On Lorentzian \( \beta \)-Kenmotsu manifolds \(^1\)

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Abstract

The present paper deals with Lorentzian \( \beta \)-Kenmotsu manifold with conformally flat and quasi conformally flat curvature tensor. It is proved that in both cases, the manifold is locally isometric with a sphere \( S^{2n+1}(c) \). Further it is shown that an Lorentzian \( \beta \)-Kenmotsu manifold with \( R(X, Y).C = 0 \) is an \( \eta \)-Einstein manifold.

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1 Introduction

In [12], S. Tanno classified connected almost contact metric manifolds whose automorphism groups possess the maximum dimension. For such a manifold, the sectional curvature of plane sections containing \( \xi \) is a constant, say c. He

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showed that they can be divided into three classes:

1. homogeneous normal contact Riemannian manifolds with $c > 0$,
2. global Riemannian products of a line or a circle with a Kaehler manifold of constant holomorphic sectional curvature if $c = 0$ and
3. a warped product space $\mathbb{R} \times f \mathbb{C}$ if $c > 0$. It is known that the manifolds of class (1) are characterized by admitting a Sasakian structure. Kenmotsu [7] characterized the differential geometric properties of the manifolds of class (3); the structure so obtained is now known as Kenmotsu structure. In general, these structures are not Sasakian [7]. In the Gray-Hervella classification of almost Hermitian manifolds [6], there appears a class, $W_4$, of Hermitian manifolds which are closely related to locally conformal Kaehler manifolds [5].

An almost contact metric structure on a manifold $M$ is called a trans-Sasakian structure [11] if the product manifold $M \times \mathbb{R}$ belongs to the class $W_4$. The class $C_6 \oplus C_5$ [8], [9] coincides with the class of the trans-Sasakian structures of type $(\alpha, \beta)$. In fact, in [9], local nature of the two subclasses, namely, $C_5$ and $C_6$ structures, of trans-Sasakian structures are characterized completely.

We note that trans-Sasakian structures of type $(0, 0)$, $(0, \beta)$ and $(\alpha, 0)$ are cosymplectic [4], $\beta$-Kenmotsu [7] and $\alpha$-Sasakian [7] respectively. In [13] it is proved that trans-Sasakian structures are generalized quasi-Sasakian [10]. Thus, trans-Sasakian structures also provide a large class of generalized quasi-Sasakian structures.

An almost contact metric structure $(\phi, \xi, \eta, g)$ on $M$ is called a trans-Sasakian structure [11] if $(M \times \mathbb{R}, J, G)$ belongs to the class $W_4$ [6], where $J$ is the almost complex structure on $M \times \mathbb{R}$ defined by

$$J(X, f \frac{d}{dt}) = (\phi X - f \xi, \eta(X) \frac{d}{dt})$$

for all vector fields $X$ on $M$ and smooth functions $f$ on $M \times \mathbb{R}$, and $G$ is the product metric on $M \times \mathbb{R}$. This may be expressed by the condition [3]

$$\nabla_X \phi Y = \alpha(g(X, Y) \xi - \eta(Y) X) + \beta(g(\phi X, Y) \xi - \eta(Y) \phi X)$$
for some smooth functions $\alpha$ and $\beta$ on $M$, and we say that the trans-Sasakian structure is of type $(\alpha, \beta)$.

**Theorem 1** A trans-sasakian structure of type $(\alpha, \beta)$ with $\beta$ a nonzero constant is always $\beta$-Kenmotsu

In this case $\beta$ becomes a constant. If $\beta = 1$, then $\beta$-Kenmotsu manifold is Kenmotsu.

In this paper, we investigate Lorentzian $\beta$-Kenmotsu manifolds in which

(1) $C = 0$

where $C$ is the Weyl conformal curvature tensor. Then we study Lorentzian $\beta$-Kenmotsu manifolds in which

(2) $\tilde{C} = 0$

where $\tilde{C}$ is the quasi conformal curvature tensor. In both cases, it is shown that Lorentzian $\beta$-Kenmotsu is isometric with a sphere $S^{2n+1}(c)$, where $c = \alpha^2$.

Finally Lorentzian $\beta$-Kenmotsu manifolds with

(3) $R(X, Y).C = 0$

has been considered, where $R(X, Y)$ is considered as a derivation of the tensor algebra at each point of the manifold of tangent vectors $X, Y$. It is shown that Lorentzian $\beta$-Kenmotsu manifold is a $\eta$-Einstein.

## 2 Preliminaries

A differentiable manifold $M$ of dimension $(2n + 1)$ is called Lorentzian $\beta$-Kenmotsu manifold if it admits a $(1, 1)$-tensor field $\phi$, a contravariant vector field $\xi$, a covariant vector field $\eta$ and Lorentzian metric $g$ which satisfy [1], [2],

(4) $\begin{align*}
(a) & \quad \eta \xi = -1, \\
(b) & \quad \phi \xi = 0, \\
(c) & \quad \eta (\phi X) = 0
\end{align*}$

(5) $\begin{align*}
(a) & \quad \phi^2 X = X + \eta (X) \xi, \\
(b) & \quad g(X, \xi) = \eta (X)
\end{align*}$
for all \(X, Y \in TM\).

