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GENERALIZED DERIVATIVES OF AN ARBITRARY ORDER AND THE BOUNDARY PROPERTIES OF DIFFERENTIATED POISSON INTEGRALS FOR THE HALF-SPACE $\mathbb{R}_+^{k+1}$ $(k > 1)$

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Abstract. The notions of a generalized differential and a generalized spherical derivative of an arbitrary order are introduced for a function of several variables and Fatou type theorems are proved on the boundary properties of partial derivatives of an arbitrary order of the Poisson integral for the half-space, when the integral density has a generalized differential or a generalized spherical derivative.

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This work continues the author's papers [1–3], in which the notions of the generalized partial derivative and the generalized Laplace operator were introduced for a function of several variables and Fatou type theorems were proved concerning the boundary behavior of partial derivatives of any order of Poisson integral on a half-space; the examples were given showing that the results obtained cannot be improved.

1. Notation and Definitions

In addition to the notation from [1–3], the following notation is used below:

$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_s)$, $\beta = (\beta_1, \beta_2, \dots, \beta_s)$, where α_i and β_i are non-negative integer numbers, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_s$, $\alpha! = \alpha_1!\alpha_2! \cdots \alpha_s!$;

if $B = \{i_1, i_2, \dots, i_s\} \subset M = \{1, 2, \dots, k\}$, $1 \le s \le k$ ($i_l < i_r$ for $l < r$), then $\overline{x}_B = (x_{i_1}, x_{i_2}, \dots, x_{i_s}) \in \mathbb{R}^s$ and $\overline{x}_B^\alpha = x_{i_1}^{\alpha_1} x_{i_2}^{\alpha_2} \cdots x_{i_s}^{\alpha_s}$;

$m(\alpha)$ is the number of elements of a set $A \subset M$;

$D^{(\alpha)} = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} \cdots \left(\frac{\partial}{\partial x_s}\right)^{\alpha_s}$;

$\widetilde{L}(\mathbb{R}^k)$ is the set of functions $f(x) = f(x_1, x_2, \dots, x_k)$ such that

$$\frac{f(x)}{(1+|x|^2)^{\frac{k+1}{2}}} \in L(\mathbb{R}^k); \quad \mathbb{R}_+^{k+1} = \Big\{(x, x_{k+1}) \in \mathbb{R}^{k+1}: \ x \in \mathbb{R}^k, \ x_{k+1} > 0\Big\};$$

$\Delta_x = \Delta = \sum\limits_{i=1}^{k} \frac{\partial^2}{x^2}$ and $\Delta^r = \Delta(\Delta^{r-1})$, where $\Delta^1 = \Delta$, $r = 2, 3, \dots$;

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$U(f;x,x_{k+1})$ is the Poisson integral of the function $f(x)$ in the half-space $\mathbb{R}^{k+1}_+$, i.e.,

$$U(f;x,x_{k+1})=\frac{\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}}\int\limits_{\mathbb{R}^k}P(t-x,x_{k+1})f(t)dt,$$
$$P(t-x,x_{k+1})=\frac{x_{k+1}}{(|t-x|^2+x_{k+1}^2)^{\frac{k+1}{2}}}.$$

In what follows it will always be assumed that the considered functions belong to $\widetilde{L}(R^k)$. The following generalized derivatives of the function $f(x)$ will be considered:

1. Let $A\subset M$; $B\subset M$. If there exist functions $a_\beta(\overline{x}_B)$ ($m(\beta)=m(A)$, $|\beta|\le r-1$; if $B=\varnothing$, then $a_\beta(\overline{x}_B)=a_\beta=\text{const}$) and numbers a_α ($m(\alpha)=m(A)$, $|\alpha|=r$) are such that there are limits $\lim_{\overline{x}_B\to\overline{x}_B^0}a_\beta(\overline{x}_B)=a_\beta$ and, in the neighborhood of the point x^0, we have the representation

$$f(x_B+x^0_{B'}+t_A)=\sum_{i=0}^{r-1}\sum_{|\beta|=i}\frac{a_\beta(\overline{x}_B)}{\beta!}\overline{t}_A^{\beta}+\sum_{|\alpha|=r}\frac{a_\alpha}{\alpha!}\overline{t}_A^{\,\alpha}+\varepsilon(\overline{t}_A,\overline{x}_B)\frac{|\overline{t}_A|^r}{r!},$$

where

$$\lim_{\substack{|\overline{t}_A|\to0\\ \overline{x}_B\to\overline{x}_B^0}}\varepsilon(\overline{t}_A,\overline{x}_B)=0,$$

then the value $r!\sum\limits_{|\alpha|=r}\frac{a_\alpha}{\alpha!}\overline{t}_A^{\,\alpha}$ is called a generalized differential of order r of the function $f(x)$ at the point x^0 with respect to variables whose indices form the set A. This differential is denoted by

$$d^{(r)}_{A(B)}f(x^0)$$
$$\Big(d^{(r)}_{A(\varnothing)}f(x^0)=d^{(r)}_{A}f(x^0),\quad d^{(r)}_{M}f(x^0)=d^{(r)}f(x^0);$$
$$d^{(r)}_{A(M)}f(x^0)=\overline{d}^{\,(r)}_{A}f(x^0),\quad \overline{d}^{\,(r)}_{M}f(x^0)=\overline{d}^{\,(r)}f(x^0)\Big),$$

