On $\beta$-Connectedness in Intuitionistic Fuzzy Topological Spaces

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Abstract

In this paper the concept of types of intuitionistic fuzzy $\beta$-connected and intuitionistic fuzzy $\beta$-extremally disconnected in intuitionistic fuzzy topological spaces are introduced and studied. Here we introduce the concepts of intuitionistic fuzzy $\beta C_5$-connectedness, intuitionistic fuzzy $\beta C_S$-connectedness, intuitionistic fuzzy $\beta C_M$-connectedness, intuitionistic fuzzy $\beta$-strongly connectedness, intuitionistic fuzzy $\beta$-super connectedness, intuitionistic fuzzy $\beta C_i$-connectedness($i=1,2,3,4$), and obtain several properties and some characterizations concerning connectedness in these spaces.

Keywords: Intuitionistic fuzzy connected, intuitionistic fuzzy $\beta$-connected, intuitionistic fuzzy $\beta$-strongly connected, intuitionistic fuzzy $\beta C_5$-connectedness, intuitionistic fuzzy $\beta C_S$-connectedness, intuitionistic fuzzy $\beta C_M$-connectedness, intuitionistic fuzzy $\beta C_i$-connectedness($i=1,2,3,4$), intuitionistic fuzzy $\beta$-super connectedness.

1 Introduction

Ever since the introduction of fuzzy sets by L.A.Zadeh [11], the fuzzy concept has invaded almost all branches of mathematics. The concept of fuzzy topological spaces was introduced and developed by C.L.Chang [2]. Atanassov[1]
introduced the notion of intuitionistic fuzzy sets, Coker [3] introduced the intuitionistic fuzzy topological spaces. Several types of fuzzy connectedness in intuitionistic fuzzy topological spaces were defined by Turnali and Coker[10]. In this paper we have introduced some types of intuitionistic fuzzy $\beta$-connected and intuitionistic fuzzy $\beta$-extremally disconnected spaces and studied their properties and characterizations.

2 Preliminaries

Definition 2.1. [1] Let $X$ be a nonempty fixed set and $I$ the closed interval $[0,1]$. An intuitionistic fuzzy set (IFS) $A$ is an object of the following form

$$A = \{<x, \mu_A(x), \nu_A(x)>; x\in X\}$$

where the mappings $\mu_A(x): X\to I$ and $\nu_A(x): X\to I$ denote the degree of membership(namely) $\mu_A(x)$ and the degree of nonmembership(namely) $\nu_A(x)$ for each element $x\in X$ to the set $A$ respectively, and $0 \leq \mu_A(x) + \nu_A(x) \leq 1$ for each $x\in X$.

Definition 2.2. [1] Let $A$ and $B$ are intuitionistic fuzzy sets of the form $A = \{<x, \mu_A(x), \nu_A(x)>; x\in X\}$ and $B = \{<x, \mu_B(x), \nu_B(x)>; x\in X\}$. Then

(i) $A \subseteq B$ if and only if $\mu_A(x) \leq \mu_B(x)$ and $\nu_A(x) \geq \nu_B(x)$;

(ii) $\bar{A}$ (or $A^c$) = $\{<x, \nu_A(x), \mu_A(x)>; x\in X\}$;

(iii) $A \cap B = \{<x, \mu_A(x) \land \mu_B(x), \nu_A(x) \lor \nu_B(x)>; x\in X\}$;

(iv) $A \cup B = \{<x, \mu_A(x) \lor \mu_B(x), \nu_A(x) \land \nu_B(x)>; x\in X\}$;

(v) $\lceil A = \{<x, \mu_A(x), 1 - \mu_A(x)>; x\in X\}$;

(vi) $\langle A = \{<x, 1 - \nu_A(x), \nu_A(x)>; x\in X\}$

We will use the notation $A = \{<x, \mu_A, \nu_A>; x\in X\}$ instead of $A = \{<x, \mu_A(x), \nu_A(x)>; x\in X\}$

Definition 2.3. [3] $0_\sim = \{<x, 0, 1>; x\in X\}$ and $1_\sim = \{<x, 1, 0>; x\in X\}$.

Let $\alpha, \beta \in [0,1]$ such that $\alpha + \beta \leq 1$. An intuitionistic fuzzy point (IFP) $p_{(\alpha,\beta)}$ is intuitionistic fuzzy set defined by $p_{(\alpha,\beta)}(x) = \begin{cases} (\alpha,\beta) & \text{if } x = p, \\ (0,1) & \text{otherwise} \end{cases}$

Definition 2.4. [3] An intuitionistic fuzzy topology (IFT) in Coker’s sense on a nonempty set $X$ is a family $\tau$ of intuitionistic fuzzy sets in $X$ satisfying the following axioms:
(i) $0_\tau, 1_\tau \in \tau$;

(ii) $G_1 \cap G_2 \in \tau$, for any $G_1, G_2 \in \tau$;

(iii) $\cup G_i \in \tau$ for any arbitrary family $\{G_i ; i \in J \} \subseteq \tau$.

In this paper by $(X, \tau)$ or simply by $X$ we will denote the intuitionistic fuzzy topological space (IFTS). Each IFS which belongs to $\tau$ is called an intuitionistic fuzzy open set (IFOS) in $X$. The complement $\bar{A}$ of an IFOS $A$ in $X$ is called an intuitionistic fuzzy closed set (IFCS) in $X$.

**Definition 2.5.** [6] Let $p_{(\alpha, \beta)}$ be an IFP in IFTS $X$. An IFS $A$ in $X$ is called an intuitionistic fuzzy neighborhood (IFN) of $p_{(\alpha, \beta)}$ if there exists an IFOS $B$ in $X$ such that $p_{(\alpha, \beta)} \in B \subseteq A$.

