Meromorphic functions concerning their differential polynomials sharing the fixed-points with finite weight

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Abstract

This paper deals with some uniqueness problems of meromorphic functions concerning their differential polynomials sharing the fixed-points or a small function with finite weight. These results in this paper greatly improve the recent results given by X.-Y. Zhang & J.-F. Chen and W.C. Lin [X.-Y. Zhang, J.-F. Chen, W.C. Lin, Entire or meromorphic functions sharing one value, Comput. Math. Appl. 56(2008), 1876-1883].

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1 Introduction and Main Results

Let $f$ be a non-constant meromorphic function in the whole complex plane. We shall use following standard notations of the value distribution theory:

$$T(r; f), m(r; f), N(r; f), \mathcal{N}(r; f), \ldots$$

(see Hayman [6], Yang [17] and Yi and Yang[14]). We denote by $S(r; f)$ any quantity satisfying $S(r; f) = o(T(r; f))$, as $r \to +\infty$, possibly outside of a set with finite measure. A meromorphic function $\alpha$ is called a small function with respect to $f$ if $T(r; \alpha) = S(r; f)$. Let $S(f)$ be the set of meromorphic functions

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in the complex plane $\mathbb{C}$ which are small functions with respect to $f$. For some complex number $a \in \mathbb{C} \cup \infty$, we define $\Theta(a, f) = 1 - \lim_{r \to \infty} \frac{N(r, a, f)}{T(r, f)}$.

Let $f$ and $g$ be two nonconstant meromorphic functions defined in the open complex plane $\mathbb{C}$. If for some $\alpha \in S(f) \cap S(g)$ the roots of $f - \alpha$ and $g - \alpha$ coincide in locations and multiplicities we say that $f$ and $g$ share the value $\alpha$ CM (counting multiplicities) and if coincide in locations only we say that $f$ and $g$ share $\alpha$ IM (ignoring multiplicities).

For $a \in \mathbb{C} \cup \infty$ and $k$ a positive integer. We denote by $N(r, a; f) = 1$ the counting function of simple $a$-points of $f$, denote by $N(r, a; f) \leq k$ $(N(r, a; f) \geq k)$ the counting functions of those $a$-points of $f$ whose multiplicities are not greater (less) than $k$ where each $a$-point is counted according to its multiplicity (see [6]). $N(r, a; f) \leq k(N(r, a; f) \geq k)$ are defined similarly, where in counting the $a$-points of $f$ we ignore the multiplicities. Set $N_k(r, a; f) = \overline{N}(r, a; f) + \overline{N}(r, a; f) \geq 2) + \cdots + \overline{N}(r, a; f) \geq k)$.

In 1997, Yang and Hua [13] proved the following result.

**Theorem A** [13] Let $f$ and $g$ be two nonconstant meromorphic functions, $n \geq 11$ an integer, and $a \in \mathbb{C} - \{0\}$. If $f^n f'$ and $g^n g'$ share the value $a$ CM, then either $f = dg$ for some $(n + 1)th$ root of unity $d$ or $g = c_1 e^{cz}$ and $f = c_2 e^{-cz}$ where $c, c_1$, and $c_2$ are constants satisfying $(c_1 c_2)^{n+1} c^2 = -a^2$.

W.C.Lin and H.X.Yi [10] obtained some unicity theorems corresponding to Theorem A.

**Theorem B** [10] Let $f$ and $g$ be two nonconstant meromorphic functions satisfying $\Theta(\infty, f) > \frac{2}{n+1}$, $n \geq 12$. If $[f^n(f - 1)] f'$ and $[g^n(g - 1)] g'$ share 1 CM, then $f \equiv g$.

W.C.Lin and H.X.Yi [11] extended Theorem B by replacing the value 1 with the function $z$ and obtained the following result.

**Theorem C** [11] Let $f$ and $g$ be two transcendental meromorphic functions, $n \geq 12$ an integer. If $f^n(f - 1)f'$ and $g^n(g - 1)g'$ share $z$ CM, then either $f \equiv g$ or $g = \frac{(n+2)(1-z^{n+1})}{(n+1)(1-h^{n+1})}$ and $f = \frac{(n+2)(1-z^{n+1})}{(n+1)(1-h^{n+1})}$, where $h$ is a nonconstant meromorphic function.

Recently, Xiao-Yu Zhang, Jun-Fan Chen and Wei-Chuan Lin [18] extended Theorem F and G and obtained the following result.

**Theorem D** [18] Let $f$ and $g$ be two nonconstant meromorphic functions, and let $n$ and $m$ be two positive integers with $n > \max\{m + 10, 3m + 3\}$, and let $P(z) = a_n z^n + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_m - 1, a_m \neq 0$
are complex constants. If $f^n P(f)f'$ and $g^n P(g)g'$ share 1 CM, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n \left( \frac{a_{m-1}\omega_1^{m-1}}{n+m+1} + \frac{a_{m-2}\omega_1^{m-2}}{n+m} + \cdots + \frac{a_0}{n+1} \right) - \omega_2^n \left( \frac{a_{m-1}\omega_2^{m-1}}{n+m+1} + \frac{a_{m-2}\omega_2^{m-2}}{n+m} + \cdots + \frac{a_0}{n+1} \right)$.