(for $B=\varnothing$ similar definitions see in [4], p. 313 and [5]). a_α is called a generalized mixed derivative of the function $f(x)$ of order $r=|\alpha|$ at the point x^0 with respect to variables whose indices form the set A. This derivative is denoted by

$$D^{(\alpha)}_{A(B)}f(x^0)$$
$$\Big(D^{(\alpha)}_{A(\varnothing)}f(x^0)=D^{(\alpha)}_{A}f(x^0),\quad D^{(\alpha)}_{M}f(x^0)=D^{(\alpha)}f(x^0);$$
$$D^{(\alpha)}_{A(M)}f(x^0)=\overline{D}^{\,(\alpha)}_{A}f(x^0),\quad \overline{D}^{\,(\alpha)}_{M}f(x^0)=\overline{D}^{\,(\alpha)}f(x^0)\Big).$$

2. Let r be an even number. If there exist functions $b_\beta(\overline{x}_B)$ ($|\beta|$ is even, $m(\beta)=m(A)$, $|\beta|\le r-2$; if $B=\varnothing$, then $b_\beta(\overline{x}_B)=b_\beta=\text{const}$) and numbers b_α ($|\alpha|=r$) such that there are limits $\lim_{\overline{x}_B\to\overline{x}_B^0}b_\beta(\overline{x}_B)=b_\beta$ and, in the

neighborhood of the point x^0 we have the equality

$$\frac{1}{2}\Big[f(x_B+x^0_{B'}+t_A)+f(x_B+x^0_{B'}-t_A)\Big]=\sum_{i=0}^{\frac{r-2}{2}}\sum_{|\beta|=2i}\frac{1}{\beta!}b_\beta(\overline{x}_B)\overline{t}_A^{\,\beta}$$
$$+\sum_{|\alpha|=r}\frac{1}{\alpha!}b_\alpha\overline{t}_A^{\,\alpha}+\varepsilon(\overline{t}_A,\overline{x}_B)\frac{\overline{t}_A|^r}{r!},$$

where

$$\lim_{\substack{|\overline{t}_A|\to 0\\ \overline{x}_B\to\overline{x}^0_B}}\varepsilon(\overline{t}_A,\overline{x}_B)=0,$$

then $r!\sum\limits_{|\alpha|=r}\frac{b_\alpha}{\alpha!}\overline{t}_A^{\,\alpha}$ is called a generalized symmetric differential of order r of the function $f(x)$ at the point x^0 with respect to variables whose indices form the set A. This differential is denoted by

$$d^{*(r)}_{A(B)}f(x^0)$$
$$\Big(d^{*(r)}_{A(\varnothing)}f(x^0)=d^{*(r)}_{A}f(x^0),\quad d^{*(r)}_{M}f(x^0)=d^{*(r)}f(x^0);$$
$$d^{*(r)}_{A(M)}f(x^0)=\overline{d}^{\,*(r)}_{A}f(x^0),\quad \overline{d}^{\,*(r)}_{M}f(x^0)=\overline{d}^{\,*(r)}f(x^0)\Big).$$

For odd r the definition is as above with the only difference that the sum $f(x_B+x^0_{B'}+t_A)+f(x_B+x^0_{B'}-t_A)$ is to be replaced by the difference $f(x_B+x^0_{B'}+t_A)-f(x_B+x^0_{B'}-t_A)$ (for $B=\varnothing$ the definitions are similar; see [4], p. 315 and [5]).

b_α is called a generalized mixed symmetric derivative of the function $f(x)$ of order $r=|\alpha|$ at the point x^0 with respect to variables whose indices form the set A. This derivative is denoted by

$$D^{*(\alpha)}_{A(B)}f(x^0)$$
$$\Big(D^{*(\alpha)}_{A(\varnothing)}f(x^0)=D^{*(\alpha)}_{A}f(x^0),\quad D^{*(\alpha)}_{M}f(x^0)=D^{*(\alpha)}f(x^0);$$
$$D^{*(\alpha)}_{A(M)}f(x^0)=\overline{D}^{\,*(\alpha)}_{A}f(x^0),\quad \overline{D}^{\,*(\alpha)}_{M}f(x^0)=\overline{D}^{\,*(\alpha)}f(x^0)\Big).$$

3. Now let us introduce the notion of a spherical Laplace Ω-operator. Assume that $\Omega(t)$, $t\in\mathbb{R}^k$, $|t|=1$ is a spherical harmonic, i.e., the restriction of a harmonic homogeneous operator $Q(x)\not\equiv 0$ of order ν, $\nu=0,1,\dots$, $x\in\mathbb{R}^k$, to the unit sphere. Let the function $f(x)$ be given in the neighborhood of the point x^0 and integrable on the spheres $|x-x^0|=\rho$ for all sufficiently small $\rho>0$. Let B be any subset of the set M. If there exist functions $a_i(\overline{x}_B)$ (if $B=\varnothing$, then $a_i(\overline{x}_B)=a_i=\text{const}$), $i=\overline{0,r-1}$, given in the neighborhood of the point x^0, and a number a_r such that there exist limits $\lim_{\overline{x}_B\to\overline{x}^0_B}a_i(\overline{x}_B)=a_i$, and, in the neighborhood of x^0, we have limits

$$\frac{1}{|S_\rho|}\int\limits_{S_\rho(x_B+x^0_{B'})}f(t)\Omega\Big(\frac{t}{|t|}\Big)dS(t)$$

$$= \frac{\Gamma(k/2)}{2\pi k/2} \int\limits_S f(x_B + x^0_{B'} + \rho t)\Omega(t)d\omega(t)$$

$$= \sum_{i=0}^{r-1} a_i(\overline{x}_B)\rho^{\nu+2i} + \Big[a_r + \varepsilon(\rho, \overline{x}_B)\Big]\rho^{\nu+2r},$$