Let $X$ and $Y$ are two non-empty sets and $f: (X, \tau) \to (Y, \sigma)$ be a function. If $B = \{<y, \mu_B(y), \nu_B(y)>; y \in Y\}$ is an IFS in $Y$, then the pre-image of $B$ under $f$ is denoted and defined by $f^{-1}(B) = \{<x, f^{-1}(\mu_B(x)), f^{-1}(\nu_B(x))>; x \in X\}$

Since $\mu_B(x), \nu_B(x)$ are fuzzy sets, we explain that $f^{-1}(\mu_B(x)) = \mu_B(x)(f(x)), f^{-1}(\nu_B(x)) = \nu_B(x)(f(x))$.

**Definition 2.6.** [3] Let $(X, \tau)$ be an IFTS and $A = \{<x, \mu_A(x), \nu_A(x)>; x \in X\}$ be an IFS in $X$. Then the intuitionistic fuzzy closure and intuitionistic fuzzy interior of $A$ are defined by

(i) $\text{cl}(A) = \bigcap \{C: C$ is an IFCS in $X$ and $C \supseteq A\}$;

(ii) $\text{int}(A) = \bigcup \{D: D$ is an IFOS in $X$ and $D \subseteq A\}$;

It can be also shown that $\text{cl}(A)$ is an IFCS, $\text{int}(A)$ is an IFOS in $X$ and $A$ is an IFCS in $X$ if and only if $\text{cl}(A) = A$; $A$ is an IFOS in $X$ if and only if $\text{int}(A) = A$.

**Proposition 2.1.** [3] Let $(X, \tau)$ be an IFTS and $A, B$ be intuitionistic fuzzy sets in $X$. Then the following properties hold:

(i) $\text{cl}A = (\text{int}(A)), \text{int}(A) = (\text{cl}(A))$;

(ii) $\text{int}(A) \subseteq A \subseteq \text{cl}(A)$.

**Definition 2.7.** [5] An IFS $A$ in an IFTS $X$ is called an intuitionistic fuzzy $\beta$-open set (IF$\beta$OS) if and only if $A \subseteq \text{cl}(\text{int}(A))$. The complement of an IF$\beta$OS $A$ in $X$ is called intuitionistic fuzzy $\beta$-closed (IF$\beta$CS) in $X$.

**Definition 2.8.** [5] Let $f$ be a mapping from an IFTS $X$ into an IFTS $Y$. The mapping $f$ is called:
Example 3.1. Let \( A \) and \( B \) are intuitionistic fuzzy \( \beta \)-open sets in \( X \), for each \( \beta \)-open sets \( B \) in \( Y \).

Definition 2.9. [9] Let \((X,\tau)\) be an IFTS and \( A = \{<x, \mu_A(x), \nu_A(x)>; x \in X\} \) be an IFS in \( X \). Then the intuitionistic fuzzy \( \beta \)-closure and intuitionistic fuzzy \( \beta \)-interior of \( A \) are defined by

(i) \( \beta \text{cl}(A) = \bigcap \{C: C \text{ is an IFCS in } X \text{ and } C \supseteq A\}; \)

(ii) \( \beta \text{int}(A) = \bigcup \{D: D \text{ is an IFOS in } X \text{ and } D \subseteq A\}. \)

Definition 2.10. [7] A function \( f:(X,\tau) \rightarrow (Y,\sigma) \) from an intuitionistic fuzzy topological space \( (X,\tau) \) to another intuitionistic fuzzy topological space \( (Y,\sigma) \) is said to be intuitionistic fuzzy \( \beta \)-irresolute if \( f^{-1}(B) \) is an IF\( \beta \)OS in \( (X,\tau) \) for each IF\( \beta \)OS \( B \) in \( (Y,\sigma) \).

3 Types of Intuitionistic Fuzzy \( \beta \)-Connectedness in Intuitionistic Fuzzy Topological Spaces

Definition 3.1. An IFTS \((X,\tau)\) is IF\( \beta \)-disconnected if there exists intuitionistic fuzzy \( \beta \)-open sets \( A \), \( B \) in \( X \), \( A \neq 0_\beta \), \( B \neq 0_\beta \) such that \( A \cup B = 1_\beta \) and \( A \cap B = 0_\beta \). If \( X \) is not IF\( \beta \)-disconnected then it is said to be IF\( \beta \)-connected.

Example 3.1. Let \( X = \{a, b\}, \tau = \{0_\beta, 1_\beta, A\} \) where

\[ A = \{<x,(\frac{a}{0.4}, \frac{b}{0.2}),(\frac{a}{0.3}, \frac{b}{0.3})>; x \in X\}, B = \{<x,(\frac{a}{0.4}, \frac{b}{0.2}),(\frac{a}{0.2}, \frac{b}{0.5})>; x \in X\}, \]

\( A \) and \( B \) are intuitionistic fuzzy \( \beta \)-open sets in \( X \), \( A \neq 0_\beta \), \( B \neq 0_\beta \) and \( A \cup B = 1_\beta \), \( A \cap B = 0_\beta \). Hence \( X \) is IF\( \beta \)-connected.

Example 3.2. Let \( X = \{a, b\}, \tau = \{0_\beta, 1_\beta, A\} \) where \( A = \{<x,(\frac{a}{0.1}, \frac{b}{0.2}),(\frac{a}{0.3}, \frac{b}{0.3})>; x \in X\}, B = \{<x,(\frac{a}{0.1}, \frac{b}{0.2}),(\frac{a}{0.1}, \frac{b}{0.2})>; x \in X\}, \)

\( B \) and \( C \) are intuitionistic fuzzy \( \beta \)-open sets in \( X \), \( B \neq 0_\beta \), \( B \neq 0_\beta \) and \( B \cap C = 0_\beta \). Hence \( X \) is IF\( \beta \)-disconnected.

