Superordination Properties for Certain Analytic Functions

H.A. Al-Kharsani, N.M. Al-Areefi

Abstract

The purpose of the present paper is to derive superordination result for functions in the class \( M_{l;m}^{1,n}(\alpha, \lambda, b) \) of normalized analytic functions in the open unit disk \( U \). A number of interesting applications of the superordination result are also considered.

2000 Mathematics Subject Classification: 30C45.
Key words and phrases: Analytic functions, Hadamard product, the Dziok-Srivastava operator, superordinating factor sequence.

1 Introduction

Let \( A \) denote the class of functions of the form

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n
\]

which are analytic in the unit disc \( U = \{ z : |z| < 1 \} \). We also denote by \( K \) the class of functions \( f \in A \) that are convex in \( U \).

Given two functions \( f, g \in A \), where \( f \) is given by (1) and \( g \) is defined by

\[
g(z) = z + \sum_{n=2}^{\infty} b_n z^n.
\]

Received 24 April, 2009
Accepted for publication (in revised form) 17 November, 2009
The Hadamard product (or convolution) $f * g$ is defined by

$$(3) \quad (f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n \quad (z \in U).$$

For $\alpha_j \in \mathbb{C}$ ($j = 1, 2, \ldots, l$) and $\beta_j \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$ ($j = 1, 2, \ldots, m$), the generalized hypergeometric function $\, _{l}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ is defined by the infinite series

$$\, _{l}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) := \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_l)_n z^n}{(\beta_1)_n \cdots (\beta_m)_n n!} \quad (l \leq m + 1; m \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}),$$

where $(\lambda)_n$ is the Pochhammer symbol defined by

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda+1)(\lambda+2) \cdots (\lambda+n-1) & (n \in \mathbb{N} \Rightarrow \{1, 2, 3, \ldots\}) \end{cases}.$$ 

Corresponding to the function

$$(4) \quad h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = z_l \, _{l}F_{m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z).$$

The Dziok-Srivastava operator [4] (see also [11]) $H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ is defined by the Hadamard product

$$(5) \quad H^{l,m}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) := h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z).$$

We note that the linear operator $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ includes various other linear operators which were introduced and studied by Carlson and Shaffer [3], Hohlov [6], Ruscheweyh [10], and so on [5], [9].

Corresponding to the function $h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$, defined by (4), we introduce a function $F_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z)$ given by

$$(6) \quad h(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * F_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) = \frac{z}{(1-z)^{\mu}} \quad (z \in U, \mu > 0).$$

Analogous to $H(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$, in [2] we define the linear operator $J_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m)$ on $A$ as follows:

$$(7) \quad J_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m) f(z) = F_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m; z) * f(z)$$

where $(\alpha_i; \beta_j \in \mathbb{C} \setminus \mathbb{Z}_0; i = 1, \ldots, l; j = 1, \ldots, m; \mu > 0; z \in U; f \in A)$. 
For convenience, we write

\[ J_{\mu}^{l;m}(\alpha_1) := J_{\mu}(\alpha_1, \ldots, \alpha_l; \beta_1, \ldots, \beta_m). \]

This operator was defined by Cho [7] special cases were studied by Noor [8] and Alkharsani [1].

**Definition 1** Let \( g \) be analytic and univalent in \( U \). If \( f \) is analytic in \( U \), \( f(0) = g(0) \), and \( f(U) \subset g(U) \), then one says that \( f \) is subordinate to \( g \) in \( U \), and we write \( f \prec g \) or \( f(z) \prec g(z) \). One also says that \( g \) is superordinate to \( f \) in \( U \).

**Definition 2** Let \( f = \sum_{k=1}^{\infty} a_k z^k \in A \). An infinite sequence \( \{a_k\}_{k=1}^{\infty} \) of complex numbers where

\[
c_k = \begin{cases} 
\frac{1}{a_k} & a_k \neq 0 \\
0 & a_k = 0 
\end{cases}
\]

will be called superordinating factor if for every \( g = z + \sum_{k=2}^{\infty} b_k z^k \) in \( K \), one has

\[ f_{-1} \ast g < g \]

where \( f_{-1} \) is defined as follows,

\[ f \ast f_{-1} \ast g < f \ast g, \]

then

\[ f_{-1} = z \ast \sum_{k=2}^{\infty} c_k z^k \]

one also says that (11) is equivalent to

\[ \sum_{k=1}^{\infty} c_k b_k z^k < g \quad (z \in U; c_1 = 1), \]