Recent years, I. Lahiri [7,8] and A. Banerjee [1,2] employ the idea of weighted sharing of values which measures how close a shared value is to being shared IM or to being shared CM. Many interesting results [3,4,7,8,12,14,15] were obtained by many mathematicians such as H.X. Yi, I. Lahiri, A.Banerjee, W.C. Lin, X.M.Li and so on.

In 2008, A. Banerjee [2] employed the idea of weighted sharing of values and obtained the following result which improved Theorem A.

**Theorem E** [2] Let $f$ and $g$ be two nonconstant meromorphic functions and $n > 22 - \left[ 5\Theta(\infty; f) + 5\Theta(\infty; g) + \min\{\Theta(\infty; f), \Theta(\infty; g)\} \right]$, is an integer. If for any $a \in \mathbb{C} - 0$, $E_2(a; f^n f') = E_2(a; g^n g')$, the conclusion of Theorem A holds.

Regarding Theorem C,D and E, it is natural to ask the following questions.

**Question 1.1** Is it possible that the value 1 can be replaced by a function $z$ or a small function in Theorems D and E?

**Question 1.2** Is it possible to relax the nature of sharing $z$ or a small function in Theorem C and D and if possible, how far?

In this paper we shall investigate the possible solutions of the above questions. We now state the following theorems which are the main results of the paper.

**Theorem 1.1** Let $f$ and $g$ be two transcendental meromorphic functions, and let $n$ and $m$ be two positive integers with $n > \max\{m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g)), 3m + 1\}$, and let $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$, where $a_0 \neq 0, a_1, \ldots, a_m, a_m \neq 0$ are complex constants. If $f^n P(f)f'$ and $g^n P(g)g'$ share $z$ CM, then either $f \equiv tg$ for a constant $t$ such that $t^d = 1$, where $d = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(\omega_1, \omega_2) = \omega_1^n \left( \frac{a_{m-1}\omega_1^{m-1}}{n+m+1} + \frac{a_{m-2}\omega_1^{m-2}}{n+m} + \cdots + \frac{a_0}{n+1} \right) - \omega_2^n \left( \frac{a_{m-1}\omega_2^{m-1}}{n+m+1} + \frac{a_{m-2}\omega_2^{m-2}}{n+m} + \cdots + \frac{a_0}{n+1} \right)$.

**Theorem 1.2** Let $f$ and $g$ be two transcendental meromorphic functions, and let $n, l$ and $m$ be three positive integers with $n > \max\{\frac{2}{3}m + \frac{38}{3} - 2(\Theta(\infty; g) +
Theorem 1.3 Let $f$ and $g$ be two transcendental meromorphic functions, and let $n, l$ and $m$ be three positive integers with $n > \max\{m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g)), 3m + 1\}$. If $\mathcal{E}_{1}(z, f^{n}P(f)f') = \mathcal{E}_{1}(z, g^{n}P(g)g')$ and $\mathcal{E}_{1}(z, f^{n}P(f)f') = \mathcal{E}_{1}(z, g^{n}P(g)g')$, where $l \geq 3$, then the conclusion of Theorem 1.1 holds.

Remark 1.1 From Theorem 1.1 and 1.3, we can get the same conclusion under the condition of Theorem 1.1 holds.

Though the standard definitions and notations of the value distribution theory are available in [6], we explain some definitions and notations which are used in the paper.

Definition 1.1 [2] Let $k$ and $r$ be two positive integers such that $1 \leq r < k - 1$ and for $a \in \mathbb{C}$, $\mathcal{E}_{k}(a; f) = \mathcal{E}_{k}(a; g)$, $E_{r}(a; f) = E_{r}(a; g)$. Let $z_{0}$ be a zero of $f - a$ of multiplicity $p$ and a zero of $g - a$ of multiplicity $q$. We denote by $N_{L}(r, a; f)(N_{L}(r, a; g))$ the reduced counting function of those $a$-points of $f$ and $g$ for which $p > q \geq r + 1(q > p \geq r + 1)$, by $N_{E}(r, a; f)$ the reduced counting function of those $a$-points of $f$ and $g$ for which $p = q \geq r + 1$, by $N_{f \geq k+1}(r, a; f|g \neq a)(N_{g \geq k+1}(r, a; g|f \neq a))$ the reduced counting functions of those $a$-points of $f$ and $g$ for which $p \geq k + 1$ and $q = 0(q \geq k + 1$ and $p = 0).

Definition 1.2 [2] If $r = 0$ in definition 1.1 then we use the same notations as in definition 1.5 except by $N_{L}(r, a; f)$ we mean the common simple $a$-points of $f$ and $g$ and by $N_{E}(r, a; f)$ we mean the reduced counting functions of those $a$-points of $f$ and $g$ for which $p = q \geq 2$.

Definition 1.3 [8] Let $a, b \in \mathbb{C} \cup \{\infty\}$. We denote by $N(r, a; f|g = b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are $b$-points of $g$; by $N(r, a; f|g \neq b)$ the counting function of those $a$-points of $f$, counted according to multiplicity, which are not the $b$-points of $g$.

