( L^1(\Omega) \) and \( T_k[u_n - \varphi_j] \rightarrow T_k[u - \varphi_j] \) weakly* in \( L^\infty(\Omega) \), we have

\[
\int_{\Omega} f_n T_k[u_n - \varphi_j] \, dx \longrightarrow \int_{\Omega} f T_k[u - \varphi_j] \, dx. \tag{3.61}
\]

Then

\[
\int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx \geq \int_{\Omega} \phi(T_k+(N+1)\|\varphi\|_{L^\infty(\Omega)}(u)) \nabla T_k[u - \varphi_j] \, dx + \int_{\Omega} f T_k[u - \varphi_j] \, dx. \tag{3.62}
\]

Now, if \( j \to \infty \) in (3.62), we get

\[
\liminf_{j \to \infty} \int_{\Omega} a(x, u, \nabla u) \nabla T_k[u - \varphi_j] \, dx \\
\geq \liminf_{j \to \infty} \int_{\Omega} [a(x, u, \nabla u) - a(x, u, \nabla \varphi_j)] \nabla T_k[u - \varphi_j] \, dx \\
+ \lim_{j \to \infty} \int_{\Omega} a(x, u, \nabla \varphi_j) \nabla T_k[u - \varphi_j] \, dx \tag{3.63}
\]

\[
\geq \int_{\Omega} [a(x, u, \nabla u) - a(x, u, \nabla \varphi)] \nabla T_k[u - \varphi] \, dx \\
+ \int_{\Omega} a(x, u, \nabla \varphi) \nabla T_k[u - \varphi] \, dx,
\]

where we have used Fatou lemma for the first integral, and for the second the convergences \( \nabla T_k[u - \varphi_j] \rightarrow \nabla T_k[u - \varphi] \) in \( (L_M(\Omega))^N \) for the modular convergence and \( a(x, u, \nabla \varphi_j) \rightarrow a(x, u, \nabla \varphi) \) in \( (L_M(\Omega))^N \) for the modular convergence,
which implies that

\[
\liminf_{j \to \infty} \int_a(x, u, \nabla u) \nabla T_k \big[ u - \varphi_j \big] \, dx \geq \int_a(x, u, \nabla u) \nabla T_k \big[ u - \varphi \big] \, dx.
\] (3.64)

On the other hand, since \( \nabla T_k \big[ u - \varphi_j \big] \to \nabla T_k \big[ u - \varphi \big] \) in \( (L_M(\Omega))^N \) for the modular convergence, then weakly for \( \sigma(\prod L_M, \prod L_M) \) and \( \phi(T_k+(N+1)\|\varphi\|_{L^\infty}(u)) \in (L_M(\Omega))^N \) we have

\[
\int_a \phi(T_k+(N+1)\|\varphi\|_{L^\infty}(u)) \nabla T_k \big[ u - \varphi_j \big] \, dx \quad \overset{j \to \infty}{\longrightarrow} \quad \int_a \phi(T_k+(N+1)\|\varphi\|_{L^\infty}(u)) \nabla T_k \big[ u - \varphi \big] \, dx
\] (3.65)

Since \( f \in L^1(\Omega) \) and \( T_k \big[ u - \varphi_j \big] \rightharpoonup T_k \big[ u - \varphi \big] \) weakly* in \( L^\infty(\Omega) \), we have

\[
\int_a f T_k \big[ u - \varphi_j \big] \, dx \quad \longrightarrow \quad \int_a f T_k \big[ u - \varphi \big] \, dx.
\] (3.66)

Then

\[
\int_a a(x, u, \nabla u) \nabla T_k \big[ u - \varphi \big] \, dx \geq \int_a \phi(u) \nabla T_k \big[ u - \varphi \big] \, dx + \int_a f T_k \big[ u - \varphi \big] \, dx
\] (3.67)

and \( u \) is an entropy solution of problem (1.1). \( \square \)

**Theorem 3.7.** Suppose, in Theorem 3.5, that the N-function \( M \) satisfies, furthermore, the \( \Delta_2 \)-condition and \( f \geq 0 \), then the entropy solution \( u \) of problem (1.1) satisfies \( u \geq 0 \).

**Proof of Theorem 3.7.** Using \( \varphi = T_l(u^+) \) as test function in the definition of entropy solution, we obtain

\[
\int_a a(x, u, \nabla u) \nabla T_k \big[ u - T_l(u^+) \big] \, dx
\leq \int_a f T_k \big[ u - T_l(u^+) \big] \, dx + \int_a \phi(u) \nabla T_k \big[ u - T_l(u^+) \big] \, dx.
\] (3.68)

We have

\[
\int_a f T_k \big[ u - T_l(u^+) \big] \, dx \leq \int_{\{u \geq l\}} f T_k \big[ u - T_l(u) \big] \, dx.
\] (3.69)
Indeed,

$$\int_{\Omega} f T_k [u - T_l (u^+)] \, dx = \int_{u \geq l} f T_k [u - T_l (u^+)] \, dx$$

$$+ \int_{0 < u < l} f T_k [u - T_l (u^+)] \, dx$$

$$+ \int_{u \leq 0} f T_k [u - T_l (u^+)] \, dx.$$  \hfill (3.70)

If $0 < u < l$ then $u - T_l (u^+) = 0$ and $\int_{0 < u < l} f T_k [u - T_l (u^+)] \, dx = 0$. If $u \leq 0$ then $u - T_l (u^+) = u$ and $\int_{u \leq 0} f T_k [u - T_l (u^+)] \, dx \leq 0$ since $f$ is positive. If $u \geq l$ then $u^+ = u$ and $\int_{u \geq l} f T_k [u - T_l (u^+)] \, dx \leq \int_{u \geq l} f T_k [u - T_l (u)] \, dx$.