where $\lim\limits_{\substack{\rho\to 0\\ \overline{x}_B\to\overline{x}^0_B}} \varepsilon(\rho, \overline{x}_B) = 0$, $dS(t)$ is an element of the $(k-1)$-dimensional area of the sphere S_ρ ($d\omega(t)$ is an element of the $(k-1)$-dimensional area of the sphere $|t| = 1$),

$$a_r = \frac{\Gamma(k/2)}{r!2^{\nu+2r}\Gamma(k/2+\nu+r)} A_r,$$

then the number A_r is called a spherical Laplace Ω-operator of order r of the function $f(x)$ at the point x^0 and denoted by $\Omega\overline{\Delta}^r_{x_B} f(x^0)$. For $B = \varnothing$ we will write $\Omega\overline{\Delta}^r_{x_B} f(x^0) = \Omega\overline{\Delta}^r f(x^0)$. If $B = M$, then it is assumed that $\Omega\overline{\Delta}^r_{x_M} f(x_0) = \Omega\overline{\Delta}^r_x f(x^0)$ (for $k = 2$ and $\Omega(t) \equiv 1$ the derivatives $\Omega\overline{\Delta}^r f(x_0)$ were considered in [6], while for $k = 2$ and $\Omega(t) = t_1 + t_2$ in [7], for $k = 2$ and $\Omega(t) = t_1 t_2$ in [8], and, in the general case, where $B = \varnothing$, in [9]).

We know (see [9], p. 222) that if the function $f(x)$, $x \in \mathbb{R}^k$, has all partial derivatives of order $\nu + 2r + 1$ in the neighborhood of the point x^0, then

$$\Omega\overline{\Delta}^r f(x_0) = Q(\mathrm{grad})\Delta^r f(x^0), \tag{1}$$

where the operator $Q(\mathrm{grad})$ is obtained by replacing the coordinates x_i of the point x in the polynomial $Q(x)$ by operators $\frac{\partial}{\partial x_i}$, $1 \le i \le k$. From (1), in particular, it follows that

$$\Omega\overline{\Delta}^r \Big[\frac{1}{(|t|^2 + x^2_{k+1})^{\frac{k+1}{2}}}\Big] = Q(t)\frac{\mathcal{I}(t_1, t_2, \dots, t_k, x_{k+1})}{(|t|^2 + x^2_{k+1})^{\frac{k+4r+2\nu+1}{2}}}, \tag{2}$$

where $\mathcal{I}(t_1, t_2, \dots, t_k, x_{k+1})$ is a homogeneous polynomial of degree $2r$ of $(t_1, t_2, \dots, t_k, x_{k+1})$.

The generalized derivatives discussed above are used when studying the boundary properties of differentiated Poisson integrals in the space $\mathbb{R}^{k+1}_+$.

2. Formulations of the Results

Theorem 1. *Let $A \subset M$, $B \subset M$ and $B' \subset A$. If $f(x)$ has a generalized differential $d^{(r)}_{A(B)} f(x^0)$ of order r at the point x^0, then*

$$\lim_{(x,x_{k+1}) \underset{x_{A\setminus B}}{\overset{\wedge}{\to}} (x^0,0)} \frac{\partial^r U(f; x, x_{k+1})}{\partial \overline{x}^\alpha_A} = D^{(\alpha)}_{A(B)} f(x^0)$$

for any α ($|\alpha| = r$; $m(\alpha) = m(A)$).

Theorem 2. *Let $A \subset M$, $B \subset M$ and $B' \subset A$. If $f(x)$ has a generalized symmetric differential $d^{*(r)}_{A(B)} f(x^0)$ of order r, then*

$$\lim_{(x_B + x^0_{B'}, x_{k+1}) \to (x^0, 0)} \frac{\partial^r U(f; x_B + x^0_{B'}, x_{k+1})}{\partial \overline{x}^{\alpha}_A} = D^{*(\alpha)}_{A(B)} f(x^0)$$

for any α ($|\alpha| = r$; $m(\alpha) = m(A)$).

Theorem 3. *If at the point x^0 there exists a symmetric differential $\overline{d}^{*(r)}_A f(x^0)$ of order r, then*

$$\lim_{(x, x_{k+1}) \to (x^0, 0)} \frac{\partial^r U(f; x, x_{k+1})}{\partial \overline{x}^{\alpha}} = \overline{D}^{*(\alpha)}_A f(x^0)$$

for any α ($|\alpha| = r$; $m(\alpha) = m(A)$).

Theorem 4. *Let $B \subset M$. If at the point x^0 there exists a finite derivative $\Omega \overline{\Delta}^r_{x_B} f(x^0)$, then*

$$\lim_{(x_B + x^0_{B'}, x_{k+1}) \to (x^0, 0)} \Omega \overline{\Delta}^r U(f; x_B + x^0_{B'}, x_{k+1}) = \Omega \overline{\Delta}^r_{x_B} f(x^0).$$

Corollary 1. *If at the point x^0 there exists a finite derivative $\Omega \overline{\Delta}^r f(x^0)$, then*

$$\lim_{x_{k+1} \to 0+} \Omega \overline{\Delta}^r U(f; x^0, x_{k+1}) = \Omega \overline{\Delta}^r f(x^0).$$

Corollary 2. *If at the point x^0 there exists a finite derivative $\Omega \overline{\Delta}^r_x f(x^0)$, then*

$$\lim_{(x, x_{k+1}) \to (x^0, 0)} \Omega \overline{\Delta}^r U(f; x, x_{k+1}) = \Omega \overline{\Delta}^r_x f(x^0).$$

Remark 1. An analogue of Theorem 4 is true if a more general spherical Laplace Q-operator is used instead of the spherical Laplace Ω-operator (with respect to the ball). The notion of a spherical Laplace Q-operator was introduced in [10], where it is shown that if the integrable function has a spherical Laplace Ω-operator of order r, then it also has a spherical Laplace Q-operator of order r with the same value. The converse statement, generally speaking, does not hold.