Also Lorentzian \(\beta\)-Kenmotsu manifold \(M\) is satisfying

\[
\nabla_X \xi = \beta[X - \eta(X)\xi],
\]

\[
(\nabla_X \eta)(Y) = \beta[g(X, Y) - \eta(X)\eta(Y)],
\]

where \(\nabla\) denotes the operator of covariant differentiation with respect to the Lorentzian metric \(g\).

Further, on Lorentzian \(\beta\)-Kenmotsu manifold \(M\) the following relations hold

\[
\eta(R(X, Y)Z) = \beta^2[g(X, Z)\eta(Y) - g(Y, Z)\eta(X)],
\]

\[
R(\xi, X)Y = \beta^2(\eta(Y)X - g(X, Y)\xi),
\]

\[
R(X, Y)\xi = \beta^2(\eta(X)Y - \eta(Y)X),
\]

\[
S(X, \xi) = -2n\beta^2\eta(X),
\]

\[
Q\xi = -2n\beta^2\xi,
\]

\[
S(\xi, \xi) = 2n\beta^2.
\]

3 Lorentzian \(\beta\)-Kenmotsu manifolds with \(C = 0\)

The conformal curvature tensor \(C\) on \(M\) is defined as

\[
C(X, Y)Z = R(X, Y)Z + \left[\frac{1}{2n-1}\right] [S(Y, Z)X - S(X, Z)Y + g(X, Z)QY - g(Y, Z)QX - g(X, Z)QY + \frac{r}{2n(2n-1)}] [g(X, Z)Y - g(Y, Z)X],
\]

where \(S(X, Y) = g(QX, Y)\).

Using (1) we get from (15)

\[
R(X, Y)Z = \left[\frac{1}{2n-1}\right] [S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY + \frac{r}{2n(2n-1)}] [g(X, Z)Y - g(Y, Z)X].
\]
On Lorentzian $\beta$-Kenmotsu manifolds

Taking $Z = \xi$ in (16) and using (5), (11) and (12), we find
\[
[\eta(X)QY - \eta(Y)QX] = \left[2n\beta^2 + \frac{r}{2n} - (2n - 1)\beta^2\right] [\eta(X)Y - \eta(Y)X].
\]
Taking $Y = \xi$ in (17) and using (4), we get
\[
QX = \left[\frac{r}{2n} + \beta^2\right] X + \left[\frac{r}{2n} + \beta^2 + 2n\beta^2\right] \eta(X)\xi.
\]
Contracting (18), we get
\[
r = -2n(2n + 1)\beta^2.
\]
Using (19) in (18), we find
\[
QX = -2n\beta^2 X.
\]
Using (20) in (16) and simplifying we get
\[
\begin{align*}
R(X, Y)Z &= \beta^2 [g(X, Z)X - g(Y, Z)Y].
\end{align*}
\]
Therefore the manifold is of constant scalar curvature $\beta^2$. Hence we can state:

**Theorem 2** A conformally flat Lorentzian $\beta$-Kenmotsu manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \beta^2$.

4 Lorentzian $\beta$-Kenmotsu manifolds with $\tilde{C} = 0$

The quasi conformal curvature tensor $\tilde{C}$ on $M$ is defined as
\[
\begin{align*}
\tilde{C}(X, Y)Z &= aR(X, Y)Z + b[S(Y, Z)X - S(X, Z)Y + g(Y, Z)QX - g(X, Z)QY] \\
&\quad- g(X, Z)QY - \left[\frac{r}{2n+1}\right] \left[\frac{a}{2n} + 2b\right] [g(Y, Z)X - g(X, Z)Y],
\end{align*}
\]
where $a, b$ are constants such that $a, b \neq 0$ and $S(Y, Z) = g(QY, Z)$.

Using (3), we find from (22) that
\[
\begin{align*}
R(X, Y)Z &= \frac{b}{a}[S(X, Z)Y - S(Y, Z)X + g(X, Z)QY - g(Y, Z)QX] \\
&\quad+ \left[\frac{r}{(2n+1)a}\right] \left[\frac{a}{2n} + 2b\right] [g(Y, Z)X - g(X, Z)Y],
\end{align*}
\]
Taking $Z = \xi$ in (23) and using (5), (11) and (12), we get
\begin{equation}
[\eta(Y)QX - \eta(X)QY] = \left[ \frac{r}{(2n+1)b} \left( \frac{a}{2n} + 2b \right) + 2n\beta^2 + \frac{a}{b}\beta^2 \right] [\eta(Y)X - \eta(X)Y].
\end{equation}

Taking $Y = \xi$ in (24) and applying (4), we have
\begin{equation}
QX = \left[ \frac{r}{(2n+1)b} \left( \frac{a}{2n} + 2b \right) + 2n\beta^2 + \frac{a}{b}\beta^2 \right] X
+ \left[ \frac{r}{(2n+1)b} \left( \frac{a}{2n} + 2b \right) + 4n\beta^2 + \frac{a}{b}\beta^2 \right] \eta(X)\xi.
\end{equation}

