Width-Integrals of Mixed Projection Bodies and Mixed Affine Surface Area

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Abstract

The main purposes of this paper are to establish some new Brunn-Minkowski inequalities for width-integrals of mixed projection bodies and affine surface area of mixed bodies, and get their inverse forms.

2000 Mathematics Subject Classification: 52A40 53A15 46B20

Key words and phrases: Width-integrals, Affine surface area, Mixed projection body, Mixed body.

1 Introduction

In recent years some authors including Ball[1], Bourgain[2], Gardner[3], Schneider[4] and Lutwak[5-10] et al have given considerable attention to the Brunn-Minkowski theory and Brunn-Minkowski-Firey theory and their various generalizations. In particular, Lutwak[7] had generalized the Brunn-Minkowski inequality (1) to mixed projection body and get inequality (2):

**The Brunn-Minkowski inequality**  If \( K, L \in K^n \), then

\[
V(K + L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n},
\]

with equality if and only if \( K \) and \( L \) are homothetic.

**The Brunn-Minkowski inequality for mixed projection bodies**  If \( K, L \in K^n \), then

\[
V(\Pi(K + L))^{1/(n-1)} \geq V(\Pi K)^{1/(n-1)} + V(\Pi L)^{1/(n-1)},
\]

Received 9 September, 2009

Accepted for publication (in revised form) 15 June, 2010
with equality if and only if $K$ and $L$ are homothetic.

On the other hand, width-integral of convex bodies and affine surface areas play an important role in the Brunn-Minkowski theory. Width-integrals were first considered by Blaschke[11] and later by Hadwiger[12]. In addition, Lutwak had established the following results for the width-integrals of convex bodies and affine surface areas.

The Brunn-Minkowski inequality for width-integrals of convex bodies[10]

If $K, L \in \mathcal{K}^n$, $i < n - 1$

\[ B_i(K + L)^{1/(n-i)} \leq B_i(K)^{1/(n-i)} + B_i(L)^{1/(n-i)} \]

with equality if and only if $K$ and $L$ have similar width.

The Brunn-Minkowski inequality for affine surface area[9]

If $K, L \in \mathcal{K}^n$, and $i \in \mathbb{R}$, then for $i < -1$

\[ \Omega_i(K + L)^{(n+1)/(n-i)} \leq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \]

with equality if and only if $K$ and $L$ are homothetic, while for $i > -1$

\[ \Omega_i(K + L)^{(n+1)/(n-i)} \geq \Omega_i(K)^{(n+1)/(n-i)} + \Omega_i(L)^{(n+1)/(n-i)} \]

with equality if and only if $K$ and $L$ are homothetic.

In this paper, there are two purposes:

Firstly, we generalize inequality (3) to mixed projection bodies and get its inverse version.

**Result A** If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$, let $C = (K_3, \ldots, K_n)$, then for $i < n - 1$

\[ B_i(\Pi(K_1 + K_2, K_3, \ldots, K_n))^{1/(n-i)} \leq B_i(\Pi(K_1), K_3, \ldots, K_n)^{1/(n-i)} + B_i(\Pi(K_2), K_3, \ldots, K_n)^{1/(n-i)} \]

with equality if and only if $\Pi(K_1, K_2)$ are homothetic.

While for $i > n$ or $n > i > n - 1$,

\[ B_i(\Pi(K_1 + K_2, K_3, \ldots, K_n))^{1/(n-i)} \geq B_i(\Pi(K_1), K_3, \ldots, K_n)^{1/(n-i)} + B_i(\Pi(K_2), K_3, \ldots, K_n)^{1/(n-i)} \]

with equality if and only if $\Pi(K_1, K_2)$ are homothetic.

Secondly, we prove that analogs of inequalities (4)-(5) for affine surface area of mixed bodies.

**Result B** If $K_1, K_2, \ldots, K_n \in \mathcal{K}^n$ and all of mixed bodies of $K_1, K_2, \ldots, K_n$ have positive continuous curvature functions, respectively, then for $i < -1$

\[ \Omega_i([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]


\[ (8) \quad \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]

with equality if and only if \([K_1, K_3, K_4, \ldots, K_n] \) and \([K_2, K_3, \ldots, K_n] \) are homothetic.