Remark 2. For the examples illustrating that the above theorems do not hold for a modified type of a generalized derivative see [1–3].

3. Auxiliary Statements

Lemma 1. *For any* $(x, x_{k+1}) \in \mathbb{R}_+^{k+1}$, $r \in N$, $A \subset M$, $D \subset M$ *and* α $(|\alpha| = r$, $m(\alpha) = m(A))$ *the following statements are true:*

1) $\mathcal{I}_\beta^{(\alpha)} = \int\limits_{\mathbb{R}^{m(A)}} \dfrac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} \overline{t}_A^\beta d\overline{t}_A = 0$, *where* $m(\beta) = m(\alpha)$, $|\beta| \le |\alpha|$, $\beta \ne \alpha$;

2) $\mathcal{I}_r^{(\alpha)} = \dfrac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \dfrac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} \dfrac{\overline{t}_A^\alpha}{\alpha!} dt = 1$;

3) $\int\limits_{\mathbb{R}^k} \left| \dfrac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^\alpha} \right| |t|^r dt < C$;[1]

4) $\int\limits_{\mathbb{R}^k} \left| \dfrac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} \right| |\overline{t}_D|^r dt < C$ *for* $\frac{x_{k+1}}{|\overline{x}_D|} \ge C > 0$;

5) $\sup\limits_{|t|>\delta>0} \left| \dfrac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^\alpha} \right| (|t|^2 + x_{k+1}^2)^{\frac{k+1}{2}} |t|^\nu < C x_{k+1}$, $\nu = \overline{0, r}$.

Proof. Relation 1) is proved by induction. One can easily prove the equality $\mathcal{I}_\beta^{(\alpha)} = 0$ when $r = 2$, $\alpha_i = \alpha_j = 1$, $i \ne j$, $|\beta| \le 2$ and $\beta \ne \alpha$. Let us assume that for any α $(m(\alpha) = m(A))$ and β $(m(\alpha) = m(\beta)$, $|\beta| \le \alpha$, $\beta \ne \alpha)$ we have

$$\mathcal{I}_\beta^{(\alpha)} = \int\limits_{\mathbb{R}^{m(A)}} \frac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} \overline{t}_A^\beta d\overline{t}_A$$

$$= \int\limits_{\mathbb{R}^{m(A)-1}} \overline{t}_{A\setminus\nu}^\delta d\overline{t}_{A\setminus\nu} \int\limits_{-\infty}^{\infty} \frac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} t_\nu^s dt_\nu = 0,$$

where $s \in \{0, 1, 2, \dots, |\beta|\}$, $|\delta| = |\beta| - s$, and show that

$$\mathcal{I}_\beta^{(\gamma)} = \int\limits_{\mathbb{R}^{m(A)}} \frac{\partial^{r+1} P(t-x, x_{k+1})}{\partial \overline{x}_A^\gamma} \overline{t}_A^\beta d\overline{t}_A = 0$$

holds, where $|\gamma| = |\alpha| + 1$, $|\beta| \le \gamma$ and $\beta \ne \gamma$).

Indeed, applying the integration by parts, we obtain

$$\int\limits_{-\infty}^{\infty} \frac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} t_\nu^s dt_\nu$$

$$= (-1)^q \int\limits_{-\infty}^{\infty} \frac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_{A\setminus\nu}^{\alpha'} \partial t_\nu^q} t_\nu^s dt_\nu$$

[1]Here and in what follows C denotes the absolute positive constants which, generally speaking, can take different values in different relations.

$$= \frac{1}{s+1} \int\limits_{-\infty}^{\infty} \frac{\partial^{r+1} P(t-x, x_{k+1})}{\partial \overline{x}_{A\setminus\nu}^{\alpha'} \partial x_\nu^{q+1}} t_\nu^{s+1} dt_\nu.$$

which implies

$$0 = \frac{1}{s+1} \int\limits_{\mathbb{R}^{m(A)-1}} \overline{t}_{A\setminus\nu}^{\delta} d\overline{t}_{A\setminus\nu} \int\limits_{-\infty}^{\infty} \frac{\partial^{r+1} P(t-x, x_{k+1})}{\partial \overline{x}_{A\setminus\nu}^{\alpha'} \partial x_\nu^{q+1}} t_\nu^{s+1} dt_\nu$$
$$= \frac{1}{s+1} \int\limits_{\mathbb{R}^{m(A)}} \frac{\partial^{r+1} P(t-x, x_{k+1})}{\partial \overline{x}_A^{\gamma}} \overline{t}_A^{\beta} d\overline{t}_A.$$

Thus $\mathcal{I}_\beta^{(\gamma)} = 0$ for $|\beta| > 0$ and also for $|\beta| = 0$.

Relation 2) is also proved by induction. For $r = 2$, $\alpha_i = \alpha_j = 1$, $i \neq j$, the equality $\mathcal{I}_2^{(\alpha)} = 1$ is obtained from

$$\mathcal{I}_2^{(\alpha)} = \frac{\Gamma(\frac{k+1}{2})(k+1)(k+3)x_{k+1}}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{(t_i - x_i)(t_j - x_j)t_i t_j}{[|t-x|^2 + x_{k+1}^2]^{\frac{k+5}{2}}} dt$$
$$= \frac{\Gamma(\frac{k+1}{2})(k+1)(k+3)x_{k+1}}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{t_i^2 t_j^2 dt}{(|t|^2 + x_{k+1}^2)^{\frac{k+5}{2}}} = 1.$$

Now assume that the equality

$$\mathcal{I}_n^{(\alpha)} = \frac{\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^n P(t-x, x_{k+1})}{\partial \overline{x}_A^{\alpha}} \cdot \frac{\overline{t}_A^{\alpha}}{\alpha!} dt = 1$$

is fulfilled for any $r = n$ and any $|\alpha| = r$.