2 Some Lemmas

For the proof of our results we need the following lemmas.
Lemma 2.1 [16] Let \( f \) be a nonconstant meromorphic function and \( P(f) = a_0 + a_1 f + a_2 f^2 + \cdots + a_n f^n \), where \( a_0, a_1, a_2, \ldots, a_n \) are constants and \( a_n \neq 0 \). Then
\[
T(r, P(f)) = nT(r, f) + S(r, f).
\]

Lemma 2.2 [9] If \( N(r, 0; f^{(k)}|f \neq 0) \) denotes the counting function of those zeros of \( f^{(k)} \) which are not the zeros of \( f \), where a zero of \( f^{(k)} \) is counted according to its multiplicity then
\[
N(r, 0; f^{(k)}|f \neq 0) \leq kN(r, \infty; f) + N(r, 0; |f| < k) + kN(r, 0; |f| \geq k) + S(r, f).
\]

Lemma 2.3 [5] Let \( F \) and \( G \) be two meromorphic functions. If \( F \) and \( G \) share \( 1 \) \( CM \), one of the following three cases holds:
(i) \( T(r, F) \leq N_2(r, \infty, \infty; F) + N_2(r, \infty, G) + N_2(0, 0, F) + N_2(0, 0, G) + S(r, F) + S(r, G) \), the same inequality holding for \( T(r, G) \);
(ii) \( F \equiv G \);
(iii) \( F \cdot G \equiv 1 \).

Lemma 2.4 [2] Let \( F, G \) be two nonconstant meromorphic functions such that \( E_{1j}(1; F) = E_{1j}(1; F) \) and \( H \neq 0 \). Then
\[
N^1_{E}(r, 1; F) \leq N(r, \infty; H) + S(r, F) + S(r, G),
\]
where \( H = (\frac{F'}{F} - \frac{2F'}{F}) - (\frac{G'}{G} - \frac{2G'}{G}) \).

Lemma 2.5 [2] Let \( \mathcal{E}_{1j}(1; F) = \mathcal{E}_{1j}(1; G), E_{1j}(1; F) = E_{1j}(1; G) \) and \( H \neq 0 \), where \( l \geq 3 \). Then
\[
N(r, \infty; H) \\
\leq N(r, 0; F, G) + N(r, 0; G, F) + N(r, \infty; F, G) + N(r, \infty; G, F) + N(r, 0; F, G) + N(r, 0; G, F) + N(r, 0; F, G) + N(r, 0; G, F),
\]
where \( N_0(r, 0; F') \) is the reduced counting function of those zeros of \( F' \) which are not the zeros of \( F(F - 1) \) and \( N_0(r, 0; G') \) is similarly defined.

Lemma 2.6 [2] Let \( \mathcal{E}_{2j}(1; F) = \mathcal{E}_{2j}(1; G) \) and \( H \neq 0 \). Then
\[
N(r, \infty; H) \\
\leq N(r, 0; F, G) + N(r, 0; G, F) + N(r, \infty; F, G) + N(r, \infty; G, F) + N(r, 0; F, G) + N(r, 0; G, F) + N(r, 0; F, G) + N(r, 0; G, F).
\]
Lemma 2.7 [2] Let $E_l(1; F) = E_l(1; G)$ and $E_1(1; F) = E_1(1; G)$ and $H \neq 0$, where $l \geq 3$. Then

$$2N_{E_l}(r, 1; F) + 2N_{E_l}(r, 1; G) + N_{E_l}^2(r, 1; F)$$

$$+ lN_{E_{G=3}}(r, 1; G | F \neq 1) - N_{E_{G=2}}(r, 1; G)$$

$$\leq N(r, 1; G) - N(r, 1; G).$$

Lemma 2.8 Let $E_l(1; F) = E_l(1; G), E_1(1; F) = E_1(1; G)$, where $l \geq 3$. Then

$$N_{E_{G=3}}(r, 1; G) + N_{E_{G=2}}(r, 1; G | F \neq 1)$$

$$\leq \frac{2}{3} N(r, 0; F) + \frac{2}{3} N(r, \infty; F) - \frac{2}{3} N_0(r, 0; F') + S(r, F).$$

Proof: We note that any 1-point of $F$ with multiplicity $\geq 3$ is counted at most twice. Hence by using Lemma 2.2 we see that

$$N_{E_{G=3}}(r, 1; G) + N_{E_{G=2}}(r, 1; G | F \neq 1)$$

$$\leq N(r, 1; F | G \geq 3; | G | = 2) + 2N(r, 1; F | G \neq 1)$$

$$\leq \frac{2}{3} N(r, 0; F' | F = 1)$$

$$\leq \frac{2}{3} N(r, 0; F' | F \neq 0) - \frac{2}{3} N_0(r, 0; F')$$

$$\leq \frac{2}{3} N(r, 0; F) + \frac{2}{3} N(r, \infty; F) - \frac{2}{3} N_0(r, 0; F') + S(r, F),$$

where by $N(r, 1; F | G \geq 3; | G | = 2)$ we mean the reduced counting function of 1 points of $F$ with multiplicity not less than 3 which are the 1-points of $G$ with multiplicity 2. This completes the proof of the lemma.