While for \(i > -1\)

\[ \Omega_i([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]

\[ (9) \quad \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]

with equality if and only if \([K_1, K_3, K_4, \ldots, K_n] \) and \([K_2, K_3, \ldots, K_n] \) are homothetic.

Please see the next section for above interrelated notations, definitions and their background materials.

2 Notations and Preliminary works

The setting for this paper is \(n\)-dimensional Euclidean space \(\mathbb{R}^n (n > 2)\). Let \(\mathbb{C}^n\) denote the set of non-empty convex figures (compact, convex subsets) and \(\mathbb{K}^n\) denote the subset of \(\mathbb{C}^n\) consisting of all convex bodies (compact, convex subsets with non-empty interiors) in \(\mathbb{R}^n\), and if \(p \in \mathbb{K}^n\), let \(\mathbb{K}_p^n\) denote the subset of \(\mathbb{K}^n\) that contains the centered (centrally symmetric with respect to \(p\)) bodies. We reserve the letter \(u\) for unit vectors, and the letter \(B\) is reserved for the unit ball centered at the origin. The surface of \(B\) is \(S^{n-1}\). For \(u \in S^{n-1}\), let \(E_u\) denote the hyperplane, through the origin, that is orthogonal to \(u\). We will use \(K^u\) to denote the image of \(K\) under an orthogonal projection onto the hyperplane \(E_u\).

2.1 Mixed volumes

We use \(V(K)\) for the \(n\)-dimensional volume of convex body \(K\). Let \(h(K, \cdot) : S^{n-1} \rightarrow \mathbb{R}\), denote the support function of \(K \in \mathbb{K}^n\); i.e.

\[ h(K, u) = \max\{u \cdot x : x \in K\}, \quad u \in S^{n-1}, \]

where \(u \cdot x\) denotes the usual inner product \(u\) and \(x\) in \(\mathbb{R}^n\).

Let \(\delta\) denote the Hausdorff metric on \(\mathbb{K}^n\); i.e., for \(K, L \in \mathbb{K}^n\),

\[ \delta(K, L) = \|h_K - h_L\|_{\infty}, \]

where \(\|\|_{\infty}\) denotes the sup-norm on the space of continuous functions, \(C(S^{n-1})\).
For a convex body $K$ and a nonnegative scalar $\lambda K$, is used to denote \( \{ \lambda x : x \in K \} \). For $K_i \in \mathcal{K}^n, \lambda_i \geq 0, (i = 1, 2, \ldots, r)$, the Minkowski linear combination $\sum_{i=1}^r \lambda_i K_i \in \mathcal{K}^n$ is defined by

$$
(11) \quad \lambda_1 K_1 + \cdots + \lambda_r K_r = \{ \lambda_1 x_1 + \cdots + \lambda_r x_r \in K^n : x_i \in K_i \}.
$$

It is trivial to verify that

$$
(12) \quad h(\lambda_1 K_1 + \cdots + \lambda_r K_r, \cdot) = \lambda_1 h(K_1, \cdot) + \cdots + \lambda_r h(K_r, \cdot).
$$

If $K_i \in \mathcal{K}^n(i = 1, 2, \ldots, r)$ and $\lambda_i (i = 1, 2, \ldots, r)$ are nonnegative real numbers, then of fundamental impotence is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in $\lambda_i$ given by [4]

$$
(13) \quad V(\lambda_1 K_1 + \cdots + \lambda_r K_r) = \sum_{i_1, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n} V_{i_1 \ldots i_n},
$$

where the sum is taken over all $n$-tuples $(i_1, \ldots, i_n)$ of positive integers not exceeding $r$. The coefficient $V_{i_1 \ldots i_n}$ depends only on the bodies $K_{i_1}, \ldots, K_{i_n}$, and is uniquely determined by (13), it is called the mixed volume of $K_{i_1}, \ldots, K_{i_n}$, and is written as $V(K_{i_1}, \ldots, K_{i_n})$. Let $K_{i_1} = \cdots = K_{n-i} = K$ and $K_{n-i+1} = \cdots = K_n = L$, then the mixed volume $V(K_1, \ldots, K_n)$ is usually written $V_i(K, L)$. If $L = B$, then $V_i(K, B)$ is the $i$th projection measure (Quermassintegral) of $K$ and is written as $W_i(K)$. With this notation, $W_0 = V(K)$, while $nW_1(K)$ is the surface area of $K$, $S(K)$.