Now we can show that

$$\mathcal{I}_{n+1}^{(\gamma)} = \frac{\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^{n+1} P(t-x, x_{k+1})}{\partial \overline{x}_A^{\gamma}} \cdot \frac{\overline{t}_A^{\gamma}}{\gamma!} dt = 1$$

holds for $|\gamma| = |\alpha| + 1$.

Indeed, applying the integration by parts, we obtain

$$1 = \mathcal{I}_n^{(\alpha)} = \frac{\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^{k-1}} \frac{\overline{t}_{A\setminus\nu}^{\alpha'}}{\alpha!} d\overline{t}_{A\setminus\nu} \int\limits_{-\infty}^{\infty} \frac{\partial^n P(t-x, x_{k+1})}{\partial \overline{x}_A^{\alpha}} \cdot t_\nu^s dt_\nu$$
$$= \frac{\Gamma(\frac{k+1}{2})}{\pi^{\frac{k+1}{2}}} (-1)^{2(s+1)} \int\limits_{\mathbb{R}^k} \frac{\partial^{n+1} P(t-x, x_{k+1})}{\partial \overline{x}_A^{\gamma}} \cdot \frac{\overline{t}_A^{\gamma}}{\gamma!} dt = \mathcal{I}_{n+1}^{(\gamma)}.$$

Now let us prove that statement 3) is valid. We have

$$|t|^r \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\alpha}} = |t|^r x_{k+1} \frac{\partial^r}{\partial \overline{t}_A^{\alpha}} \Big[\frac{1}{(|t|^2 + x_{k+1}^2)^{\frac{k+1}{2}}} \Big]$$
$$= |t|^r x_{k+1} \frac{\mathcal{I}(t_1, t_2, \ldots, t_k, x_{k+1})}{(|t|^2 + x_{k+1}^2)^{\frac{k+2r+1}{2}}},$$

where $\mathcal{I}(t_1,t_2,\dots,t_k,x_{k+1})$ is a homogeneous polynomial of degree r of $(t_1,t_2,\dots,t_k,x_{k+1})$.

Passing to the spherical coordinates, we obtain

$$\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||t|^r dt < cx_{k+1}\int\limits_0^\infty \frac{T(\rho,x_{k+1})\rho^{r+k-1}}{(\rho^2+x_{k+1}^2)^{\frac{k+2r+1}{2}}}d\rho,$$

where $T(\rho,x_{k+1})>0$ is a homogeneous polynomial of degree r of (ρ,x_{k+1}).

Applying the substitution of $\rho=x_{k+1}\rho_1$, we have

$$\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||t|^r dt < C\sum_{\nu=0}^{r}\int\limits_0^\infty \frac{\rho_1^{k+r+\nu-1}d\rho_1}{(1+\rho_1^2)^{\frac{k+2r+1}{2}}} = O(1).$$

Statement 4) follows from 3) and from the fact that

$$|\bar{t}_D+\overline{x}_D|^r \le 2^r\Big(|\bar{t}_D|^r+|\overline{x}_D|^r\Big).$$

Indeed,

$$\begin{aligned}\int\limits_{\mathbb{R}^k}&\Big|\frac{\partial^r P(t-x,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||\bar{t}_D|^r dt\\ &=\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||\bar{t}_D+\overline{x}_D|^r dt \le 2^r\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||t|^r dt\\ &+2^r\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||\overline{x}_D|^r dt \le C + C\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||\overline{x}_D|^r dt.\end{aligned}$$

Passing to the spherical coordinates, we obtain

$$\mathcal{I}=\int\limits_{\mathbb{R}^k}\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big||\overline{x}_D|^r dt \le Cx_{k+1}|\overline{x}_D|^r\int\limits_0^\infty \frac{T(\rho,x_{k+1})\rho^{k-1}d\rho}{(\rho^2+x_{k+1}^2)^{\frac{k+2r+1}{2}}},$$

where $T(\rho,x_{k+1})>0$ is a homogeneous polynomial of degree r of (ρ,x_{k+1}).

Applying the substitution of $\rho=x_{k+1}\rho_1$, we get

$$\begin{aligned}\mathcal{I}&\le C|\overline{x}_D|^r x_{k+1}^{r+k+1}\sum_{\nu=0}^{r}\int\limits_0^\infty \frac{\rho_1^{k+\nu-1}d\rho_1}{x_{k+1}^{k+2r+1}(1+\rho_1^2)^{\frac{k+2r+1}{2}}}\\ &=C\Big(\frac{|\overline{x}_D|}{x_{k+1}}\Big)^r\sum_{\nu=0}^{r}\int\limits_0^\infty \frac{\rho_1^{k+\nu-1}d\rho_1}{1+\rho_1^2)^{\frac{k+2r+1}{2}}} < C\end{aligned}$$

for $\frac{x_{k+1}}{|\overline{x}_D|}\ge C>0$.

Therefore statement 4) is proved.