Lemma 2.9 Let $E_l(1; F) = E_l(1; G), E_1(1; F) = E_1(1; G)$ and $H \neq 0$, where $l \geq 3$. Then

$$T(r, F)$$

$$\leq N_2(r, 0; F) + N_2(r, \infty; F) + N_2(r, 0; G) + N_2(r, \infty; G)$$

$$+ \frac{2}{3} N(r, 0; F) + \frac{2}{3} N(r, \infty; F) + S(r, F) + S(r, G).$$
**Proof:** Using Lemmas 2.4, 2.5 and 2.7, we get
\[
\overline{N}(r, 1; F) + \overline{N}(r, 1; G)
\]
\[
\leq N(r, 1; F) = 1 + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F)
\]
\[
+ \overline{N}_{F\geq l+1}(r, 1; F|G \neq 1) + \overline{N}(r, 1; G)
\]
\[
\leq N(r, 0; F|\geq 2) + \overline{N}(r, \infty; F|\geq 2) + \overline{N}(r, 0; G|\geq 2)
\]
\[
+ \overline{N}(r, \infty; G|\geq 2) + \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G)
\]
\[
+ \overline{N}_{F\geq l+1}(r, 1; F|G \neq 1) + \overline{N}_{G\geq l+1}(r, 1; G|F \neq 1)
\]
\[
+ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_E^2(r, 1; F)
\]
\[
+ \overline{N}_{F\geq l+1}(r, 1; F|G \neq 1) + T(r, G) - m(r, 1; G)
\]
\[
+ O(1) - 2\overline{N}_L(r, 1; F) - 2\overline{N}_L(r, 1; G) - \overline{N}_E^2(r, 1; F)
\]
\[
- l(\overline{N}_{G\geq l+1}(r, 1; G|F \neq 1) + \overline{N}_{F\geq 2}(r, 1; G) + \overline{N}_0(r, 0; F')
\]
\[
+ \overline{N}_0(r, 0; G') + S(r, F) + S(r, G).
\]

From Lemma 2.8, we can get
\[
\overline{N}(r, 1; F) + \overline{N}(r, 1; G)
\]
\[
\leq N(r, 0; F|\geq 2) + \overline{N}(r, \infty; F|\geq 2) + \overline{N}(r, 0; G|\geq 2)
\]
\[
+ \overline{N}(r, \infty; G|\geq 2) + T(r, G) - m(r, 1; G)
\]
\[
+ 2\overline{N}_{F\geq l+1}(r, 1; F|G \neq 1) + \overline{N}_{F\geq 2}(r, 1; G)
\]
\[
- (l - 1)\overline{N}_{G\geq l+1}(r, 1; G|F \neq 1) + \overline{N}_0(r, 0; F')
\]
\[
+ \overline{N}_0(r, 0; G') + S(r, F) + S(r, G).
\]

By the second fundamental theorem, we have
\[
(2) \quad T(r, F) \leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, 1; F) - \overline{N}_0(r, 0; F') + S(r, F),
\]
\[
(3) \quad T(r, G) \leq \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; G') + S(r, G).
\]

Adding (2) and (3) and from (1), we get
\[
T(r, F) + T(r, G)
\]
\[
\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G)
\]
\[
+ \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; F') - \overline{N}_0(r, 0; G')
\]
\[
+ S(r, F) + S(r, G)
\]
\[
\leq N_2(r, 0; F) + N_2(r, \infty; F) + N_2(r, 0; G) + N_2(r, \infty; G)
\]
\[
+ T(r, G) - m(r, 1; G) + \frac{2}{3}\overline{N}(r, 0; F) + \frac{2}{3}\overline{N}(r, \infty; F)
\]
\[
- (l - 1)\overline{N}_{G\geq l+1}(r, 1; G|F \neq 1) + S(r, F) + S(r, G).
\]
Thus, we can get

\[
T(r, F) \\
\leq N_2(r, 0; F) + N_2(r, \infty; F) + N_2(r, 0; G) + N_2(r, \infty; G) \\
+ \frac{2}{5}N(r, 0; F) + \frac{2}{5}N(r, \infty; F) + S(r, F) + S(r, G).
\]

Therefore, we complete the proof of Lemma 2.9.

**Lemma 2.10** Let \(E_{l}(1; F) = E_{l}(1; G), E_{2}(1; F) = E_{2}(1; G)\) and \(H \neq 0\), where \(l \geq 4\). Then

\[
T(r, F) + T(r, G) \leq 2N_2(r, \infty; F) + 2N_2(r, \infty; G) + 2N_2(r, 0; F) \\
+ 2N_2(r, 0; G) + S(r, F) + S(r, G).
\]

**Proof:** By Lemma 2.6, we can get

\[
T(r, F) + T(r, G) \\
\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) \\
+ \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; F') \\
- N_0(r, 0; G') + S(r, F) + S(r, G) \\
\leq \overline{N}(r, \infty; F) + \overline{N}(r, 0; F) + \overline{N}(r, \infty; G) + \overline{N}(r, 0; G) \\
+ \overline{N}(r, 1; F) + \overline{N}(r, 1; G) - \overline{N}_0(r, 0; G') + S(r, F) + S(r, G) \\
\leq N_2(r, \infty; F) + N_2(r, 0; F) + N_2(r, \infty; G) + N_2(r, 0; G) \\
+ \overline{N}_L(r, 1; F) + \overline{N}_L(r, 1; G) + \overline{N}_{F \geq l+1}(r, 1; F \neq 1) \\
+ \overline{N}(r, 1; G) + \overline{N}_{G \geq l+1}(r, 1; G|F \neq 1) + \overline{N}(r, 1; F \geq 2) \\
+ S(r, F) + S(r, G).
\]