Definition 3.2. An IFTS \((X,\tau)\) is IF\( \beta \)C\(_5\)-disconnected if there exists IFS \( A \) in \( X \), which is both IF\( \beta \)OS and IF\( \beta \)CS such that \( A \neq 0_\beta \), and \( A \neq 1_\beta \). If \( X \) is not IF\( \beta \)C\(_5\)-disconnected then it is said to be IF\( \beta \)C\(_5\)-connected.

Example 3.3. Let \( X = \{a, b\}, \tau = \{0_\beta, 1_\beta, A\} \) where \( A = \{<x,(\frac{a}{0.4}, \frac{b}{0.2}),(\frac{a}{0.3}, \frac{b}{0.3})>; x \in X\} \) A is an IF\( \beta \)OS in \( X \), But \( A \) is not IF\( \beta \)CS since \( \text{int}(\text{cl}(\text{int}A)) \not\subseteq A \), and \( 1_\beta \neq A \neq 0_\beta \). Thus \( X \) is IF\( \beta \)C\(_5\)-connected.
Example 3.4. Let $X = \{a, b\}$, $\tau = \{0_-, 1_-, A\}$ where

$A = \{<x, (\frac{a}{0.5}, \frac{b}{0.5})>, (\frac{a}{0.5}, \frac{b}{0.5})>; x \in X\}$, $B = \{<x, (\frac{a}{0.4}, \frac{b}{0.4})>, (\frac{a}{0.4}, \frac{b}{0.4})>; x \in X\}$,

$B$ is an intuitionistic fuzzy $\beta$-open sets in $X$. Also $B$ is IF$\beta$CS since $\text{int} (\text{cl}(\text{int} B)) = 0_-$ $\subseteq B$. Hence there exists an IFS $B$ in $X$ such that $1_- \neq B \neq 0_-$ which is both IF$\beta$OS and IF$\beta$CS in $X$. Thus $X$ is IF$\beta$C$\beta_\circ$-disconnected.

Proposition 3.1. IF$\beta$C$\beta_\circ$-connectedness implies IF$\beta$-connectedness.

Proof. Suppose that there exists nonempty intuitionistic fuzzy $\beta$-open sets $A$ and $B$ such that $A \cup B = 1_-$ and $A \cap B = 0_-$ (IF$\beta$-disconnected) then $\mu_A \lor \mu_B = 1$, $\nu_A \land \nu_B = 0$ and $\mu_A \lor \mu_B = 1$, $\nu_A \land \nu_B = 1$. In other words $\bar{B} = A$. Hence $A$ is IF$\beta$-clopen which implies $X$ is IF$\beta$C$\beta_\circ$-disconnected.

But the converse need not be true by the following example.

Example 3.5. Let $X = \{a, b\}$, $\tau = \{0_-, 1_-, A\}$ where

$A = \{<x, (\frac{a}{0.5}, \frac{b}{0.5})>, (\frac{a}{0.5}, \frac{b}{0.5})>; x \in X\}$, $B = \{<x, (\frac{a}{0.4}, \frac{b}{0.4})>, (\frac{a}{0.4}, \frac{b}{0.4})>; x \in X\}$,

$A$ is an IF$\beta$OS in $X$. And $B$ is an IF$\beta$OS in $X$ since $B \subseteq \text{cl}(\text{int} B)$. Also $1_- \neq A \cup B = 0_-$ and $A \cap B = 0_-$ which are intuitionistic fuzzy $\beta$-open sets in $X$. And $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(0_-) = 1_-$ which implies $C \cup D = 1_-$.

Proposition 3.2. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a IF$\beta$-irresolute surjection, $(X, \tau)$ is an IF$\beta$-connected, then $(Y, \sigma)$ is IF$\beta$-connected.

Proof. Assume that $(Y, \sigma)$ is not IF$\beta$-connected then there exists nonempty intuitionistic fuzzy $\beta$-open sets $A$ and $B$ in $(Y, \sigma)$ such that $A \cup B = 1_-$ and $A \cap B = 0_-$. Since $f$ is IF$\beta$-irresolute mapping, $C = f^{-1}(A) \neq 0_-$, $D = f^{-1}(B) \neq 0_-$ which are intuitionistic fuzzy $\beta$-open sets in $X$. And $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(0_-) = 1_-$ which implies $C \cap D = 1_-$. Thus $X$ is IF$\beta$-disconnected, which is a contradiction to our hypothesis. Hence $Y$ is IF$\beta$-connected.

Proposition 3.3. $(X, \tau)$ is IF$\beta$C$\beta_\circ$-connected iff there exists no nonempty intuitionistic fuzzy $\beta$-open sets $A$ and $B$ in $X$ such that $A = \bar{B}$.

Proof. Suppose that $A$ and $B$ are intuitionistic fuzzy $\beta$-open sets in $X$ such that $A \neq 0_- \neq B$ and $A = \bar{B}$. Since $A = \bar{B}$, $\bar{B}$ is an IF$\beta$OS and $B$ is an IF$\beta$CS. And $A \neq 0_-$ implies $B \neq 1_-$. But this is a contradiction to the fact that $X$ is IF$\beta$C$\beta_\circ$-connected.

