Approximation of common fixed points for a finite family of Zamfirescu operators

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Abstract

In this paper we introduce a new composite implicit iteration scheme with errors and a strong convergence theorem is established for a finite family of Zamfirescu operators in arbitrary normed spaces. As a corollary we observe that the iteration scheme introduced by Su and Li (18) converges to the common fixed point of a finite family of Zamfirescu operators.

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1 Introduction and preliminary definitions

In recent years, iterative techniques for approximating the common fixed points of a finite family of pseudocontractive mappings, asymptotically nonexpansive mappings, asymptotically quasi-nonexpansive mappings or nonexpansive mappings in Hilbert spaces, uniformly convex Banach spaces or arbitrary Banach spaces have been considered by several authors. [eg., 4, 9, 12, 17, 19, 20, 21]. In 2001, Xu and Ori [22] introduced an implicit iteration process for a finite family of nonexpansive mappings as follows:

Let $K$ be a nonempty closed convex subset of a normed space $E$. Let

\[ X = \bigcap_{i=1}^{n} F(T_i), \]

where $F(T_i) = \{x \in K : d(x, T_i(x)) = d(x, T_i)\}$ for $i = 1, 2, \ldots, n$. Let

\[ T_i : K \rightarrow K \]

be nonexpansive mappings for $i = 1, 2, \ldots, n$. Then $T_i$ has a unique fixed point $x_i$ in $K$. Let

\[ x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T_i(x_n), \]

where $\alpha_n$ is a sequence in $(0, 1)$ for all $n = 0, 1, 2, \ldots$. If $\alpha_n = 1/n$, then the sequence $\{x_n\}$ converges strongly to a common fixed point of $T_1, T_2, \ldots, T_n$.
\{T_1, T_2, ..., T_N\} be \( N \) nonexpansive self-maps of \( K \). Then for an arbitrary point \( x_0 \in K \), and \( \{\alpha_n\} \subset (0, 1) \), the sequence \( \{x_n\} \) generated can be written in the compact form as follows:

\[(1) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n x_n, \quad \forall n \geq 1,\]

where \( T_n = T_{n(modN)} \) (the \( modN \) function takes values in \( I = \{1, 2, 3, ..., N\} \)).

Xu and Ori proved the weak convergence of this process to a common fixed point of a finite family of nonexpansive mappings defined in a Hilbert space.

In 2004, Osilike [12] extended the results of Xu and Ori from nonexpansive mappings to strictly pseudocontractive mappings.

Inspired by the above facts, in 2006 Su and Li [18] introduced a new two-step implicit iteration process which is defined as follows:

Let \( E \) be a real Banach space and \( K \) a nonempty closed convex subset of \( E \). Let \( \{T_i\}_{i=1}^N \) be \( N \) strictly pseudocontractive self-maps of \( K \). From arbitrary \( x_0 \in K \), define the sequence \( \{x_n\} \) by

\[(2) \quad x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n, \quad y_n = \beta_n x_{n-1} + (1 - \beta_n) T_n x_n,\]

where \( T_n = T_{n(modN)} \) and \( \{\alpha_n\}, \{\beta_n\} \subset [0, 1] \).

Using this iteration they proved a convergence theorem for a finite family of strictly pseudocontractive maps. It is observed that the class of Zamfirescu operators is independent (see Rhoades [16]) of the class of strictly pseudocontractive operators.

Consideration of error terms in iterative processes is an important part of the theory. Several authors have introduced and studied one-step, two-step as well as multi-step iteration schemes with errors to approximate fixed points of various classes of mappings in Banach spaces [2, 5, 6, 7, 8, 10, 13, 14].

Let \( K \) be a nonempty closed convex subset of a normed space \( E \). Motivated by the above facts, we introduce the following composite implicit iteration processes with errors for a finite family of Zamfirescu operators \( \{T_i\}_{i=1}^N : K \to K \), and define the sequences \( \{x_n\} \subset K \) as follows:

\[x_0 \in K,\]

\[x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n + u_n,\]

\[y_n = \beta_n x_{n-1} + (1 - \beta_n) T_n x_n + v_n,\]
where $T_n = T_{n(modN)}$ (the modN function takes values in $I = \{1, 2, 3, ..., N\}$),

$\{u_n\}$ and $\{v_n\}$ are two summable sequences in $E$, i.e., $\sum_{n=0}^{\infty} \|u_n\| < \infty$, $\sum_{n=0}^{\infty} \|v_n\| < \infty$, and $\{\alpha_n\}$ and $\{\beta_n\}$ are two sequences in $[0, 1]$, satisfying certain restrictions.