Since

\[
\overline{N}(r, 1; F) = l; G| = l - 1 \geq \overline{N}(r, 1; F = l; G| = 3) \leq \overline{N}(r, 1; F = l);
\]

and

\[
\overline{N}(r, 1; G) = l; F| = l - 1 \geq \overline{N}(r, 1; G = l; F| = 3) \leq \overline{N}(r, 1; G = l),
\]
we see that
\[
\bar{N}_L(r, 1; F) + \bar{N}_L(r, 1; G) + \bar{N}_{F \geq l + 1}(r, 1; F|G \neq 1) \\
+ \bar{N}_{G \geq l + 1}(r, 1; G|F \neq 1) + \bar{N}(r, 1; F) \geq l + 2 + \bar{N}(r, 1; G) \\
\leq \bar{N}(r, 1; F) = l; G| = l - 1 + \cdots + \bar{N}(r, 1; F) = l; G| = 3 \\
+ \bar{N}(r, 1; F) \geq l + 2 + \bar{N}(r, 1; G| = l; F) = l - 1 + \cdots \\
+ \bar{N}(r, 1; G| = l; F) = 3 + \bar{N}(r, 1; G| \geq l + 2 \\
+ \bar{N}(r, 1; G| \geq l + 2 + \bar{N}(r, 1; F) \geq l + 1 \\
+ \bar{N}(r, 1; G| \geq l + 1 + \bar{N}(r, 1; F) = 2 + \cdots \\
(6) \\
+ \bar{N}(r, 1; F) = l + \bar{N}(r, 1; F) \geq l + 1 + \bar{N}(r, 1; G| = 1) \\
+ \cdots + \bar{N}(r, 1; G| = l + \bar{N}(r, 1; G| \geq l + 1) \\
\leq \frac{1}{2} N(r, 1; F) = 1 + \bar{N}(r, 1; F) = 2 + \cdots + 2 \bar{N}(r, 1; F) = l \\
+ 2 \bar{N}(r, 1; F) \geq l + 1 + \bar{N}(r, 1; F) \geq l + 2 + \frac{1}{2} N(r, 1; G| = 1 \\
+ \bar{N}(r, 1; G| = 2 + \cdots + 2 \bar{N}(r, 1; F) = l + 2 \bar{N}(r, 1; G| \geq l + 1 \\
+ \bar{N}(r, 1; G| \geq l + 2 \\
\leq \frac{1}{2} [N(r, 1; F) + N(r, 1; G)] \\
\leq \frac{1}{2} [T(r, F) + T(r, G)].
\]

From (5) and (6), we can get
\[
T(r, F) + T(r, G) \leq 2N_2(r, \infty; F) + 2N_2(r, \infty; G) + 2N_2(r, 0; F) \\
+ 2N_2(r, 0; G) + S(r, F) + S(r, G).
\]

Thus, we complete the proof of Lemma 2.10.

**Lemma 2.11** Let \( f \) and \( g \) be two transcendental meromorphic functions, and let \( n \) and \( m \) be three positive integers with \( n \geq 7 \), and let \( P(z) = a_m z_m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 \), where \( a_0 \neq 0, a_1, \ldots, a_{m-1}, a_m \neq 0 \) are complex constants. If \( f^n P(f) f' \) and \( g^n P(g) g' \) share \( z \) IM, then \( S(r, f) = S(r, g) \).

**Proof** Let \( F_1 = f^n P(f) f' \) and \( G_1 = g^n P(g) g' \), by Lemma 2.1, we have
\[
(n + m)T(r, f) = T(r, \frac{F_1}{f}) + O(1) \leq T(r, F_1) + T(r, f') + S(r, f).
\]

Hence,
\[
(7) \quad (n + m - 2)T(r, f) + S(r, f) \leq T(r, F_1).
\]

Since
\[
T(r, F_1) \leq T(r, f^n P(f)) + T(r, f') + S(r, f) \\
\leq (n + m + 2)T(r, f) + S(r, f).
\]
From (7) and (8), we have $S(r, F_1) = S(r, f)$. From the condition of Lemma 2.11 and the second fundamental theory, we have

\[
T(r, F_1) \leq \overline{N}(r, \infty; F_1) + \overline{N}(r, 0; F_1) + \overline{N}(r, z; F_1) = S(r, F_1)
\]

\[
\leq \overline{N}(r, \infty; f) + \overline{N}(r, 0; f^n P(f)) + \overline{N}(r, z; G_1) + S(r, f)
\]

\[
\leq (m + 4)T(r, f) + T(r, G_1) + S(r, f)
\]

\[
\leq (m + 4)T(r, f) + (n + m + 2)T(r, g) + S(r, f) + S(r, g).
\]

Thus, we have

\[
(n - 6)T(r, f) \leq (n + m + 2)T(r, g) + S(r, g) + S(r, f).
\]

Since $n \geq 7$, we can get the conclusion of Lemma 2.11.

**Lemma 2.12** Let $f$ and $g$ be two transcendental meromorphic functions. Then

\[
f^n P(f) f' g^n P(g) g' \neq z^2,
\]

where $n > 3m + 1$ is a positive integer.

**Proof:** Let

\[
f^n P(f) f' g^n P(g) g' \equiv z^2.
\]

Now we rewrite $P(z) = a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0$ as

\[
P(z) = a_m(z - \beta_1)^{\gamma_1} \cdots (z - \beta_s)^{\gamma_s},
\]

where $\gamma_1 + \cdots + \gamma_i + \cdots + \gamma_s = m$, $1 \leq s \leq m$, $\beta_i \neq \beta_j$, $i \neq j$, $1 \leq i, j \leq s$ and $\beta_1, \ldots, \beta_i, \ldots, \beta_s$ are nonzero constants and $\gamma_1, \ldots, \gamma_i, \ldots, \gamma_s$ are positive integers.