Statement 5) follows from the inequality

$$\Big|\frac{\partial^r P(t,x_{k+1})}{\partial\bar{t}_A^{\,\alpha}}\Big|(|t|^2+x_{k+1}^2)^{\frac{k+1}{2}}|t|^\nu$$

$$< x_{k+1}\frac{|\mathcal{I}(t_1,t_2,\dots,t_k,x_{k+1})|}{(|t|^2+x_{k+1}^2)^{r-\frac{\nu}{2}}},\quad \nu=\overline{0,r},$$

since $\mathcal{I}(t_1,t_2,\dots,t_k,x_{k+1})$ is a homogeneous polynomial of degree r of $(t_1,t_2,\dots,t_k,x_{k+1})$. □

By virtue of equality (2) our next lemma is proved similarly to Lemma 1.

Lemma 2. *For any integer numbers $\nu\ge 0$, $r\ge 1$ and $k\ge 1$ the following statements are valid:*

1) $\displaystyle\int_0^\infty\Big(\frac{\Omega\overline{\Delta}^r_{\rho\theta}P}{\Omega(\frac{t}{|t|})}\Big)_{x=0}\rho^{\nu+k+2i-1}d\rho=0,\ i=\overline{0,r-1};$[2]

2) $\displaystyle\frac{2\Gamma\big(\frac{k+1}{2}\big))}{\sqrt{\pi}r!2^{\nu+2r}\Gamma\big(\frac{k}{2}+\nu+r\big)}\int_0^\infty\Big(\frac{\Omega\overline{\Delta}^r_{\rho\theta}P}{\Omega(\frac{t}{|t|})}\Big)_{x=0}\rho^{\nu+k+2r-1}d\rho=1;$

3) $\displaystyle\int_0^\infty\Big|\Big(\frac{\Omega\overline{\Delta}^r_{\rho\theta}P}{\Omega(\frac{t}{|t|})}\Big)_{x=0}\Big|\rho^{\nu+k+2r-1}d\rho<C;$

4) $\displaystyle\sup_{\rho\ge\delta>0}\Big|\Big(\frac{\Omega\overline{\Delta}^r_{\rho\theta}P}{\Omega(\frac{t}{|t|})}\Big)_{x=0}\Big|\rho^{k+1}<Cx_{k+1};$

5) $\displaystyle\int_\delta^\infty\Big|\Big(\frac{\Omega\overline{\Delta}^r_{\rho\theta}P}{\Omega(\frac{t}{|t|})}\Big)_{x=0}\Big|\rho^{\nu+k+2i-1}d\rho<Cx_{k+1}$ *holds for any $\delta>0$, $i=\overline{0,r}$.*

4. Proofs of the Theorems

Proof of Theorem 1. Let $x^0=0$. By virtue of statements 1) and 2) of Lemma 1 we have

$$\frac{\partial^2U(f;x,x_{k+1})}{\partial\overline{x}_A^\alpha}=\frac{\Gamma\big(\frac{k+1}{2}\big)}{\pi^{\frac{k+1}{2}}}\int_{\mathbb{R}^k}\frac{\partial^rP(t-x,x_{k+1})}{\partial\overline{x}_A^\alpha}f(t)dt$$

$$=\frac{\Gamma\big(\frac{k+1}{2}\big)}{\pi^{\frac{k+1}{2}}}\int_{\mathbb{R}^k}\frac{\partial^rP(t-x+x_{A\cap B},x_{k+1})}{\partial\overline{x}_A^\alpha}f(t+x_{A\cap B})dt$$

$$=\frac{\Gamma\big(\frac{k+1}{2}\big)}{\pi^{\frac{k+1}{2}}}\int_{\mathbb{R}^k}\frac{\partial^rP(t-x+x_{A\cap B},x_{k+1})}{\partial\overline{x}_A^\alpha}|\overline{t}_A|^r\Big[f(t+x_{A\cap B})$$

$$-\sum_{i=0}^{r-1}\sum_{|\beta|=i}\frac{a_\beta(\overline{x}_{A\cap B},\overline{t}_{A'})}{\beta!}\overline{t}_A^{\,\beta}-\sum_{|\beta|=r}\frac{a_\beta}{\beta!}\overline{t}_A^{\,\beta}\Big]\frac{dt}{|\overline{t}_A|^r}$$

$$+D^{(\alpha)}_{A(B)}f(0)=C\Big(\int_{V_\delta}+\int_{CV_\delta}\Big)+D^{(\alpha)}_{A(B)}f(0)$$

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[2] $\Delta_{\rho\theta}$ is Laplace operator in terms of spherical coordinates $(\rho,\vartheta_1,\vartheta_2,\dots,\vartheta_{k-2},\varphi)$.

$$= C(\mathcal{I}_1 + \mathcal{I}_2) + D^{(\alpha)}_{A(B)} f(0),$$

where V_δ is the ball with center at the point 0 and radius δ. Let $\varepsilon > 0$. Choose $\delta > 0$ such that

$$\Big| f(t + x_{A\cap B}) - \sum_{i=0}^{r-1} \sum_{|\beta|=i} \frac{a_\beta(\overline{x}_{A\cap B}, \overline{t}_{A'})}{\beta!} \overline{t}_A^{\,\beta} - \sum_{|\beta|=r} \frac{a_\beta}{\beta!} \overline{t}_A^{\,\beta} \Big| \frac{1}{|\overline{t}_A|^r} < \varepsilon$$

for $|t + x_{A\cap B}| < \delta$.