or the sequence \( \{c_k\}_{k=1}^{\infty} \) is a superordinating factor if and only if

\[ \text{Re} \left\{ 1 + 2 \sum_{k=1}^{\infty} c_k z^k \right\} > 0 \quad (z \in U). \]
Definition 3 Suppose that $f \in A$. Then the function $f_{-1}$ is said to be a member of the class $L^{l;m}_{\mu}(\alpha, \lambda, b)$ if it satisfies

$$
\left| \frac{\lambda \mu \left( J^{l;m}_{\mu+1}(\alpha_1) f_{-1}(z) \right)}{z} + (1 - \lambda \mu) \left( \frac{J^{l;m}_{\mu}(\alpha_1) f_{-1}(z)}{z} \right) - 1 \right| < 1
$$

$$
\left| \frac{\lambda \mu \left( J^{l;m}_{\mu+1}(\alpha_1) f_{-1}(z) \right)}{z} + (1 - \lambda \mu) \left( \frac{J^{l;m}_{\mu}(\alpha_1) f_{-1}(z)}{z} \right) + 2b(1 - \alpha) - 1 \right| < 1
$$

$$(z \in U; 0 \leq \alpha < 1; \lambda \geq 0; b \in \mathbb{C} \setminus \{0\}; \mu > 0),$$

Now, we prove the following lemma which gives a sufficient condition for functions belonging to the class $L^{l;m}_{\mu}(\alpha, \lambda, b)$.

Lemma 1 If the function $f_{-1}$ satisfies the following conditions:

$$
\sum_{k=2}^{\infty} |1 + \lambda (k - 1)| C(\mu, k) |c_k| \leq (1 - \alpha) |b|
$$

$$(0 \leq \alpha < 1; \lambda \geq 0; b \in \mathbb{C} \setminus \{0\}; \mu > 0),$$

where

$$
C(\mu, k) = \prod_{j=2}^{k} \frac{(j + \mu - 2)}{(k - 1)!} \psi_{k-1, \psi_{k-1}}
$$

$$
= \frac{(\beta_1)_{k-1} \cdots (\beta_m)_{k-1}}{(\alpha_1)_{k-1} \cdots (\alpha_l)_{k-1}}, \quad (\mu > 0, k = 1, 2, 3, \ldots),
$$

then $f_{-1} \in L^{l;m}_{\mu}(\alpha, \lambda, b)$.

Proof. Supposes that the inequality (12) holds. Using the identity

$$
z \left( J^{l;m}_{\mu}(\alpha_1) f_{-1}(z) \right)' = \mu J^{l;m}_{\mu+1}(\alpha_1) f_{-1}(z) - (\mu - 1) J^{l;m}_{\mu}(\alpha_1) f_{-1}(z),$$
we have for \( z \in U \),

\[
\left| (1 - \lambda) \frac{J_{\mu}^{l,m}(\alpha_1 f^{-1}(z))}{z} + \lambda(J_{\mu}^{l,m}(\alpha_1 f^{-1}(z)))' - 1 \right|
\]

\[
- 2b(1 - \alpha) + (1 - \lambda) \frac{J_{\mu}^{l,m}(\alpha_1 f^{-1}(z))}{z} + \lambda(J_{\mu}^{l,m}(\alpha_1 f^{-1}(z)))' - 1 \right|
\]

\[
= \sum_{k=2}^{\infty} \left( 1 + \lambda(k-1) C(\mu, k) c_k z^{k-1} \right)
\]

\[
- 2b(1 - \alpha) + \sum_{k=2}^{\infty} \left( 1 + \lambda(k-1) C(\mu, k) c_k z^{k-1} \right)
\]

\[
\leq \sum_{k=2}^{\infty} \left( 1 + \lambda(k-1) C(\mu, k) c_k z^{k-1} \right)
\]

\[
- \left\{ 2b(1 - \alpha) - \sum_{k=2}^{\infty} \left( 1 + \lambda(k-1) C(\mu, k) c_k z^{k-1} \right) \right\}
\]

\[
\leq 2 \left\{ \sum_{k=2}^{\infty} \left( 1 + \lambda(k-1) C(\mu, k) c_k \right) - b(1 - \alpha) \right\} \leq 0,
\]

which shows that \( f^{-1} \) belongs to \( L_{\mu}^{l,m}(\alpha, \lambda, b) \).

Let \( M_{\mu}^{l,m}(\alpha, \lambda, b) \) denote the class of functions \( f \) in \( A \) whose Taylor-Maclaurin coefficients \( a_k \) satisfy the condition (12).