Conversely, let $A$ be both IF$\beta$OS and IF$\beta$CS in $X$ such that $0_- \neq A \neq 1_-$. Now take $B = A$. $B$ is an IF$\beta$OS and $A \neq 1_-$ which implies $B = A \neq 0_-$ which is a contradiction.
**Definition 3.3.** An IFTS \((X, \tau)\) is IF\(\beta\)-strongly connected if there exists no nonempty IF\(\beta\)CS A and B in X such that \(\mu_A + \mu_B \subseteq 1, \nu_A + \nu_B \supseteq 1\).

In otherwords, an IFTS \((X, \tau)\) is IF\(\beta\)-strongly connected if there exists no nonempty IF\(\beta\)CS A and B in X such that \(A \cap B = 0_\sim\).

**Proposition 3.4.** An IFTS \((X, \tau)\) is IF\(\beta\)-strongly connected if there exists no IF\(\beta\)OS A and B in X, \(A \neq 1_\sim \neq B\) such that \(\mu_A + \mu_B \supseteq 1, \nu_A + \nu_B \subseteq 1\).

**Example 3.6.** Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A\}\) where
\[
A = \{x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5}); x \in X\}, \quad B = \{x, (\frac{a}{0.4}, \frac{b}{0.2}), (\frac{a}{0.6}, \frac{b}{0.8}); x \in X\},
\]
A is an IF\(\beta\)OS in X. And B is an IF\(\beta\)OS in X since B \(\subseteq\) cl(int(clB)). Also \(\mu_A + \mu_B \subseteq 1, \nu_A + \nu_B \supseteq 1\). Hence X is IF\(\beta\)-strongly connected.

**Proposition 3.5.** Let \(f:(X, \tau) \rightarrow (Y, \sigma)\) be a IF\(\beta\)-irresolute surjection. If X is an IF\(\beta\)-strongly connected, then so is Y.

**Proof.** Suppose that Y is not IF\(\beta\)-strongly connected then there exists IF\(\beta\)CS C and D in Y such that \(C \neq 0_\sim, D \neq 0_\sim, C \cap D = 0_\sim\). Since f is IF\(\beta\)-irresolute, \(f^{-1}(C), f^{-1}(D)\) are IF\(\beta\)CSs in X and \(f^{-1}(C) \cap f^{-1}(D) = 0_\sim, f^{-1}(C) \neq 0_\sim, f^{-1}(D) \neq 0_\sim\). (If \(f^{-1}(C) = 0_\sim\) then \(f(f^{-1}(C)) = C\) which implies \(f(0_\sim) = C\). So C = \(0_\sim\) a contradiction) Hence X is IF\(\beta\)-strongly disconnected, a contradiction. Thus (Y, \(\sigma\)) is IF\(\beta\)-strongly connected.

IF\(\beta\)-strongly connected does not imply IF\(\beta\)C\(_5\)-connected, and IF\(\beta\)C\(_5\)-connected does not imply IF\(\beta\)-strongly connected. For this purpose we see the following examples:

**Example 3.7.** Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A\}\) where
\[
A = \{x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5}); x \in X\}, \quad B = \{x, (\frac{a}{0.6}, \frac{b}{0.8}), (\frac{a}{0.8}, \frac{b}{0.6}); x \in X\},
\]
A is an IF\(\beta\)OS in X. And B is an IF\(\beta\)OS in X since B \(\subseteq\) cl(int(clB)). Also \(\mu_A + \mu_B \subseteq 1, \nu_A + \nu_B \supseteq 1\). Hence X is IF\(\beta\)-strongly connected. But X is not IF\(\beta\)C\(_5\)-connected, since A is both IF\(\beta\)OS and IF\(\beta\)CS in X.

**Example 3.8.** Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A, B, A \cup B, A \cap B\}\) where
\[
A = \{x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5}); x \in X\}, \quad B = \{x, (\frac{a}{0.4}, \frac{b}{0.2}), (\frac{a}{0.4}, \frac{b}{0.2}); x \in X\},
\]
X is IF\(\beta\)C\(_5\)-connected. But X is not IF\(\beta\)-strongly connected since A and B are intuitionistic fuzzy \(\beta\)-open sets in X such that \(\mu_A + \mu_B \supseteq 1, \nu_A + \nu_B \subseteq 1\).

**Lemma 3.1.** \([10](i)\) \(A \cap B = 0_\sim \Rightarrow A \subseteq \bar{B}\). \((ii)\) \(A \nsubseteq \bar{B} \Rightarrow A \cap B \neq 0_\sim\)
Definition 3.4. A and B are non-zero intuitionistic fuzzy sets in \((X, \tau)\). Then A and B are said to be
(i) IF\(\beta\)-weakly separated if \(\beta\text{cl}A \subseteq \overline{B}\) and \(\beta\text{cl}B \subseteq \overline{A}\)
(ii) IF\(\beta\)-q-separated if \((\beta\text{cl}A) \cap B = 0_\sim = A \cap (\beta\text{cl}B)\).

Definition 3.5. An IFTS \((X, \tau)\) is said to be IF\(\beta\)C\(S\)-disconnected if there exists IF\(\beta\)-weakly separated non-zero intuitionistic fuzzy sets A and B in \((X, \tau)\) such that \(A \cup B = 1_\sim\).