Now we have

$$|\mathcal{I}_1| \le C\varepsilon \int\limits_{V_\delta} \Big| \frac{\partial^r P(t - x + x_{A\cap B}, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| |\overline{t}_A|^r dt$$
$$\le C\varepsilon \int\limits_{\mathbb{R}^k} \Big| \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| \Big(\sqrt{\sum_{i\in A\cap B} t_i^2 + \sum_{i\in B'} (t_i + x_i)^2} \Big)^r dt$$
$$\le C\varepsilon \int\limits_{\mathbb{R}^k} \Big| \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| \Big(\sqrt{\sum_{i\in A\cap B} t_i^2} + \sqrt{\sum_{i\in B'} (t_i + x_i)^2} \Big)^r dt$$
$$\le C\varepsilon \int\limits_{\mathbb{R}^k} \Big| \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| \Big[\Big(\sqrt{\sum_{i\in A\cap B} t_i^2} \Big)^r + \Big(\sqrt{\sum_{i\in B'} (t_i + x_i)^2} \Big)^r \Big] dt$$
$$\le C\varepsilon \Big(\int\limits_{\mathbb{R}^k} \Big| \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| |t|^r dt + \int\limits_{\mathbb{R}^k} \Big| \frac{\partial^r P(t, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| \Big| \overline{(t + x)}_{B'} \Big|^r dt,$$

which by virtue of statements 3) and 4) of Lemma 1 implies

$$|\mathcal{I}_1| < C\varepsilon \quad \text{for} \quad \frac{x_{k+1}}{|\overline{x}_{B'}|} \ge C > 0. \tag{3}$$

Further,

$$|\mathcal{I}_2| \le \int\limits_{CV_\delta} \Big| \frac{\partial^r P(t - x + x_{A\cap B}, x_{k+1})}{\partial \overline{t}_A^{\,\alpha}} \Big| \Big[|f(t + x_{A\cap B})|$$
$$+ \sum_{i=0}^{r-1} \sum_{|\beta|=i} \frac{|a_\beta(\overline{x}_{A\cap B}, \overline{t}_{A'})|}{\beta!} |t|^i + \Big(\sum_{|\beta|=r} \frac{|a_\beta|}{\beta!} \Big) |t|^r \Big] dt.$$

Applying statement 5) of Lemma 1, the latter inequality yields

$$|\mathcal{I}_2| < C x_{k+1} \quad \text{for} \quad |x| < \frac{\delta}{2}. \tag{4}$$

By virtue of (3) and (4) (ε is assumed to be arbitrarily small) we conclude that Theorem 1 is valid. □

Proof of Theorem 2. Let $x^0 = 0$ and r be an even number. We have

$$\frac{\partial^r U(f; x_B, x_{k+1})}{\partial \overline{x}_A^{\,\alpha}} = \frac{\Gamma\big(\frac{k+1}{2}\big)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t - x + x_{B'}, x_{k+1})}{\partial \overline{x}_A^{\,\alpha}} f(t) dt$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x+x_A, x_{k+1})}{\partial \overline{x}_A^\alpha} f(t+x_{A\cap B})dt.$$

Note that $\frac{\partial^r P(t-x+x_{B'}, x_{k+1})}{\partial \overline{x}_A^\alpha}$ for even r is an even function of variables $\overline{t}_{B'}$ and thus, applying the substitution of first $t_{B'} = -\tau_{B'}$ and then $t = \tau + x$, we obtain

$$\frac{\partial^r U(f; x_B, x_{k+1})}{\partial \overline{x}_A^\alpha} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x+x_{B'}, x_{k+1})}{\partial \overline{x}_A^\alpha} f(t-2t_{B'})dt$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x+x_A, x_{k+1})}{\partial \overline{x}_A^\alpha} f(t+x_{A\cap B}-2t_{B'})dt.$$

These equalities give

$$\frac{\partial^r U(f; x_B, x_{k+1})}{\partial \overline{x}_A^\alpha} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x+x_A, x_{k+1})}{\partial \overline{x}_A^\alpha}$$

$$\times \frac{f(t+x_{A\cap B}) + f(t+x_{A\cap B}-2t_{B'})}{2} dt.$$

By virtue of statements 1) and 2) of Lemma 1 we obtain

$$\frac{\partial^r U(f; x_B, x_{k+1})}{\partial \overline{x}_A^\alpha} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x+x_A, x_{k+1})}{\partial \overline{x}_A^\alpha} \cdot |\overline{t}_A|^r$$

$$\times \Bigg[\frac{f(t+x_{A\cap B}) + f(t+x_{A\cap B}-2t_{B'})}{2} - \sum_{i=0}^{\frac{r-2}{2}} \sum_{|\beta|=i} \frac{b_\beta(\overline{x}_{A\cap B}, \overline{t}_{A'})}{\beta!} \overline{t}_A^\beta$$

$$- \sum_{|\beta|=r} \frac{b_\beta}{\beta!} \overline{t}_A^\beta \Bigg] \frac{dt}{|\overline{t}_A|^r} + D_{A(B)}^{*(\alpha)} f(0).$$

Applying the same arguments as in prooving Theorem 1, from the latter equality we obtain the validity of Theorem 2. □

Proof of Theorem 3. Let $x^0 = 0$ and r be an even number. We have

$$\frac{\partial^r U(f; x, x_{k+1})}{\partial \overline{x}_A^\alpha} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t-x, x_{k+1})}{\partial \overline{x}_A^\alpha} f(t)dt$$

$$= \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t, x_{k+1})}{\partial \overline{x}_A^\alpha} f(t+x)dt,$$

and also

$$\frac{\partial^r U(f; x, x_{k+1})}{\partial \overline{x}_A^\alpha} = \frac{\Gamma\left(\frac{k+1}{2}\right)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t, x_{k+1})}{\partial \overline{x}_A^\alpha} f(x-t)dt.$$

These equalities yield

$$\frac{\partial^r U(f;x,x_{k+1})}{\partial \overline{x}_A^{\alpha}} = \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t,x_{k+1})}{\partial \overline{x}_A^{\alpha}} \times \frac{f(x+t)+f(x-t)}{2} dt.$$