We note that

\[
M_{\mu}^{l,m}(\alpha, \lambda, b) \subseteq L_{\mu}^{l,m}(\alpha, \lambda, b).
\]

\[\blacksquare\]

**Example 1** (i) For \( 0 \leq \alpha < 1, \lambda > 0, b \in \mathbb{C} \setminus \{0\} \) and \( \mu > 0 \), the following function defined by

\[
f_0(z) = z + \frac{\mu(\lambda + 1)}{2b(1 - \alpha)} \psi_1 z^2 \left( 1, 2, 1 + \frac{1}{\lambda}, 2 + \frac{1}{\lambda}, \mu + 1; z \right) \quad (z \in U)
\]

is in the class \( L_{\mu}^{l,m}(\alpha, \lambda, b) \).

(ii) For \( 0 \leq \alpha < 1, \lambda > 0, b \in \mathbb{C} \setminus \{0\} \), and \( \mu > 0 \), the following functions
defined by
\[ f_1(z) = z + \frac{\mu(\lambda + 1)\psi_1}{(1 - \alpha)|b|} z^2 \quad (z \in U), \]
\[ f_2(z) = z + \frac{\mu(\mu + 1)(2\lambda + 1)\psi_2}{(1 - \alpha)|b|} z^3 \quad (z \in U), \]
\[ f_3(z) = z + \mu(\lambda + 1)\psi_1 z^2 + \frac{\mu(\mu + 1)(2\lambda + 1)\psi_2}{2(1 - \alpha)|b| - 1} z^3 \quad (z \in U). \]

are in the \( M_{\mu}^{l,m}(\alpha, \lambda, b) \).

In this paper, we obtain a sharp superordination result associated with the class \( M_{\mu}^{l,m}(\alpha, \lambda, b) \). Some applications of the main result which give important results of analytic functions are also investigated.

2 Main Theorem

Theorem 1 Let \( f_{-1} \in M_{\mu}^{l,m}(\alpha, \lambda, b) \). Then
\[ (18) \quad \frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} (f_{-1} \ast g)(z) - g(z) \quad (z \in U) \]
for every function \( g \) in \( K \), and
\[ (19) \quad \text{Re} \ f_{-1}(z) > -\frac{\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)}{\mu(\lambda + 1)\psi_1}. \]
The constant \( \frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} \) cannot be replaced by a larger one.

Proof. Let \( f_{-1} \in M_{\mu}^{l,m}(\alpha, \lambda, b) \) and let
\[ (20) \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k \]
be any function in the class \( K \). Then we readily have
\[ (21) \quad \frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} (f_{-1} \ast g)(z) = \frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} \left( z + \sum_{k=2}^{\infty} b_k c_k z^k \right). \]
Thus, by Definition 3, the superordination result (18) will hold true if the sequence
\[
(22) \quad \left\{ \frac{\mu(\lambda + 1)c_k \psi_1}{2[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} \right\}_{k=1}^\infty
\]
is a superordinating factor sequence, with \(c_1 = 1\). In view of Definition 2, this is equivalent to the following inequality:
\[
(23) \quad \text{Re} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\mu(\lambda + 1)\psi_1}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} c_k z^k \right\} > 0, \quad (z \in U).
\]
Now, since
\[
(24) \quad [1 + \lambda(k - 1)]C(\mu, k) \quad (\lambda \geq 0, \mu > 0)
\]
is an increasing function of \(K\), we have
\[
\text{Re} \left\{ 1 + \sum_{k=1}^{\infty} \frac{\mu(\lambda + 1)\psi_1}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} c_k z^k \right\} = \text{Re} \left\{ 1 + \frac{\mu(\lambda + 1)\psi_1}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} z \right. \\
+ \frac{1}{\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)} \sum_{k=2}^{\infty} \mu(\lambda + 1)\psi_1 c_k z^k \\
\left. > 1 - \frac{\mu(\lambda + 1)\psi_1}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} r \\
- \frac{1}{\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)} \sum_{k=2}^{\infty} (1 + \lambda(k - 1)C(\mu, k)) c_k r^k \\
\right.
\]
\[
> 1 - \frac{\mu(\lambda + 1)\psi_1}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} r \\
- \frac{|b|(1 - \alpha)}{[\mu(\lambda + 1)\psi_1 + |b|(1 - \alpha)]} r > 0 \quad (|z| = r).
\]
This proves the inequality (23), and hence also subordination result (18) asserted by Theorem 1. The inequality (19) follows from (18) by taking
\[
(25) \quad g(z) = \frac{z}{1 - z} \in K.
\]
Next, we consider the function