Example 3.9. Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A\}\) where \(A = \{<x, (\frac{a}{0.4}, \frac{b}{0.2})>, (\frac{a}{0.3}, \frac{b}{0.5})>\}; \ x \in X\}, B = \{<x, (\frac{a}{1}, \frac{b}{0})>, (\frac{a}{0.5}, \frac{b}{0.3})>; \ x \in X\}, C = \{<x, (\frac{a}{0.6}, \frac{b}{0.4})>, (\frac{a}{1}, \frac{b}{0.5})>; \ x \in X\}, B and C are intuitionistic fuzzy \(\beta\)-open sets in \(X, \beta\text{cl}B \subseteq C\) and \(\beta\text{cl}C \subseteq B\). Hence B and C are IF\(\beta\)-weakly separated and \(B \cup C = 1_\sim\). So \(X\) is IF\(\beta\)C\(S\)-disconnected.

Definition 3.6. An IFTS \((X, \tau)\) is said to be IF\(\beta\)C\(M\)-disconnected if there exists IF\(\beta\)-q-separated non-zero IF's A and B in \((X, \tau)\) such that \(A \cup B = 1_\sim\).

Example 3.10. Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A\}\) where \(A = \{<x, (\frac{a}{0.5}, \frac{b}{0.3})>, (\frac{a}{0.3}, \frac{b}{0.5})>\}; \ x \in X\}, B = \{<x, (\frac{a}{1}, \frac{b}{0})>, (\frac{a}{0.5}, \frac{b}{0.3})>; \ x \in X\}, C = \{<x, (\frac{a}{0.6}, \frac{b}{0.4})>, (\frac{a}{1}, \frac{b}{0.5})>; \ x \in X\}, B and C are intuitionistic fuzzy \(\beta\)-open sets in \(X, (\beta\text{cl}B) \cap C = 0_\sim\) and \(B \cap (\beta\text{cl}C) = 0_\sim\) which implies B and C are IF\(\beta\)-q-separated and \(B \cup C = 1_\sim\). Hence \(X\) is IF\(\beta\)C\(M\)-disconnected.

Remark 3.1. An IFTS \((X, \tau)\) is be IF\(\beta\)C\(S\)-connected if and only if \((X, \tau)\) is IF\(\beta\)C\(M\)-connected.

Definition 3.7. An IFS A in \((X, \tau)\) is said to be IF\(\beta\)-regular open set if \(\beta\text{int}(\beta\text{cl}(A)) = A\) and IF\(\beta\)-regular closed set if \(\beta\text{cl}(\beta\text{int}(A)) = A\).

Definition 3.8. An IFTS \((X, \tau)\) is said to be IF\(\beta\)-super disconnected if there exists an IF\(\beta\)-regular open set A in X such that \(0_\sim \neq A \neq 1_\sim\). X is called IF\(\beta\)-super connected if X is not IF\(\beta\)-super disconnected.

Example 3.11. Let \(X = \{a, b\}, \tau = \{0_\sim, 1_\sim, A\}\) where \(A = \{<x, (\frac{a}{0.3}, \frac{b}{0.2})>, (\frac{a}{0.3}, \frac{b}{0.5})>\}; \ x \in X\}, B = \{<x, (\frac{a}{1}, \frac{b}{0})>, (\frac{a}{0.5}, \frac{b}{0.3})>; \ x \in X\}, C = \{<x, (\frac{a}{0.6}, \frac{b}{0.4})>, (\frac{a}{1}, \frac{b}{0.5})>; \ x \in X\}, B and C are intuitionistic fuzzy \(\beta\)-open sets in \(X\), and \(\beta\text{int}(\beta\text{cl}B) = B\). This implies B is an IF\(\beta\)-regular open set in X. Hence X is an IF\(\beta\)-super disconnected.

Proposition 3.6. Let \((X, \tau)\) be an IFTS. Then the following are equivalent:
(a) X is IF\(\beta\)-super connected.
(b) For each IF\(\beta\)OS A \(\neq 0_\sim\) in X, we have \(\beta\text{cl}A = 1_\sim\)
(c) For each IF\(\beta\)CS A \(\neq 1_\sim\) in X, we have \(\beta\text{int}A = 0_\sim\)
(d) There exists no IF\(\beta\)OS s A and B in X such that \(A \neq 0_\sim \neq B\) and \(A \subseteq \overline{B}\).
(e) There exists no IF\(\beta\)OS s A and B in X such that \(A \neq 0_\sim \neq B, B = \beta\text{cl}A\) and \(A = \overline{\beta\text{int}B}\).
(f) There exists no IF\(\beta\)CS s A and B in X such that \(A \neq 1_\sim \neq B, B = \overline{\beta\text{int}A}\) and \(A = \overline{\beta\text{int}B}\).
Proof. Proof: (a) ⇒ (b) Assume that there exists an \( A \neq 0_\sim \) such that \( \beta cl A \neq 1_\sim \). Take \( A = \beta int(\beta cl A) \). Then \( A \) is proper \( \beta \)-regular open set in \( X \) which contradicts that \( X \) is IF\( \beta \)-super connectedness.

(b) ⇒ (c) Let \( A \neq 1_\sim \) be an IF\( \beta \)CS in \( X \). If we take \( B = \bar{A} \) then \( B \) is an IF\( \beta \)OS in \( X \) and \( B \neq 0_\sim \). Hence by (b) \( \beta cl B = 1_\sim \Rightarrow \beta cl \bar{B} = 0_\sim \Rightarrow \beta int(\bar{B}) = 0_\sim \Rightarrow \beta int A = 0_\sim \).

(c) ⇒ (d) Let \( A \) and \( B \) are IF\( \beta \)OS in \( X \) such that \( A \neq 0_\sim \neq B \) and \( A \subseteq \bar{B} \). Since \( \bar{B} \) is an IF\( \beta \)CS in \( X \), \( \bar{B} \neq 1_\sim \) by (c) \( \beta int \bar{B} = 0_\sim \). But \( A \subseteq \bar{B} \) implies \( 0_\sim \neq A = \beta int(A) \subseteq \beta int(\bar{B}) = 0_\sim \) which is a contradiction.