### 2.2 Width-integrals of convex bodies

For $u \in S^{n-1}$, $b(K, u)$ is defined to be half the width of $K$ in the direction $u$. Two convex bodies $K$ and $L$ are said to have similar width if there exists a constant $\lambda > 0$ such that $b(K, u) = \lambda b(L, u)$ for all $u \in S^{n-1}$. For $K \in \mathcal{K}^n$ and $p \in \text{int} K$, we use $K^p$ to denote the polar reciprocal of $K$ with respect to the unit sphere centered at $p$. The width-integral of index $i$ is defined by Lutwak [10]: For $K \in \mathcal{K}^n, i \in \mathbb{R}$

$$
(14) \quad B_i(K) = \frac{1}{n} \int_{S^{n-1}} b(K, u)^{n-i} dS(u),
$$

where $dS$ is the $(n - 1)$-dimensional volume element on $S^{n-1}$.

The width-integral of index $i$ is a map

$$
B_i : \mathcal{K}^n \to \mathbb{R}.
$$
It is positive, continuous, homogeneous of degree \( n - i \) and invariant under motion. In addition, for \( i \leq n \) it is also bounded and monotone under set inclusion.

The following results\(^{[10]} \) will be used later

\begin{equation}
 b(K + L, u) = b(K, u) + b(L, u),
\end{equation}

with equality if and only if \( K \) is symmetric with respect to \( p \).

### 2.3. The radial function and the Blaschke linear combination

The radial function of convex body \( K, \rho(K, \cdot) : S^{n-1} \to \mathbb{R} \), defined for \( u \in S^{n-1} \), by

\[ \rho(K, \cdot) = \text{Max}\{\lambda \geq 0 : \lambda \mu \in K\}. \]

If \( \rho(K, \cdot) \) is positive and continuous, \( K \) will be called a star body. Let \( \varphi^n \) denote the set of star bodies in \( \mathbb{R}^n \).

A convex body \( K \) is said to have a positive continuous curvature function\(^{[5]} \),

\[ f(K, \cdot) : S^{n-1} \to [0, \infty), \]

if for each \( L \in \varphi^n \), the mixed volume \( V_1(K, L) \) has the integral representation

\[ V_1(K, L) = \frac{1}{n} \int_{S^{n-1}} f(K, u) h(L, u) dS(u). \]

The subset of \( \mathcal{K}^n \) consisting of bodies which have a positive continuous curvature function will be denoted by \( \kappa^n \). Let \( \kappa^n_c \) denote the set of centrally symmetric member of \( \kappa^n \).

The following result is true\(^{[6]} \), for \( K \in \kappa^n \)

\[ \int_{S^{n-1}} u f(K, u) dS(u) = 0. \]

Suppose \( K, L \in \kappa^n \) and \( \lambda, \mu \geq 0 \) (not both zero). From above it follows that the function \( \lambda f(K, \cdot) + \mu f(L, \cdot) \) satisfies the hypothesis of Minkowski’s existence theorem(see [13]). The solution of the Minkowski problem for this function is denoted by \( \lambda \cdot K + \mu \cdot L \) that is

\begin{equation}
 f(\lambda \cdot K + \mu \cdot L, \cdot) = \lambda f(K, \cdot) + \mu f(L, \cdot),
\end{equation}

where the linear combination \( \lambda \cdot K + \mu \cdot L \) is called a Blaschke linear combination.
The relationship between Blaschke and Minkowski scalar multiplication is given by