Let $z_0(\neq 0, \infty)$ be a zero of $f$ of order $p(\geq 1)$ and it be a pole of $g$. Suppose that $z_0$ is an order $q(\geq 1)$. Then $np + p - 1 = (n + m)q + q + 1$ i.e., $mq = (n + 1)(p - q) - 2 \geq n - 1$ i.e., $q \geq \frac{n-1}{m}$. So $p \geq \frac{n+m-1}{m}$.

Let $z_1(\neq 0, \infty)$ be a zero of $P(f)$ of order $p_1$ and be a zero of $f - \beta_i$ of order $q_i$ for $i = 1, 2, \ldots, s$. Then $p_1 = \gamma_i q_i$ for $i = 1, 2, \ldots, s$. Suppose that $z_1$ is a pole of $g$ of order $q$. Again by (9) we can obtain $q_i \gamma_i + q_i - 1 = nq + mq + q + 1$, i.e., $q_i \geq \frac{n+m-1}{\gamma_i+1}$ for $i = 1, 2, \ldots, s$.

Let $z_2(\neq 0, \infty)$ be a zero of $f'$ of order $p_2$ that is not a zero of $f P(f)$. Similarly, we get $p_2 \geq n + m + 2$.

Therefore we can get that a pole of $f$ is either a zero of $g P(g)$ or a zero of $g'$, we get

\[
\overline{N}(r, \infty; f) \leq \overline{N}(r, 0; g) + \overline{N}(r, \beta_1; g) + \cdots + \overline{N}(r, \beta_s; g) + \cdots
\]

\[
+ \overline{N}(r, \beta_i; g) + \overline{N}_0(0; g')
\]

\[
\leq \frac{m}{n+m+1}T(r, g) + \frac{m+s}{n+m+3}T(r, g) + \overline{N}_0(r, 1/g'),
\]

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where $N_0(r;0;g')$ is the reduced counting function of those zeros of $g'$ which are not the zeros of $gP(g)$.

By the second fundamental theorem we obtain

$$sT(r,f) \leq N(r,\infty;f) + N(r,0;f) + N(r,\beta_1;f) + \cdots + N(r,\beta_k;f)$$
$$+ \cdots + N(r,\beta_l;g) - N_0(r,0;f') + S(r,f)$$
$$\leq \left( \frac{m}{n+m+1} + \frac{m+s}{n+m+3} \right) T(r,g) + \left( \frac{m}{n+m+3} + \frac{m+s}{n+m+7} \right) T(r,f)$$
$$+ N_0(r,0;g') - N_0(r,0;f') + 2 \log r + S(r,f).$$

Similarly we get

$$sT(r,g) \leq \left( \frac{m}{n+m+1} + \frac{m+s}{n+m+3} \right) T(r,f) + \left( \frac{m}{n+m+3} + \frac{m+s}{n+m+7} \right) T(r,g)$$
$$+ N_0(r,0;f') - N_0(r,0;g') + 2 \log r + S(r,g).$$

Adding (10) and (11) we get

$$(s - \frac{2m}{n + m - 1} - \frac{2m + 2s}{n + m + 3}) \{T(r,f) + T(r,g)\} \leq 4 \log r + S(r,f) + S(r,g).$$

From $1 \leq s \leq m$ and $n \geq 3m + 2$, we can get a contradiction.

Thus, we can get the conclusion of this lemma.

**Lemma 2.13** Let $f$ and $g$ be two transcendental meromorphic functions, and let $n$ and $m$ be three positive integers with $n \geq m + 3$, $F = \frac{f^n P(f)}{z}$ and $G = \frac{g^n P(g)}{z}$, where $n \geq 4$ is a positive integer. If $F \equiv G$, then either $f \equiv t g$ for a constant $t$ such that $t^d = 1$, where $d = (n+m+1, \ldots, n+m+1-i, \ldots, n+1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$, or $f$ and $g$ satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(\omega_1, \omega_2)$ is the definition of Theorem 1.4.

**Proof:** Let

$$F^* = \frac{a_m f^{n+m+1}}{n+m+1} + \frac{a_{m-1} f^{n+m}}{n+m} + \cdots + \frac{a_0 f^{n+1}}{n+1},$$

and

$$G^* = \frac{a_m g^{n+m+1}}{n+m+1} + \frac{a_{m-1} g^{n+m}}{n+m} + \cdots + \frac{a_0 g^{n+1}}{n+1}.$$

From $F \equiv G$, we can get

$$(12) \quad F^* \equiv G^* + C,$$

where $C$ is a constant. Then we have $T(r,f) = T(r,g) + S(r,f)$.
Suppose that $C \neq 0$, by the second fundamental theorem, we have

\begin{equation}
(n + m + 1)T(r, f) = T(r, F^*) \leq N(r, 0; F^*) + N(r, \infty; F^*) + N(r, C; F^*) + S(r, f) \leq (2m + 3)T(r, g) + S(r, g).
\end{equation}

By $n \geq m + 3$, we can get a contradiction. Thus, we can get $F^* \equiv G^*$, i.e.,

\begin{equation}
\frac{amg^{n+m+1}(h^{n+m+1} - 1)}{n + m + 1} + \frac{a_{m-1}f^{n+m}(h^{n+m} - 1)}{n + m} + \cdots + \frac{a_0f^{n+1}(h^{n+1} - 1)}{n + 1} = 0,
\end{equation}

which implies $h^\mu = 1$, where $\mu = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1)$, $a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$. Thus $f \equiv tg$ for a constant $t$ such that $t^\mu = 1$, where $\mu = (n + m + 1, \ldots, n + m + 1 - i, \ldots, n + 1), a_{m-i} \neq 0$ for some $i = 0, 1, \ldots, m$.