(d) ⇒ (a) Let \( \bar{A} \neq A \neq 1_\sim \) be an IF\( \beta \)-regular open set in \( X \). If we take \( B = \overline{\beta cl A} \), we get \( B \neq 0_\sim \). (If not \( B = 0_\sim \) implies \( \beta cl A = 0_\sim \Rightarrow A = \beta int(\beta cl A) = \beta int(1_\sim ) = 1_\sim \Rightarrow A = 1_\sim \) a contradiction to \( A \neq 1_\sim \).) We also have \( A \subseteq \bar{B} \) which is also a contradiction. Therefore \( X \) is IF\( \beta \)-super connected.

(a) ⇒ (e) Let \( A \) and \( B \) be two IF\( \beta \)OS in \( (X, \tau) \) such that \( A \neq 0_\sim \neq B \), \( B = \beta cl A \) and \( A = \beta cl B \). Now we have \( \beta int(\beta cl A) = \beta int(\bar{B}) = \beta cl \bar{B} = A \neq 0_\sim \) and \( A \neq 1_\sim \), since if \( A = 1_\sim \) then \( 1_\sim = \beta cl \bar{B} \Rightarrow \beta cl B = 0_\sim \Rightarrow B = 0_\sim \). But \( B \neq 0_\sim \). Therefore \( A \neq 1_\sim \Rightarrow A \) is proper IF\( \beta \)-regular open set in \( (X, \tau) \) which is contradiction to (a). Hence (e) is true.

(e) ⇒ (a) Let \( A \) be IF\( \beta \)OS in \( X \) such that \( A = \beta int(\beta cl A) \), \( 0_\sim \neq A \neq 1_\sim \). Now take \( B = \overline{\beta cl A} \). In this case, we get \( B \neq 0_\sim \) and \( B \) is an IF\( \beta \)OS in \( X \) and \( B = \overline{\beta cl A} \) and \( \beta cl B = \beta cl(\overline{\beta cl A}) = (\overline{\beta int(\beta cl A)}) = \beta int(\beta cl A) = A \). But this is a contradiction to (e). Therefore \( (X, \tau) \) is IF\( \beta \)-super connected space.

(e) ⇒ (f) Let \( A \) and \( B \) be IF\( \beta \)-closed sets in \( (X, \tau) \) such that \( A \neq 1_\sim \neq B \), \( B = \beta int A \) and \( A = \beta int B \). Taking \( C = \bar{A} \) and \( D = \bar{B} \), \( C \) and \( D \) become IF\( \beta \)-open sets in \( (X, \tau) \) and \( C \neq 0_\sim \neq D \), \( \beta cl C = \beta cl(\bar{A}) = (\overline{\beta int A}) = \beta int A = \bar{B} = D \) and similarly \( \beta cl D = C \). But this is a contradiction to (e). Hence (f) is true.

(f) ⇒ (e) We can prove this by the similar way as in (e) ⇒(f).

\[ \square \]

**Proposition 3.7.** Let \( f:(X,\tau)\to(Y,\sigma) \) be a IF\( \beta \)-irresolute surjection. If \( X \) is an IF\( \beta \)-super connected, then so is \( Y \).

**Proof.** Suppose that \( Y \) is IF\( \beta \)-super disconnected. Then there exists IF\( \beta \)OS’s \( C \) and \( D \) in \( Y \) such that \( C \neq 0_\sim \neq D \), \( C \subseteq \bar{D} \). Since \( f \) is IF\( \beta \)-irresolute, \( f^{-1}(C) \) and \( f^{-1}(D) \) are IF\( \beta \)OSs in \( X \) and \( C \subseteq \bar{D} \). Since \( f \) is IF\( \beta \)-irresolute, \( f^{-1}(C) \subseteq f^{-1}(D) \) which means that \( X \) is IF\( \beta \)-super disconnected which is a contradiction.

\[ \square \]

**Definition 3.9.** [4] Two intuitionistic fuzzy sets \( A \) and \( B \) are said to be \( q \) coincident (\( A \cap B \)) if and only if there exists an element \( x \in X \) such that \( \mu_A(x) > \nu_B(x) \) or \( \nu_A(x) < \mu_B(x) \).

**Definition 3.10.** [4] Two intuitionistic fuzzy sets \( A \) and \( B \) in \( X \) are said to be not \( q \) coincident (\( A \nsubseteq B \)) if and only if \( A \subseteq \bar{B} \).
Definition 3.11. [8] An IFTS \((X, \tau)\) is called intuitionistic fuzzy \(C_5\)-connected between two intuitionistic fuzzy sets \(A\) and \(B\) if there is no IFOS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B\).

Definition 3.12. An IFTS \((X, \tau)\) is called intuitionistic fuzzy \(\beta\)-connected between two intuitionistic fuzzy sets \(A\) and \(B\) if there is no IF\(\beta\)OS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B\).

Example 3.12. Let \(X = \{a, b\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\) where \(M = \{<x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.6})>; x \in X\}\), \((X, \tau)\) be IFTS. Consider the intuitionistic fuzzy sets \(A = \{<x, (\frac{a}{0.2}, \frac{b}{0.4}), (\frac{a}{0.5}, \frac{b}{0.6})>; x \in X\}, B = \{<x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.3})>; x \in X\}\) \(M\) is IF\(\beta\)OS in \((X, \tau)\). Then \((X, \tau)\) is intuitionistic fuzzy \(\beta\)-connected between \(A\) and \(B\).