In particular if $u_n = 0, v_n = 0$ for all $n > 0$, then the iteration scheme obtained is the scheme introduced by Su and Li.

We recall the following definitions in a metric space $(X, d)$, from Berinde [1, p.6, 50-51, 131] and Ciric [3, p.268].

A mapping $T : X \to X$ is called an $a$-contraction if
\[(z_1) \quad d(Tx, Ty) \leq ad(x, y) \text{ for all } x, y \in X, \text{ where } a \in [0, 1).\]

The map $T$ is called a Kannan mapping if there exists $b \in [0, \frac{1}{2})$ such that
\[(z_2) \quad d(Tx, Ty) \leq b[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.\]

A similar definition is due to Chatterjea : there exists $c \in [0, \frac{1}{2})$ such that
\[(z_3) \quad d(Tx, Ty) \leq c[d(x, Ty) + d(y, Tx)] \text{ for all } x, y \in X.\]

It is known, see Rhoades [15] that $(z_1), (z_2)$, and $(z_3)$ are independent contractive conditions. An operator $T$ which satisfies at least one of the contractive conditions $(z_1), (z_2)$ and $(z_3)$ is called a Zamfirescu operator or a $Z$-operator. Alternatively we say that $T$ satisfies Condition $Z$.

The main purpose of this paper is to establish a strong convergence theorem to approximate common fixed points of a finite family of Zamfirescu operators in normed spaces using the new iteration scheme defined above.

We need the following lemma.

**Lemma 1** [11]. Let $\{r_n\}, \{s_n\}, \{t_n\}$ and $\{k_n\}$ be sequences of nonnegative numbers satisfying
\[r_{n+1} \leq (1 - s_n)r_n + s_nt_n + k_n, \quad \text{for all } n \geq 1.\]

If $\sum_{n=1}^{\infty} s_n = \infty$, $\lim_{n \to \infty} t_n = 0$ and $\sum_{n=1}^{\infty} k_n < \infty$ hold, then $\lim_{n \to \infty} r_n = 0$.

2 Main result

**Theorem 2** Let $K$ be a nonempty closed convex subset of a normed space $E$. Let $\{T_1, T_2, T_3, ..., T_N\} : K \to K$ be $N$, Zamfirescu operators with $F = \ldots$
\( \cap_{i=1}^{N} F(T_i) \neq \emptyset \) (\( F \) denotes the set of common fixed points of \( \{T_1, T_2, T_3, \ldots, T_N\} \)).

Let \( \{u_n\} \) and \( \{v_n\} \) be two summable sequences in \( E \), and \( \{\alpha_n\} \) and \( \{\beta_n\} \) be two real sequences in \([0, 1]\) satisfying the following conditions:

(i) \( \sum_{n=1}^{\infty} \beta_n (1 - \alpha_n) = \infty \);

(ii) \( \|v_n\| = o(\beta_n) \).

For any \( x_0 \in K \), let the sequence \( \{x_n\} \subset K \) be defined by

\[
\begin{align*}
{x_n} & = \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n + u_n \\
y_n & = \beta_n x_{n-1} + (1 - \beta_n) T_n x_n + v_n
\end{align*}
\]

where \( T_n = T_{n(\text{mod}N)} \) (the mod\( N \) function takes values in \( I = \{1, 2, 3, \ldots, N\} \)). Then \( \{x_n\} \) converges strongly to a common fixed point of \( \{T_1, T_2, T_3, \ldots, T_N\} \).

**Proof.** It follows from the assumption \( F = \cap_{i=1}^{N} F(T_i) \neq \emptyset \), that the operators \( \{T_1, T_2, T_3, \ldots, T_N\} \) have a common fixed point in \( K \), say \( p \). Consider \( x, y \in K \).

Since each \( T_i \) is a Zamfirescu operator, each \( T_i \) satisfies at least one of the conditions \((z_1), (z_2)\) and \((z_3)\).

If \((z_2)\) holds, then for any \( x, y \in K \)

\[
\|T_i x - T_i y\| \leq b [\|x - T_i x\| + \|y - T_i y\|] \\
\leq b [\|x - T_i x\| + \|y - x\| + \|x - T_i x\| + \|T_i x - T_i y\|],
\]

which implies

\[
(1 - b) \|T_i x - T_i y\| \leq b \|x - y\| + 2b \|x - T_i x\|,
\]

since \( 0 \leq b < \frac{1}{2} \) we get

\[
(4) \quad \|T_i x - T_i y\| \leq \frac{b}{1 - b} \|x - y\| + \frac{2b}{1 - b} \|x - T_i x\|.
\]