If $h$ is not a constant, then we can get that $f$ and $g$ satisfy the algebraic equation $R(f, g) = 0$, where $R(\omega_1, \omega_2) = \omega^1_1 \left( \frac{a_m \omega^{m-1}_1}{n + m + 1} + \frac{a_{m-1} \omega^{m-2}_1}{n + m} + \cdots + \frac{a_0}{n + 1} \right)$.

Thus, we complete the proof of Lemma 2.13.

**Lemma 2.14** [15] Let $f$ and $g$ be two nonconstant meromorphic functions. If $h \equiv 0$ where $h \equiv \left( \frac{f'}{f} - 2\frac{f'}{Fz} \right) - \left( \frac{g'}{g} - \frac{2g'}{Gz} \right)$, then $f, g$ share 1 CM.

## 3 The Proofs of Theorems

Let $F, G, F^*$ and $G^*$ be the definition of Lemma 2.13, and $F_1, G_1$ be the definition of Lemma 2.11.

**The Proof of Theorem 1.1:** From the condition of Theorem 1.1, we have $F, G$ share 1 CM.

By Lemma 2.1, we have

\begin{equation}
\begin{aligned}
T(r, F^*) &= (n + m + 1)T(r, f) + S(r, f), \\
T(r, G^*) &= (n + m + 1)T(r, g) + S(r, g).
\end{aligned}
\end{equation}

Since $(F^*)' = Fz$, we deduce

\[m(r, \frac{1}{F^*}) \leq m(r, \frac{1}{Fz}) + S(r, f) \leq m(r, \frac{1}{F}) + \log r + S(r, f),\]
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and by the first fundamental theorem

\[
T(r, F^*) \leq T(r, F) + N(r, 0; F^*) - N(r, 0; F) + \log r + S(r, f)
\]
\[
\leq T(r, F) + N(r, 0; f) + N(r, b_1; f) + \cdots + N(r, b_m; f)
\]
\[
- N(r, c_1; f) - \cdots - N(r, c_m; f) - N(r, 0; f')
\]
\[
+ \log r + S(r, f),
\]

where \(b_1, b_2, \ldots, b_m\) are roots of the algebraic equation \(\frac{a_m z^m}{n+m+1} + \frac{a_{m-1} z^{m-1}}{n+m} + \cdots + \frac{a_0}{n+1} = 0\), and \(c_1, c_2, \ldots, c_m\) are roots of the algebraic equation \(a_m z^m + a_{m-1} z^{m-1} + \cdots + a_1 z + a_0 = 0\).

By the definition of \(F, G\), we have

\[
N_2(r, 0; F) + N_2(r, \infty; F)
\]
\[
\leq 2N(r, \infty; f) + 2N(r, 0; f) + N(r, c_1; f)
\]
\[
+ \cdots + N(r, c_m; f) + N(r, 0; f') + 2 \log r.
\]

Similarly, we obtain

\[
N_2(r, 0; G) + N_2(r, \infty; G)
\]
\[
\leq 2N(r, \infty; g) + 2N(r, 0; g) + N(r, c_1; g)
\]
\[
+ \cdots + N(r, c_m; g) + N(r, 0; g') + 2 \log r.
\]

If Lemma 2.3(i) holds, from (17),(18), we have for \(\varepsilon > 0\)

\[
T(r, F^*) \leq (m + 3)T(r, f) + (m + 4)T(r, g) + 2N(r, \infty; f)
\]
\[
+ 2N(r, \infty; g) + 5 \log r + S(r, f) + S(r, g)
\]
\[
\leq (m + 3)T(r, f) + (m + 4)T(r, g) + (2 - 2\Theta(\infty; f) + \varepsilon)T(r, f)
\]
\[
+ (2 - 2\Theta(\infty; g) + \varepsilon)T(r, g) + 5 \log r + S(r, f) + S(r, g)
\]
\[
\leq [2m + 11 - (2\Theta(\infty; f) + 2\Theta(\infty; g) + 2\varepsilon)]T(r) + 5 \log r + S(r),
\]

where \(T(r) = \max\{T(r, f), T(r, g)\}\) and \(S(r) = \max\{S(r, f), S(r, g)\}\).

Similarly, we obtain

\[
T(r, G^*) \leq [2m + 11 - (2\Theta(\infty; f) + 2\Theta(\infty; g) + 2\varepsilon)]T(r) + 5 \log r + S(r).
\]

By (15),(19) and (20), we have

\[
[n - m - 10 + (2\Theta(\infty; f) + 2\Theta(\infty; g) + 2\varepsilon)]T(r) \leq 5 \log r + S(r).
\]

Since \(f, g\) are two transcendental meromorphic functions and \(n > m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g))\), we can obtain a contradiction.