Contracting (25), we get
\begin{equation}
r = -2n(2n+1)\beta^2.
\end{equation}

Using (26) in (25), we find
\begin{equation}
QX = -2n\beta^2 X.
\end{equation}

Using (27) in (23), we get
\begin{equation}
R(X, Y)Z = \beta^2[g(X, Z)Y - g(Y, Z)X].
\end{equation}

Thus we can state

**Theorem 3** A quasi conformally flat Lorentzian $\beta$-Kenmotsu manifold is locally isometric to a sphere $S^{2n+1}(c)$, where $c = \beta^2$.

5 Lorentzian $\beta$-Kenmotsu manifold satisfying
\begin{equation}
R(X, Y).C = 0
\end{equation}

In view of (5) and (9), we obtained from (15)
\begin{equation}
(29)\eta(C(X, Y)Z) = \left[ -\beta^2 + \frac{2n\beta^2}{2n-1} + \frac{r}{2n(2n-1)} \right] [g(Y, Z)\eta(X)
- g(X, Z)\eta(Y)] - \left[ \frac{1}{2n-1} \right] [S(Y, Z)\eta(X) - S(X, Z)\eta(Y)].
\end{equation}
Putting $Z = \xi$ in (29) and using (5) and (12), we get

$$ \eta(C(X, Y)\xi) = 0. $$

Again taking $X = \xi$ in (29) we have

$$ \eta(C(\xi, Y)Z) = \beta^2 - \frac{2n\beta^2}{2n-1} - \frac{r}{2n(2n-1)} [g(Y, Z) + \eta(Y)\eta(Z)] $$

$$ + \left[ \frac{1}{2n-1} \right] [S(Y, Z) - 2n\beta^2 \eta(Y)\eta(Z)]. $$

Now

$$ (R(X, Y)C(U, V)Z = R(X, Y)C(U, V)Z - C(R(X, Y)U, V)Z $$

$$ - C(U, R(X, Y)V)Z - C(U, V)R(X, Y)Z. $$

By virtue of $R(X, Y).C = 0$, we have

$$ R(X, Y)C(U, V)Z - C(R(X, Y)U, V)Z - C(U, R(X, Y)V)Z - C(U, V)R(X, Y)Z = 0. $$

Therefore,

$$ g[R(\xi, Y)C(U, V)Z, \xi] - g[C(R(\xi, Y)U, V)Z, \xi] $$

$$ - g[C(U, R(\xi, Y)V)Z, \xi] - g[C(U, V)R(\xi, Y)Z, \xi] = 0. $$

From this it follows that

$$ \beta^2 \hat{C}(U, V, Z, Y) + \beta^2 \eta(Y)\eta(C(U, V)Z) $$

$$ + \beta^2 g(U, Y)\eta(C(\xi, V)Z) - \beta^2 \eta(Y)\eta(C(U, Y)Z) $$

$$ + \beta^2 g(Y, V)\eta(C(U, \xi)Z) - \beta^2 \eta(Z)\eta(C(U, V)Y) = 0. $$

where $\hat{C}(U, V, Z, Y) = g(C(U, V)Z, Y)$. Putting $Y = U$ in (33), we get

$$ \hat{C}(U, V, Z, U) + g(U, U)\eta(C(\xi, V)Z) + g(U, V)\eta(C(U, \xi)Z) $$

$$ - \eta(V)\eta(C(U, U)Z) - \eta(Z)\eta(C(U, V)U) = 0. $$

Let $\{e_i\}, i = 1, 2, ..., (2n + 1)$ be an orthonormal basis of the tangent space at any point. Then the sum for $1 \leq i \leq (2n + 1)$ of the relation (34) for $U = e_i$,
yields

\[
\eta(C(\xi, V)Z) = \left[ \frac{1}{(2n + 1)(2n - 1)} \right] S(V, Z) \\
+ \left[ \frac{\beta^2}{2n + 1} - \frac{2n\beta^2}{(2n + 1)(2n - 1)} - \frac{r}{2n(2n + 1)(2n - 1)} \right] g(V, Z) \\
+ \left[ \frac{\beta^2}{2n + 1} - \frac{4n\beta^2}{(2n + 1)(2n - 1)} - \frac{r}{2n(2n + 1)(2n - 1)} \right] \eta(V)\eta(Z).
\]

From (31) and (35), we have

\[
S(V, Z) = \left[ \frac{r}{2n} + \beta^2 \right] g(V, Z) + \left[ \frac{r}{2n} + \beta^2 + 2n\beta^2 \right] \eta(V)\eta(Z).
\]

Hence we can state the following:

**Theorem 4** In a Lorentzian $\beta$-Kenmotsu manifold $M$, if the relation $R(X, Y) \cdot C = 0$ holds then the manifold is $\eta$-Einstein.

**References**


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