Theorem 3.1. If an IFTS \((X, \tau)\) is an intuitionistic fuzzy \(\beta\)-connected between two intuitionistic fuzzy sets \(A\) and \(B\), then it is intuitionistic fuzzy \(C_5\)-connected between two intuitionistic fuzzy sets \(A\) and \(B\).

Proof. Suppose \((X, \tau)\) is not intuitionistic fuzzy \(C_5\)-connected between two intuitionistic fuzzy sets \(A\) and \(B\). Then there exists an IFOS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B\). Since every IFOS is IF\(\beta\)OS, there exists an IF\(\beta\)OS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B\). This implies \((X, \tau)\) is not intuitionistic fuzzy \(\beta\)-connected between \(A\) and \(B\), a contradiction to our hypothesis. Therefore, \((X, \tau)\) is intuitionistic fuzzy \(C_5\)-connected between \(A\) and \(B\). \(\square\)

Example 3.13. Let \(X = \{a, b\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}\) where \(M = \{<x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.4}, \frac{b}{0.3})>; x \in X\}\), \((X, \tau)\) be IFTS. Consider the intuitionistic fuzzy sets \(A = \{<x, (\frac{a}{0.3}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.6})>; x \in X\}, B = \{<x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.3}, \frac{b}{0.4})>; x \in X\}\) \(M\) is IFOS in \((X, \tau)\). Then \((X, \tau)\) is intuitionistic fuzzy \(C_5\)-connected between \(A\) and \(B\). Consider IFS \(C = \{<x, (\frac{a}{0.5}, \frac{b}{0.5}), (\frac{a}{0.5}, \frac{b}{0.5})>; x \in X\}\) \(C\) is an IF\(\beta\)OS such that \(A \subseteq C\) and \(C \subseteq B\) which implies \((X, \tau)\) is intuitionistic fuzzy \(\beta\)-disconnected between \(A\) an \(B\).

Theorem 3.2. Let \((X, \tau)\) be an IFTS and \(A\) and \(B\) be intuitionistic fuzzy sets in \((X, \tau)\). If \(A \sqsubset B\) then \((X, \tau)\) is intuitionistic fuzzy \(\beta\)-connected between \(A\) and \(B\).

Proof. Suppose \((X, \tau)\) is not intuitionistic fuzzy \(\beta\)-connected between \(A\) and \(B\). Then there exists an IF\(\beta\)OS \(E\) in \((X, \tau)\) such that \(A \subseteq E\) and \(E \subseteq B\). This implies \((X, \tau)\) is not intuitionistic fuzzy \(\beta\) connected between \(A\) and \(B\).

However, the converse of the above Theorem is need not be true, as shown by the following example.
Example 3.14. Let \( X = \{a, b\}, \tau = \{0, 1\}, M \) where \( M = \{<x, (\frac{a}{0.4} \cdot \frac{b}{0.4}), (\frac{a}{0.4} \cdot \frac{b}{0.4})>; x \in X\} \), \((X, \tau)\) be IFTS. Consider the intuitionistic fuzzy sets \( A = \{<x, (\frac{a}{0.2} \cdot \frac{b}{0.7}), (\frac{a}{0.2} \cdot \frac{b}{0.7})>; x \in X\} \), \( B = \{<x, (\frac{a}{0.5} \cdot \frac{b}{0.5}), (\frac{a}{0.5} \cdot \frac{b}{0.5})>; x \in X\} \). \( M \) is IF/βOS in \((X, \tau)\). Then \((X, \tau)\) is intuitionistic fuzzy \( β \)-connected between \( A \) an \( B \). But \( A \) is not \( q \)-coincident with \( B \), since \( \mu_A(x) < \nu_B(x) \).

Definition 3.13. Let \( N \) be an IFS in IFTS \((X, \tau)\)

(a) If there exists intuitionistic fuzzy \( β \)-open sets \( M \) and \( W \) in \( X \) satisfying the following properties, then \( N \) is called IF/β\( C_i \)-disconnected \((i = 1, 2, 3, 4)\):

\[ C_1 : N \subseteq M \cup W, M \cap W \subseteq \bar{N}, N \cap M \neq 0, \bar{N} \cap W \neq 0, \]
\[ C_2 : N \subseteq M \cup W, N \cap M \cap W = 0, N \cap M \neq 0, \bar{N} \cap W \neq 0, \]
\[ C_3 : N \subseteq M \cup W, M \cap W \subseteq \bar{N}, M \nsubseteq \bar{N}, W \nsubseteq \bar{N}, \]
\[ C_4 : N \subseteq M \cup W, N \cap M \cap W = 0, M \nsubseteq \bar{N}, W \nsubseteq \bar{N}, \]

(b) \( N \) is said to be IF/β\( C_i \)-connected \((i = 1, 2, 3, 4)\) if \( N \) is not IF/β\( C_i \)-disconnected \((i = 1, 2, 3, 4)\).