Similarly, if \((z_3)\) holds, then we have for any \( x, y \in K \)

\[
\|T_i x - T_i y\| \leq c [\|x - T_i y\| + \|y - T_i x\|] \\
\leq c [\|x - T_i x\| + \|T_i x - T_i y\| + \|y - x\| + \|x - T_i x\|]
\]

which implies

\[
(1 - c) \|T_i x - T_i y\| \leq c \|x - y\| + 2c \|x - T_i x\|,
\]
since $0 \leq c < \frac{1}{2}$ we get

\begin{equation}
\|T_i x - T_i y\| \leq \frac{c}{1 - c} \|x - y\| + \frac{2c}{1 - c} \|x - T_i x\|.
\end{equation}

Denote

\begin{equation}
\delta = \max \left\{ a, \frac{b}{1 - b}, \frac{c}{1 - c} \right\}.
\end{equation}

Then we have $0 \leq \delta < 1$ and, in view of (5), (4), (5) and (6), it results that the inequality

\begin{equation}
\|T_i x - T_i y\| \leq \delta \|x - y\| + 2\delta \|x - T_i x\|
\end{equation}

holds for all $x, y \in K$ and for every $i \in \{1, 2, 3, \ldots, N\}$.

Now, since $T_i p = p$, $T_n = T_n^{(\text{mod}N)}$ and the modN function takes values in $\{1, 2, 3, \ldots, N\}$, for $y = x_n$ and $x = p$, the above inequality (7) gives the following result

\begin{equation}
\|T_n x_n - p\| \leq \delta \|x_n - p\|
\end{equation}

Again, with $y = y_n$ and $x = p$, in (7) we get

\begin{equation}
\|T_n y_n - p\| \leq \delta \|y_n - p\|.
\end{equation}

Now, let $\{x_n\}$ be the implicit iteration process with errors defined by (3) and $x_0 \in K$ be arbitrary.

Then

\begin{align*}
\|x_n - p\| &= \|\alpha_n x_{n-1} + (1 - \alpha_n)T_n y_n + u_n - p\| \\
&= \|\alpha_n x_{n-1} + (1 - \alpha_n)T_n y_n + u_n - (\alpha_n + 1 - \alpha_n)p\| \\
&= \|\alpha_n (x_{n-1} - p) + (1 - \alpha_n)(T_n y_n - p) + u_n\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|T_n y_n - p\| + \|u_n\|.
\end{align*}

Using (9) in the above inequality we obtain that

\begin{equation}
\|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n) \|y_n - p\| + \|u_n\|
\end{equation}
Substitute for $y_n$ from (3) we get
\[
\|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\delta \|\beta_n x_{n-1} - 1\|
+ (1 - \beta_n)T_n x_n + v_n - p\| + \|u_n\|
\]
\[
= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\delta \|\beta_n x_{n-1} - 1\|
+ (1 - \beta_n)T_n x_n + v_n - (\beta_n + 1 - \beta_n)p\| + \|u_n\|
\]
\[
= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\delta \|\beta_n(x_{n-1} - p)\|
+ (1 - \beta_n)(T_n x_n - p) + v_n\| + \|u_n\|
\]
\[
\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\delta \left\{ \beta_n \|x_{n-1} - p\| \right. 
+ (1 - \beta_n) \|T_n x_n - p\| + \|v_n\| \left\} + \|u_n\|.
\]
Using (8) in the above inequality we get that
\[
\|x_n - p\| \leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)\delta \left\{ \beta_n \|x_{n-1} - p\| 
+ (1 - \beta_n) \| \|x_{n-1} - p\| + (1 - \alpha_n)\delta \| \|x_{n-1} - p\| + (1 - \alpha_n)\delta \|v_n\| + \|u_n\|
\]
that is
\[
(1 - (1 - \alpha_n)(1 - \beta_n)\delta^2) \|x_n - p\| \leq \left[ \alpha_n + (1 - \alpha_n)\beta_n \delta \right] \|x_{n-1} - p\|
+ (1 - \alpha_n)\delta \|v_n\| + \|u_n\|
\]
since $0 \leq (1 - \alpha_n)(1 - \beta_n)\delta^2 < 1$, we have
\[
(10) \|x_n - p\| \leq \frac{\left[ \alpha_n + (1 - \alpha_n)\beta_n \delta \right] \|x_{n-1} - p\| + (1 - \alpha_n)\delta \|v_n\| + \|u_n\|}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}
\]
\[
= \frac{\alpha_n + (1 - \alpha_n)\beta_n \delta}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2} \|x_{n-1} - p\| 
+ (1 - \alpha_n)\delta \|v_n\| + \|u_n\|
\]
Let
\[
A_n = \alpha_n + (1 - \alpha_n)\beta_n \delta
\]
\[
B_n = 1 - (1 - \alpha_n)(1 - \beta_n)\delta^2.
\]
Consider
\[
1 - \frac{A_n}{B_n} = 1 - \frac{\alpha_n + (1 - \alpha_n)\beta_n \delta}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}
\]
\[
= \frac{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2 - [\alpha_n + (1 - \alpha_n)\beta_n \delta]}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}
\]
\begin{equation}
1 - \frac{[1 - \alpha_n](1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\beta_n\delta]}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}
\end{equation}