\( f_1(z) = z - \frac{\mu(\lambda + 1)\psi_1}{(1 - \alpha)b^2}z^2(0 \leq \alpha < 1; \lambda \geq 0; b \in \mathbb{C} \setminus \{0\}, \mu > 0) \)

which is a member of the class \( M^1_m(a, \lambda, b) \). Then by using (18), we have

\[
\frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b(1 - \alpha)|]}f_{-1}(z) < \frac{z}{1 - z} \quad (z \in U).
\]

It can be easily verified for the function \( f_1(z) \) defined by (26) that

\[
\inf_{z \in U} \left\{ \text{Re} \left( \frac{\mu(\lambda + 1)\psi_1}{2[\mu(\lambda + 1)\psi_1 + |b(1 - \alpha)|]}f_{-1}(z) \right) \right\} = -\frac{1}{2} \quad (z \in U)
\]

which completes the proof of Theorem 1.

### 3 Some Applications

Taking \( \mu = 1 \) in Theorem 1, we obtain the following:

**Corollary 1** If the function \( f_{-1} \) satisfies

\[
\sum_{k=2}^{\infty} [1 + \lambda(k - 1)]|\psi_{k-1}|c_k| \leq m \quad (\lambda \geq 0, \ m > 0),
\]

then for every function \( g \) in \( K \), one has

\[
\frac{(\lambda + 1)\psi_1}{2[(\lambda + 1)\psi_1 + m]}(f_{-1} * g)(z) < g(z) \quad (z \in U)
\]

\[
\text{Re} \ f_{-1}(z) > -\left[ 1 + \frac{m}{(\lambda + 1)\psi_1} \right].
\]

The constant \( \frac{(\lambda + 1)\psi_1}{2[(\lambda + 1)\psi_1 + m]} \) cannot be replaced by larger one.

Putting \( \lambda = 0 \) in Theorem 1, we have the following corollary.

**Corollary 2** If the function \( f_{-1} \) satisfies

\[
\sum_{k=2}^{\infty} C(\mu, k)|c_k| \leq m, \quad m > 0,
\]
where \( C(\mu, k) \) is defined by (13), then for every function \( g \) in \( K \), one has

\[
\frac{\mu \psi_1}{2[\mu \psi_1 + m]} (f_{-1} \ast g)(z) < g(z) \quad (z \in U), \quad \text{Re } f_{-1}(z) > -\left(1 + \frac{m}{\mu \psi_1}\right).
\]

The constant \( \frac{\mu \psi_1}{2[\mu \psi_1 + m]} \) cannot be replaced by larger one.

Next, letting \( \lambda = 1 \) and \( \mu = 1 \), in Theorem 1, we obtain the following corollary.

**Corollary 3** If the function \( f_{-1} \) satisfies

\[
\sum_{k=2}^{\infty} k|c_k| \psi_{k-1} \leq m \quad (m > 0),
\]

then for every function \( g \) in \( K \), one has

\[
\frac{\psi_1}{2\psi_1 + m} (f_{-1} \ast g)(z) < g(z) \quad (z \in U), \quad \text{Re } f_{-1}(z) > -\left(1 + \frac{m}{2\psi_1}\right).
\]

The constant \( \frac{\psi_1}{2\psi_1 + m} \) cannot be replaced by larger one.

Also, by taking \( \lambda = 0 \) and \( \mu = 1 \), in Theorem 1, we have the following:

**Corollary 4** If the function \( f \) satisfies

\[
\sum_{k=2}^{\infty} \psi_{k-1} |c_k| \leq m \quad (m > 0),
\]

then for every function \( g \) in \( K \), one has

\[
\frac{\psi_1}{2(\psi_1 + m)} (f_{-1} \ast g)(z) < g(z) \quad (z \in U), \quad \text{Re } f_{-1} < -\left(1 + \frac{m}{\psi_1}\right).
\]

The constant \( \frac{\psi_1}{2(\psi_1 + m)} \) cannot be replaced by larger one.
References


Superordination Properties for Certain Analytic Functions

H.A. Al-Kharsani, N.M. Al-Areefi
Faculty of Science
Department of Mathematics
P.O. Box 838, Dammam 31113, Saudi Arabia
e-mail: najarifi@hotmail.com