Obviously, we can obtain the following implications between several types of IF/β\( C_i \)-connected \((i = 1, 2, 3, 4)\):

\[
\text{IF/β} C_1 \text{-connectedness} \quad \rightarrow \quad \text{IF/β} C_2 \text{-connectedness} \\
\quad \downarrow \\
\text{IF/β} C_3 \text{-connectedness} \quad \rightarrow \quad \text{IF/β} C_4 \text{-connectedness}
\]

Example 3.15. Let \( X = \{a, b, c\}, \tau = \{0, 1\}, M, W \)
where \( M = \{<x, (\frac{a}{0.3} \cdot \frac{b}{0.1} \cdot \frac{c}{0.3}), (\frac{a}{0.6} \cdot \frac{b}{0.9} \cdot \frac{c}{0.7})>; x \in X\} \), \( W = \{<x, (\frac{a}{0.5} \cdot \frac{b}{0.3} \cdot \frac{c}{0.4}), (\frac{a}{0.5} \cdot \frac{b}{0.7} \cdot \frac{c}{0.8})>; x \in X\} \), \((X, \tau)\) be IFTS. Consider the IFS \( N = \{<x, (\frac{a}{0.3} \cdot \frac{b}{0.1} \cdot \frac{c}{0.2}), (\frac{a}{0.7} \cdot \frac{b}{0.9} \cdot \frac{c}{0.8})>; x \in X\} \), \( N \) is IF/β\( C_2 \)-connected IF/β\( C_3 \)-connected, IF/β\( C_4 \)-connected but IF/β\( C_1 \)-disconnected.

Example 3.16. Let \( X = \{a, b\}, \tau = \{0, 1\}, M, W, M \cup W, M \cap W \)
where \( M = \{<x, (\frac{a}{0.3} \cdot \frac{b}{0.9}), (\frac{a}{0.7} \cdot \frac{b}{0.1})>; x \in X\} \), \( W = \{<x, (\frac{a}{0.5} \cdot \frac{b}{0.7}), (\frac{a}{0.1} \cdot \frac{b}{0.3})>; x \in X\} \), \((X, \tau)\) be IFTS. Consider the IFS \( N = \{<x, (\frac{a}{0.2} \cdot \frac{b}{0.2}), (\frac{a}{0.8} \cdot \frac{b}{0.8})>; x \in X\} \), \( N \) is IF/β\( C_4 \)-connected but IF/β\( C_3 \)-disconnected.

Example 3.17. Let \( X = \{a, b\}, \tau = \{0, 1\}, M, W, M \cup W \)
where \( M = \{<x, (\frac{a}{0.3} \cdot \frac{b}{0.2}), (\frac{a}{0.7} \cdot \frac{b}{0.8})>; x \in X\} \), \( W = \{<x, (\frac{a}{0.5} \cdot \frac{b}{0.5}), (\frac{a}{0.5} \cdot \frac{b}{0.5})>; x \in X\} \), \((X, \tau)\) be IFTS. Consider the IFS \( N = \{<x, (\frac{a}{0.1} \cdot \frac{b}{0.1}), (\frac{a}{0.9} \cdot \frac{b}{0.9})>; x \in X\} \), \( M \) and \( W \) are intuitionistic fuzzy \( β \)-open sets in \( X \). Then \( N \) is IF/β\( C_4 \)-connected but IF/β\( C_2 \)-disconnected.
4 Intuitionistic Fuzzy $\beta$-Extremally Disconnect-
edness in Intuitionistic Fuzzy Topological Spaces

**Definition 4.1.** Definition. Let $(X, \tau)$ be any IFTS. $X$ is called IF$\beta$-extremally disconnected if the $\beta$-closure of every IF$\beta$OS in $X$ is IF$\beta$OS.

**Theorem 4.1.** For an IFTS $(X, \tau)$ the following are equivalent:
(i) $(X, \tau)$ is an IF$\beta$-extremally disconnected space.
(ii) For each IF$\beta$CS $A$, $\beta\text{int}(A)$ is an IF$\beta$CS.
(iii) For each IF$\beta$OS $A$, $\beta\text{cl}(A) = \beta\text{cl}(\beta\text{cl}(A))$
(iv) For each intuitionistic fuzzy $\beta$-open sets $A$ and $B$ with $\beta\text{cl}(A) = \overline{B}$, $\beta\text{cl}(A) = \beta\text{cl}\beta B$

**Proof.** (i)$\Rightarrow$(ii) Let $A$ be any IF$\beta$CS. Then $\overline{A}$ is an IF$\beta$OS. So $\beta\text{cl}(\overline{A}) = \overline{\beta\text{int}A}$ is an IF$\beta$OS. Thus $\beta\text{int}(A)$ is an IF$\beta$CS in $(X, \tau)$.

(ii)$\Rightarrow$(iii) Let $A$ be an IF$\beta$OS. Then $\beta\text{cl}(\beta\text{cl}(A)) = \beta\text{cl}(\beta\text{int}(A))$. Since $A$ is an IF$\beta$OS, $\overline{A}$ is an IF$\beta$CS. So by (ii) $\beta\text{int}(\overline{A})$ is an IF$\beta$CS. That is $\beta\text{cl}(\beta\text{int}(\overline{A})) = \beta\text{int}(\overline{A})$. Hence $\beta\text{cl}(\beta\text{int}(\overline{A})) = \beta\text{int}(\overline{A}) = \beta\text{cl}(A)$.

(iii)$\Rightarrow$(iv) Let $A$ and $B$ be any two intuitionistic fuzzy $\beta$-open sets in $(X, \tau)$ such that $\beta\text{cl}(A) = B$. (iii) implies $\beta\text{cl}(A) = \beta\text{cl}(\beta\text{cl}(A)) = \beta\text{cl}(\overline{B}) = \beta\text{cl}\beta B$.

(iv)$\Rightarrow$(i) Let $A$ be any IF$\beta$OS in $(X, \tau)$. Put $B = \overline{\beta\text{cl}A}$. Then $\beta\text{cl}(A) = \overline{B}$. Hence by (iv) $\beta\text{cl}(A) = \beta\text{cl}\beta B$. Therefore $\beta\text{cl}(A)$ is IF$\beta$OS in $(X, \tau)$. That is $(X, \tau)$ is an IF$\beta$-extremally disconnected space.

**References**