$$\lambda \cdot K = \lambda^{1/(n-1)} K.$$  

2.4 Mixed affine area and mixed bodies

The affine surface area of $K \in \mathbb{R}^n$, $\Omega(K)$, is defined by

$$\Omega(K) = \int_{S^{n-1}} f(K, u)^{n/(n+1)} dS(u).$$

It is well known that this functional is invariant under unimodular affine transformations. For $K, L \in \mathbb{R}^n$, and $i \in \mathbb{R}$, the $i$th mixed affine surface area of $K$ and $L$, $\Omega_i(K, L)$, was defined in$^5$ by

$$\Omega_i(K, L) = \int_{S^{n-1}} f(K, u)^{(n-i)/(n+1)} f(L, u)^{i/(n+1)} dS(u).$$

Now, we define the $i$th affine area of $K \in \mathbb{R}^n$, $\Omega_i(K)$, to be $\Omega_i(K, B)$, since $f(B, \cdot) = 1$ one has

$$\Omega_i(K) = \int_{S^{n-1}} f(K, u)^{(n-i)/(n+1)} dS(u), \quad i \in \mathbb{R}.$$

Lutwak$^8$ defined mixed bodies of convex bodies $K_1, \ldots, K_{n-1}$ as $[K_1, \ldots, K_{n-1}]$. The following property will be used later:

$$[K_1 + K_2, K_3, \ldots, K_n] = [K_1, K_3, \ldots, K_n] + [K_2, K_3, \ldots, K_n]$$

2.5 Mixed projection bodies and their polars

If $K$ is a convex that contains the origin in its interior, we define the polar body of $K$, $K^*$, by

$$K^* := \{ x \in \mathbb{R}^n | x \cdot y \leq 1, y \in K \}.$$  

If $K_i (i = 1, 2, \ldots, n-1) \in \mathbb{R}^n$, then the mixed projection body of $K_i (i = 1, 2, \ldots, n-1)$ is denoted by $\Pi(K_1, \ldots, K_{n-1})$, and whose support function is given, for $u \in S^{n-1}$, by$^7$

$$h(\Pi(K_1, \ldots, K_{n-1}), u) = v(K_1^u, \ldots, K_{n-1}^u).$$

It is easy to see, $\Pi(K_1, \ldots, K_{n-1})$ is centered.
We use $\Pi^*(K_1, \ldots, K_{n-1})$ to denote the polar body of $\Pi(K_1, \ldots, K_{n-1})$, and is called polar of mixed projection body of $K_i$ $(i = 1, 2, \ldots, n-1)$. If $K_1 = \cdots = K_{n-1-i} = K$ and $K_{n-i} = \cdots = K_{n-1} = L$, then $\Pi(K_1, \ldots, K_{n-1})$ will be written as $\Pi_i(K, L)$. If $L = B$, then $\Pi_i(K, B)$ is called the $i$th projection body of $K$ and is denoted $\Pi_i K$. We write $\Pi_0 K$ as $\Pi K$. We will simply write $i(K)$ and $i(K)$ rather than $(i(K))$ and $(i(K))$, respectively.

The following property will be used:

(25) $\Pi(K_3, \ldots, K_n, K_1 + K_2) = \Pi(K_3, \ldots, K_n, K_1) + \Pi(K_3, \ldots, K_n, K_2)$

## 3 Main results and their proofs

Our main results are the following Theorems which were stated in the introduction.

**Theorem 1** If $K_1, K_2, \ldots, K_n \in \mathbb{K}^n$, let $C = (K_3, \ldots, K_n)$, then for $i < n - 1$

(26) $B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \leq B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2))^{1/(n-i)},$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.

While for $i > n$,

(27) $B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} \geq B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2))^{1/(n-i)},$

with equality if and only if $\Pi(C, K_1)$ and $\Pi(C, K_2)$ are homothetic.