If Lemma 2.3(ii) holds, then \(F \equiv G\). By Lemma 2.13, we can get the conclusion of Theorem 1.1.
If Lemma 2.2(iii) holds, then $F \cdot G \equiv 1$. By Lemma 2.12 and $n > \max\{m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g)), 3m + 1\}$, we can get a contradiction.

Therefore, we complete the proof of Theorem 1.1.

The Proof of Theorem 1.2: From the condition of Theorem 1.2 and the definition of $F, G$, we have $E_{(1)}(1, F) = E_{(1)}(1, G)$, $E_{(1)}(1, F) = E_{(1)}(1, G)$ where $l \geq 3$ and

$$
\begin{align*}
\mathcal{N}(r, \infty; F) & \leq \mathcal{N}(r, \infty; f) + \log r, \\
\mathcal{N}(r, 0; F) & \leq \mathcal{N}(r, 0; f) + N(r, c_1; f) + \cdots + N(r, c_m; f) + N(r, 0; f') + \log r,
\end{align*}
$$

where $c_1, c_2, \ldots, c_m$ are the definition of Subsection 3.1.

Suppose that $H \neq 0$. From (16)-(18), (22) and Lemma 2.9, we have

$$
\begin{align*}
[\frac{5}{3}m - \frac{38}{3} + \frac{8}{3}\Theta(\infty; f) + 2\Theta(\infty; g)]T(r) & \leq \frac{19}{3}\log r + S(r),
\end{align*}
$$

or

$$
\begin{align*}
[\frac{5}{3}m - \frac{38}{3} + \frac{8}{3}\Theta(\infty; g) + 2\Theta(\infty; f)]T(r) & \leq \frac{19}{3}\log r + S(r).
\end{align*}
$$

From $n > \frac{5}{3}m + \frac{38}{3} - 2(\Theta(\infty; g) + \Theta(\infty; f) - \frac{2}{3}\min\{\Theta(\infty; g), \Theta(\infty; f)\})$ and $f, g$ are two transcendental meromorphic functions, we can get a contradiction.

Therefore, we can get $H \equiv 0$. From Lemma 2.14, we have that $F, G$ share $1\, CM$. By $n > \max\{\frac{5}{3}m + \frac{38}{3} - 2(\Theta(\infty; g) + \Theta(\infty; f) - \frac{2}{3}\min\{\Theta(\infty; g), \Theta(\infty; f)\}), 3m + 1\}$ and Theorem 1.1, we can obtain the conclusion of Theorem 1.2.

Therefore, we complete the proof of Theorem 1.2

The Proof of Theorem 1.3: From the condition of Theorem 1.2 and the definition of $F, G$, we have $E_{(1)}(1, F) = E_{(1)}(1, G)$, and $E_{(2)}(1, F) = E_{(2)}(1, G)$ where $l \geq 4$.

Suppose that $H \neq 0$. From (16)-(18) and Lemma 2.10, we have

$$
\begin{align*}
[\frac{1}{3}m - 10 + (2\Theta(\infty; f) + 2\Theta(\infty; g) + 2\varepsilon)]T(r) & \leq 5\log r + S(r).
\end{align*}
$$

Since $f, g$ are two transcendental meromorphic functions and $n > m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g))$, we can get a contradiction.

Therefore, we can get $H \equiv 0$. From Lemma 2.14, we have that $F, G$ share $1\, CM$. From Theorem 1.1 and $n > \max\{m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g)), 3m + 1\}$, we can obtain the conclusion of Theorem 1.3.

Therefore, we complete the proof of Theorem 1.3.
4 Remarks

It follows from the proof of Theorem 1.1 that if the condition \( f^n P(f)f' \) and \( g^n P(g)g' \) share \( z \) CM is replaced by the condition \( f^n P(f)f' \) and \( g^n P(g)g' \) share \( \alpha(z) \) CM, where \( \alpha(z) \) is a meromorphic function such that \( \alpha(z) \neq 0, \infty \) and \( \alpha(z) \in S(f) \cap S(g) \), the conclusion of Theorem 1.1 still holds.

Similarly, we can get the following results.

Theorem 4.1 Let \( f \) and \( g \) be two transcendental meromorphic functions, and let \( n, l \) and \( m \) be three positive integers with \( n > \max \{ \frac{2}{3}m + \frac{2h}{3} - 2(\Theta(\infty; f) - \frac{1}{3} \min \{ \Theta(\infty; g), \Theta(\infty; f) \}, 3m+1 \} \). If \( E_l(\alpha(z), f^n P(f)f') = E_l(\alpha(z), g^n P(g)g') \) and \( E_1(\alpha(z), f^n P(f)f') = E_1(\alpha(z), g^n P(g)g') \), where \( l \geq 3 \), then the conclusion of Theorem 1.1 holds.

Theorem 4.2 Let \( f \) and \( g \) be two transcendental meromorphic functions, and let \( n, l \) and \( m \) be three positive integers with \( n > \max \{ m + 10 - (2\Theta(\infty; f) + 2\Theta(\infty; g)), 3m+1 \} \). If \( E_l(\alpha(z), f^n P(f)f') = E_l(\alpha(z), g^n P(g)g') \) and \( E_2(\alpha(z), f^n P(f)f') = E_2(\alpha(z), g^n P(g)g') \), where \( l \geq 4 \), then the conclusion of Theorem 1.1 holds.

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