By virtue of statements 1) and 2) of Lemma 1 we have

$$\frac{\partial^r U(f;x,x_{k+1})}{\partial \overline{x}_A^{\alpha}} = \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \frac{\partial^r P(t,x_{k+1})}{\partial \overline{x}_A^{\alpha}} |\overline{t}_A|^r \times \Big[\frac{f(x+t)+f(x-t)}{2} - \sum_{i=0}^{\frac{r-2}{2}} \sum_{|\beta|=i} \frac{b_\beta(x)}{\beta!} \overline{t}_A^{\,\beta} - \sum_{|\beta|=r} \frac{b_\beta}{\beta!} \overline{t}_A^{\,\beta}\Big] \frac{dt}{|\overline{t}_A|^r} + \overline{D}_A^{*(\alpha)} f(0).$$

Hence by virtue of statements 3) and 5) of Lemma 1 we conclude that Theorem 3 is valid. □

Proof of Theorem 4. We have

$$\Omega\overline{\Delta}^r U(f;x_B+x_{B'}^0,x_{k+1}) = \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \Omega\overline{\Delta}^r P(t-x_B-x_{B'}^0,x_{k+1}) f(t)dt$$

$$= \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int\limits_{\mathbb{R}^k} \Omega\overline{\Delta} P|_{x=0} f(t+x_B+x_{B'}^0)dt$$

$$= \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int_0^\infty \overbrace{\int_0^\pi \cdots \int_0^\pi}^{k-2} \int_0^{2\pi} \left(\frac{\Omega\overline{\Delta}_{\rho,\theta}^{r} P}{\Omega\Big(\frac{t}{|t|}\Big)}\right)_{x=0} \Omega\Big(\frac{t}{|t|}\Big) f(t+x_B+x_{B'}^0)\rho^{k-1} \times \sin^{k-1}\theta_1 \cdots \sin\theta_{k-2}\, d\rho\, d\theta_1 \cdots d\theta_{k-2}\, d\varphi$$

$$= \frac{\Gamma\Big(\frac{k+1}{2}\Big)}{\pi^{\frac{k+1}{2}}} \int_0^\infty \left(\frac{\Omega\overline{\Delta}_{\rho,\theta}^{r} P}{\Omega\Big(\frac{t}{|t|}\Big)}\right)_{x=0} |S_\rho| d\rho \frac{1}{|S_\rho|} \int\limits_{S_\rho(x_B+x_{B'}^0)} \Omega\Big(\frac{t}{|t|}\Big) f(t)ds(t)$$

$$= \frac{2\Gamma\Big(\frac{k+1}{2}\Big)}{\sqrt{\pi}\Gamma\Big(\frac{k}{2}\Big)} \int_0^\infty \left(\frac{\Omega\overline{\Delta}_{\rho,\theta}^{r} P}{\Omega\Big(\frac{t}{|t|}\Big)}\right)_{x=0} \rho^{k-1} d\rho \frac{1}{|S_\rho|} \int\limits_{S_\rho(x_B+x_{B'}^0)} \Omega\Big(\frac{t}{|t|}\Big) f(t)ds(t).$$

By virtue of statements 1) and 2) of lemma 2 we obtain

$$\Omega\overline{\Delta}^r U(f;x_B+x_{B'}^0,x_{k+1}) - \Omega\overline{\Delta}_{x_B}^r f(x^0)$$

$$= \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi}\Gamma\left(\frac{k}{2}\right)} \int_0^\infty \left(\frac{\Omega\overline{\Delta}^r_{\rho,\theta} P}{\Omega\left(\frac{t}{|t|}\right)}\right)_{x=0} \rho^{k-1}$$

$$\times \left\{ \frac{r! 2^{\nu+2r}\Gamma\left(\frac{k}{2}+\nu+r\right)}{\Gamma\left(\frac{k}{2}\right)\rho^{\nu+2r}} \left[\frac{1}{|S_\rho|} \int_{S_\rho(x_B+x^0_{B'})} \Omega\left(\frac{t}{|t|}\right) f(t)ds(t) \right.\right.$$

$$\left.\left. - \sum_{i=0}^{r-1} a_i(\overline{x}_B)\rho^{\nu+2i} \right] - \Omega\overline{\Delta}^r_{x_B} f(x^0) \right\} \frac{\Gamma\left(\frac{k}{2}\right)\rho^{\nu+2r} d\rho}{r! 2^{\nu+2r}\Gamma\left(\frac{k}{2}+\nu+r\right)}$$

$$= \frac{2\Gamma\left(\frac{k+1}{2}\right)}{\sqrt{\pi} r! 2^{\nu+2r}\Gamma\left(\frac{k}{2}+\nu+r\right)} \int_0^\infty \left(\frac{\Omega\overline{\Delta}^r_{\rho,\theta} P}{\Omega\left(\frac{t}{|t|}\right)}\right)_{x=0}$$

$$\times \left\{ \frac{r! 2^{\nu+2r}\Gamma\left(\frac{k}{2}+\nu+r\right)}{\Gamma\left(\frac{k}{2}\right)\rho^{\nu+2r}} \left[\frac{1}{|S_\rho|} \int_{S_\rho(x_B+x^0_{B'})} \Omega\left(\frac{t}{|t|}\right) f(t)ds(t) \right.\right.$$

$$\left.\left. - \sum_{i=0}^{r-1} a_i(\overline{x}_B)\rho^{\nu+2i} \right] - \Omega\overline{\Delta}^r_{x_B} f(x^0) \right\} \rho^{\nu+2r+k-1} d\rho.$$

By statements 3), 4) and 5) of Lemma 2 this equality implies that Theorem 4 is valid. □

The results presented in this paper are partly announced in [11, 12].

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