**Proof** Here, we only give the proof of (27).

From (12), (14), (15), (25) and notice for $i > n$ to use inverse the Minkowski inequality for integral [14, P. 147], we obtain that

$$B_i(\Pi(C, K_1 + K_2))^{1/(n-i)} = \left(\frac{1}{n} \int_{S^{n-1}} b(\Pi(C, K_1 + K_2), u)^{n-i} dS(u)\right)^{1/(n-i)}$$

$$= \left(\frac{1}{n} \int_{S^{n-1}} b(\Pi(C, K_1) + \Pi(C, K_2), u)^{n-i} dS(u)\right)^{1/(n-i)}$$

$$= \left(\frac{1}{n} \int_{S^{n-1}} (b(\Pi(C, K_1), u) + b(\Pi(C, K_2), u))^{n-i} dS(u)\right)^{1/(n-i)}$$

$$\geq \left(\frac{1}{n} \int_{S^{n-1}} b(\Pi(C, K_1), u)^{n-i} dS(u)\right)^{1/(n-i)} +$$

$$\left(\frac{1}{n} \int_{S^{n-1}} b(\Pi(C, K_2), u)^{n-i} dS(u)\right)^{1/(n-i)}.$$
\[ B_i(\Pi(C, K_1))^{1/(n-i)} + B_i(\Pi(C, K_2))^{1/(n-i)}, \]

with equality if and only if \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) have similar width, in view of \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) are centered (centrally symmetric with respect to origin), then with equality if and only if \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) are homothetic.

The proof of inequality (27) is complete.

Taking \( i = 0 \) to (26), inequality (26) changes to the following result

**Corollary 1** If \( K_1, K_2, \ldots, K_n \in K^n \), let \( C = (K_3, \ldots, K_n) \), then

\[ B(\Pi(C, K_1 + K_2))^{1/n} = B(\Pi(C, K_1))^{1/n} + B(\Pi(C, K_2))^{1/n} \]

with equality if and only if \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) are homothetic.

Taking \( i = 2n \) to (27), inequality (27) changes to the following result

**Corollary 2** If \( K_1, K_2, \ldots, K_n \in K^n \), let \( C = (K_3, \ldots, K_n) \), then

\[ B_{2n}(\Pi(C, K_1 + K_2))^{-1/n} = B_{2n}(\Pi(C, K_1))^{-1/n} + B_{2n}(\Pi(C, K_2))^{-1/n} \]

with equality if and only if \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) are homothetic.

From (16),(29) and notice that projection body is centered (centrally symmetric with respect to origin), we get

**Corollary 3** If \( K_1, K_2, \ldots, K_n \in K^n \), let \( C = (K_3, \ldots, K_n) \), then

\[ V(\Pi^*(C, K_1 + K_2))^{-1/n} \leq V(\Pi^*(C, K_1))^{-1/n} + V(\Pi^*(C, K_2))^{-1/n} \]

with equality if and only if \( \Pi(C, K_1) \) and \( \Pi(C, K_2) \) are homothetic.

This is just Brunn-Minkowski inequality of polars of mixed projection bodies. This result first is given in here.

**Theorem 2** If \( K_1, K_2, \ldots, K_n \in K^n \) and all of mixed bodies of \( K_1, K_2, \ldots, K_n \) have positive continuous curvature functions, then for \( i < -1 \)

\[ \Omega_i([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \leq \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]

with equality if and only if \( [K_1, K_3, K_4, \ldots, K_n] \) and \( [K_2, K_3, \ldots, K_n] \) are homothetic.

While for \( i > -1 \)

\[ \Omega_i([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \geq \Omega_i([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])^{(n+1)/(n-i)} \]
with equality if and only if \([K_1, K_3, K_4, \ldots, K_n]\) and \([K_2, K_3, \ldots, K_n]\) are homothetic.