Since $1 - (1 - \alpha_n)(1 - \beta_n)\delta^2 \leq 1$, from (11) we have

$$1 - \frac{A_n}{B_n} \geq 1 - \frac{[1 - \alpha_n](1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\beta_n\delta]}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}$$

that is

$$A_n \leq (1 - \alpha_n)(1 - \beta_n)\delta^2 + \alpha_n + (1 - \alpha_n)\beta_n\delta.$$

Using the facts that $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\delta < 1$, we get

\begin{equation}
\frac{A_n}{B_n} \leq (1 - \alpha_n)(1 - \beta_n) + \alpha_n + (1 - \alpha_n)\beta_n\delta
\end{equation}

Hence from (10) and (12) we have

$$\|x_n - p\| \leq \left[1 - (1 - \alpha_n)\beta_n(1 - \alpha_n)\right]\|x_{n-1} - p\|$$

$$+ \frac{(1 - \alpha_n)\delta \|v_n\|}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2} + \frac{(1 - \alpha_n)\beta_n\delta d_n}{1 - (1 - \alpha_n)(1 - \beta_n)\delta^2}\|u_n\|$$

which, by the inequality

$$1 - \delta \leq 1 - (1 - \alpha_n)(1 - \beta_n)\delta^2,$$

implies that

$$\|x_n - p\| \leq \left[1 - (1 - \alpha_n)\beta_n(1 - \alpha_n)\right]\|x_{n-1} - p\| + \frac{(1 - \alpha_n)\delta}{1 - \delta} \|v_n\| + \frac{1}{1 - \delta} \|u_n\|.$$

Since $\|v_n\| = o(\beta_n)$ by assumption, let $\|v_n\| = d_n\beta_n$ and $d_n \to 0$. Therefore from the above inequality we obtain that

$$\|x_n - p\| \leq \left[1 - (1 - \delta)\beta_n(1 - \alpha_n)\right]\|x_{n-1} - p\|$$

$$+ \frac{(1 - \delta)(1 - \alpha_n)d_n\beta_n}{(1 - \delta)^2} + \frac{1}{1 - \delta} \|u_n\|.$$
Setting $r_n = \|x_{n-1} - p\|$, $s_n = (1 - \delta)\beta_n(1 - \alpha_n)$, $t_n = \frac{\delta}{(1 - \delta)^2}d_n$, $k_n = \frac{1}{1 - \delta}\|u_n\|$, and using the facts that $0 \leq \delta < 1$, $0 \leq \alpha_n \leq 1$, $0 \leq \beta_n \leq 1$, $\sum_{n=1}^{\infty} \beta_n(1 - \alpha_n) = \infty$, $d_n \to 0$ and $\sum_{n=1}^{\infty} \|u_n\| < \infty$, it follows from Lemma 1 that
$$\lim_{n \to \infty} \|x_n - p\| = 0$$
which implies that $x_n \to p \in F$. Hence the proof.

**Corollary 3** Let $K$ be a nonempty closed convex subset of a normed space $E$, and let $\{T_1, T_2, T_3, ..., T_N\} : K \to K$ be $N$ Zamfirescu operators with $F = \bigcap_{i=1}^{N} F(T_i) \neq \phi$ ($F(T_i)$ denotes the set of fixed points of $T_i$). Let $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ be two real sequences satisfying the condition $\sum_{n=1}^{\infty} (1 - \alpha_n)\beta_n = \infty$. For $x_0 \in K$, let the sequence $\{x_n\}$ be defined by
$$x_n = \alpha_n x_{n-1} + (1 - \alpha_n) T_n y_n$$
$$y_n = \beta_n x_{n-1} + (1 - \beta_n) T_n x_n$$
where $T_n = T_{n(\text{mod} N)}$. Then $\{x_n\}$ converges strongly to a common fixed point of $\{T_1, T_2, T_3, ..., T_N\}$.

**Remark 4** Chatterjea’s and Kannan’s contractive conditions $(z_2)$ and $(z_3)$ are both included in the class of Zamfirescu operators and so their convergence theorems for the implicit iteration process with errors defined by (3) are obtained in Theorem 2.

**References**


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