On the other hand, we claim that

$$\int_{\Omega} \phi (u) \nabla T_k [u - T_l (u^+)] \, dx = 0.$$  \hfill (3.71)

Indeed, if $0 < u < l$, then $u - T_l (u^+) = 0$, and $\int_{0 < u < l} \phi (u) \nabla T_k [u - T_l (u^+)] \, dx = 0$. If $u \leq 0$, then $u - T_l (u^+) = u$,

$$\int_{u \leq 0} \phi (u) \nabla T_k [u - T_l (u^+)] \, dx = \int_{-k \leq u \leq 0} \phi (u) \nabla u \, dx$$

$$= \int_{\Omega} \phi (u) \nabla u \chi_{[-k \leq u \leq 0]} \, dx.$$  \hfill (3.72)

We verify that the third integral of the last inequality vanishes. For this, define $\theta (t) = \phi (t) \chi_{[-k \leq t \leq 0]}$, and $\bar{\theta} (t) = \int_0^t \theta (\tau) \, d\tau$ we have, by Lemma 2.2, $\bar{\theta} (u) \in (W_0^1 L^M (\Omega))^N$ which implies

$$\int_{\Omega} \phi (u) \nabla \chi_{[-k \leq u \leq 0]} \, dx = \int_{\Omega} \theta (u) \nabla u \, dx$$

$$= \int_{\Omega} \text{div} (\bar{\theta} (u)) \, dx = 0 \quad \text{(by Lemma 3.2).}$$

If $u \geq l$ then $u^+ = u$ and

$$\int_{u \geq l} \phi (u) \nabla T_k [u - T_l (u^+)] \, dx = \int_{l \leq u \leq l+k} \phi (u) \nabla u \, dx$$

$$= \int_{\Omega} \phi (u) \nabla u \chi_{[l \leq u \leq l+k]} \, dx.$$  \hfill (3.74)

Similarly, we verify that

$$\int_{\Omega} \phi (u) \nabla u \chi_{[l \leq u \leq l+k]} \, dx = 0.$$  \hfill (3.75)
This completes the proof of the claim which implies that
\[
\int_{\Omega} a(x,u,\nabla u) \nabla T_k [u - T_l(u^+) - T_l(u)] \, dx \leq \int_{u \geq l} f T_k [u - T_l(u)] \, dx \quad (3.76)
\]
or
\[
\int_{\Omega} a(x,u,\nabla u) \nabla T_k [u - T_l(u^+)] \, dx \\
= \int_{l \leq u \leq l+k} a(x,u,\nabla u) \nabla u \, dx + \int_{-k \leq u \leq 0} a(x,u,\nabla u) \nabla u \, dx \\
\geq \int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx + \int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx,
\]
which gives
\[
\int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx + \int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \leq \int_{u \geq l} f T_k [u - T_l(u)] \, dx. \quad (3.78)
\]
Letting \( l \to \infty \) in (3.78) we have
\[
\int_{u \geq l} f T_k [u - T_l(u)] \, dx \to 0 \quad \text{since} \quad f T_k [2u] \in L^1(\Omega),
\]
\[
\int_{l \leq u \leq l+k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \geq \int_{l \leq u \leq k} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx \\
\quad = \int_{l \leq u} M \left( \frac{|\nabla T_k(u)|}{\lambda} \right) \, dx \\
\quad \to 0, \quad \text{when} \quad l \to \infty, \quad (3.79)
\]
since \( M(|\nabla T_k(u)|/\lambda) \in L^1(\Omega) \) and \( M \) satisfies the \( \Delta_2 \)-condition. Then
\[
\int_{-k \leq u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx = 0 \quad \forall k, \quad (3.80)
\]
which implies that,
\[
\int_{u \leq 0} M \left( \frac{|\nabla u|}{\lambda} \right) \, dx = \int_{\Omega} M \left( \frac{|\nabla u^-|}{\lambda} \right) \, dx = 0, \quad (3.81)
\]
\[
\nabla u^- = 0, \quad u^- = c \quad \text{a.e. in} \ \Omega.
\]
Or \( u^- \in W^1_0 L_Q(\Omega) \) then \( u^- = 0 \) a.e. in \( \Omega \) which proves that
\[
u \geq 0 \quad \text{a.e. in} \ \Omega. \quad (3.82)
\]
\( \square \)
References


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Mathematical Problems in Engineering

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Thinking about nonlinearity in engineering areas, up to the 70s, was focused on intentionally built nonlinear parts in order to improve the operational characteristics of a device or system. Keying, saturation, hysteretic phenomena, and dead zones were added to existing devices increasing their behavior diversity and precision. In this context, an intrinsic nonlinearity was treated just as a linear approximation, around equilibrium points.

Inspired on the rediscovering of the richness of nonlinear and chaotic phenomena, engineers started using analytical tools from “Qualitative Theory of Differential Equations,” allowing more precise analysis and synthesis, in order to produce new vital products and services. Bifurcation theory, dynamical systems and chaos started to be part of the mandatory set of tools for design engineers.

This proposed special edition of the Mathematical Problems in Engineering aims to provide a picture of the importance of the bifurcation theory, relating it with nonlinear and chaotic dynamics for natural and engineered systems. Ideas of how this dynamics can be captured through precisely tailored real and numerical experiments and understanding by the combination of specific tools that associate dynamical system theory and geometric tools in a very clever, sophisticated, and at the same time simple and unique analytical environment are the subject of this issue, allowing new methods to design high-precision devices and equipment.

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