**Proof** Firstly, we give the proof of (31).

From (17), (21), (22) and in view of the Minkowski inequality for integral\(^{14, P.147}\), we obtain that

\[
\Omega_i([K_1 + K_2, K_3, K_4, \ldots, K_n])\frac{(n+1)}{(n-i)}
\]

\[
= \left( \int_{S^{n-1}} f([K_1 + K_2, K_3, K_4, \ldots, K_n], u) \frac{(n-i)}{(n+1)} dS(u) \right)^{(n+1)/ (n-i)}
\]

\[
= \left( \int_{S^{n-1}} f([K_1, K_3, K_4, \ldots, K_n], u) \frac{(n-i)}{(n+1)} dS(u) \right)^{(n+1)/ (n-i)}
\]

\[
= \left( \int_{S^{n-1}} (f([K_1, K_3, K_4, \ldots, K_n], u) + f([K_2, K_3, \ldots, K_n], u)) \frac{(n-i)}{(n+1)} dS(u) \right)^{(n+1)/ (n-i)}
\]

\[
\leq \left( \int_{S^{n-1}} f([K_1, K_3, K_4, \ldots, K_n], u) \frac{(n-i)}{(n+1)} dS(u) \right)^{(n+1)/ (n-i)}
\]

\[
+ \left( \int_{S^{n-1}} f([K_2, K_3, \ldots, K_n], u) \frac{(n-i)}{(n+1)} dS(u) \right)^{(n+1)/ (n-i)}
\]

\[
= \Omega_i([K_1, K_3, K_4, \ldots, K_n])\frac{(n+1)}{(n-i)} + \Omega_i([K_2, K_3, \ldots, K_n])\frac{(n+1)}{(n-i)},
\]

with equality if and only if \([K_1, K_3, K_4, \ldots, K_n]\) and \([K_2, K_3, \ldots, K_n]\) are homothetic.

Similarly, from (17), (21), (22) and in view of inverse Minkowski inequality\(^{14, P.147}\), we can also prove (32).

The proof of Theorem 2 is complete.

Taking \(i = 0\) to (32), we have

**Corollary 4** If \(K_1, K_2, \ldots, K_n \in \mathcal{K}^n\) and all of mixed bodies of \(K_1, K_2, \ldots, K_n\) have positive continuous curvature functions, then

\[
\Omega([K_1 + K_2, K_3, \ldots, K_n])\frac{(n+1)}{n}
\]

\[
\geq \frac{(n+1)}{n} \Omega([K_1, K_3, K_4, \ldots, K_n]) + \frac{(n+1)}{n} \Omega([K_2, K_3, \ldots, K_n])
\]

with equality if and only if \([K_1, K_3, K_4, \ldots, K_n]\) and \([K_2, K_3, \ldots, K_n]\) are homothetic.

Taking \(i = 2n\) to (32), inequality (32) changes to the following result

**Corollary 5** If \(K_1, K_2, \ldots, K_n \in \mathcal{K}^n\) and all of mixed bodies of \(K_1, K_2, \ldots, K_n\) have positive continuous curvature functions, then

\[
\Omega_{2n}([K_1 + K_2, K_3, \ldots, K_n])\frac{(n+1)}{n}
\]

\[
\geq \frac{(n+1)}{n} \Omega_{2n}([K_1, K_3, K_4, \ldots, K_n]) + \frac{(n+1)}{n} \Omega_{2n}([K_2, K_3, \ldots, K_n])
\]

\[
\geq \frac{(n+1)}{n} \Omega_{2n}([K_1, K_3, K_4, \ldots, K_n]) + \frac{(n+1)}{n} \Omega_{2n}([K_2, K_3, \ldots, K_n])
\]
with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ and $[K_2, K_3, \ldots, K_n]$ are homothetic.

Taking $i = -n$ to (31), we have

**Corollary 6** If $K_1, K_2, \ldots, K_n \in K^n$ and all of mixed bodies of $K_1, K_2, \ldots, K_n$ have positive continuous curvature functions, then

$$\Omega_{-n}([K_1 + K_2, K_3, \ldots, K_n])^{(n+1)/2n}$$

(35) \leq \Omega_{-n}([K_1, K_3, K_4, \ldots, K_n])^{(n+1)/2n} + \Omega_{-n}([K_2, K_3, \ldots, K_n])^{(n+1)/2n},

with equality if and only if $[K_1, K_3, K_4, \ldots, K_n]$ and $[K_2, K_3, \ldots, K_n]$ are homothetic.

**Acknowledgments**

This research is supported by National Natural Sciences Foundation of China (10971205) and Zhejiang Natural Sciences Foundation of China (Z6100369).

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