On integral operators of meromorphic functions

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Abstract

Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1, p]$, $\Phi(z)\varphi(z) \neq 0$, $z \in U$, $\alpha, \beta, \gamma, \delta \in \mathbb{C}$ with $\beta \neq 0$, and let $\Sigma_p$ denote the class of meromorphic functions of the form $g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots$, $z \in \dot{U}$, $a_{-p} \neq 0$.

We consider the integral operator $J_{p, \alpha, \beta, \gamma, \delta}^{\Phi, \varphi} : K \subset \Sigma_p \to \Sigma_p$ defined by

$$J_{p, \alpha, \beta, \gamma, \delta}^{\Phi, \varphi}(g)(z) = \left[ \frac{\gamma - p \beta}{z^\gamma \Phi(z)} \int_0^z g^\alpha(t)\varphi(t)t^{\delta-1}dt \right]^{\frac{1}{\delta}}, \quad g \in K, \ z \in \dot{U}.$$ 

The first result of this paper gives us the conditions for which $J_{p, \alpha, \beta, \gamma, \delta}^{\Phi, \varphi}$ has some important properties. Furthermore, we study the image of the set $\Sigma_p^\alpha(\alpha, \delta)$ through the operator $J_{p, \beta, \gamma} = J_{p, 1, \beta, \gamma}^{1, 1}$ and the image of the sets $\Sigma K_p(\alpha, \delta)$, $\Sigma C_p(\alpha, \delta ; \varphi)$ through the operator $J_{p, \gamma} = J_{p, 1, \gamma}$.

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1 Introduction and preliminaries

Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$ be the unit disc in the complex plane, $\dot{U} = U \setminus \{0\}$, $H(U) = \{ f : U \to \mathbb{C} : f$ is holomorphic in $U \}$, $\mathbb{N} = \{0, 1, 2, \ldots \}$ and $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$.

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For $p \in \mathbb{N}^*$, let $\Sigma_p$ denote the class of meromorphic functions of the form

$$g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1 z + \cdots, \quad z \in \bar{U}, \quad a_{-p} \neq 0.$$  

We will also use the following notations:

$\Sigma_{p,0} = \{g \in \Sigma_p : a_{-p} = 1\}$, $\Sigma_0 = \{g \in \Sigma_{p,0} : g(z) \neq 0, \ z \in \bar{U}\}$,

$\Sigma^*_p(\alpha) = \left\{ g \in \Sigma_p : \Re \left[ -\frac{zg'(z)}{g(z)} \right] > \alpha, \ z \in U \right\}$, where $\alpha < p$,

$\Sigma^*_p(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \Re \left[ -\frac{zg'(z)}{g(z)} \right] < \delta, \ z \in U \right\}$, where $\alpha < p < \delta$,

$\Sigma K_p(\alpha) = \left\{ g \in \Sigma_p : \Re \left[ 1 + \frac{zg''(z)}{g'(z)} \right] < -\alpha, \ z \in U \right\}$, where $\alpha < p$,

$\Sigma K_{p,0}(\alpha) = \Sigma K_p(\alpha) \cap \Sigma_{p,0}$,

$\Sigma K_p(\alpha, \delta) = \left\{ g \in \Sigma_p : \alpha < \Re \left[ -1 - \frac{zg''(z)}{g'(z)} \right] < \delta, \ z \in U \right\}$, where $\alpha < p < \delta$,

$\Sigma K_{p,0}(\alpha, \delta) = \Sigma K_p(\alpha, \delta) \cap \Sigma_{p,0}$,

$\Sigma C_{p,0}(\alpha, \delta, \varphi) = \left\{ g \in \Sigma_{p,0} : \alpha < \Re \left[ \frac{g'(z)}{\varphi'(z)} \right] < \delta, \ z \in U \right\}$, where $\alpha < 1 \leq p < \delta$, $\varphi \in \Sigma K_{p,0}(\alpha, \delta)$.

We remark that $\Sigma^*_p(\alpha), \ 0 \leq \alpha < 1$, is the classes of meromorphic starlike functions of order $\alpha$ and $\Sigma K_{1,0}(\alpha) \cap \Sigma_0$ is the classes of meromorphic convex functions of order $\alpha$. These classes are classes of univalent functions.

$$H[a, n] = \{ f \in H(U) : f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots \} \text{ for } a \in \mathbb{C}, \ n \in \mathbb{N}^*,$$

$A_n = \{ f \in H(U) : f(z) = z + a_{n+1} z^{n+1} + a_{n+2} z^{n+2} + \ldots \}, \ n \in \mathbb{N}^*$, and for $n = 1$ we denote $A_1$ by $A$ and this set is called the class of analytic functions normalized at the origin.

**Definition 1.** [3, p.4], [4, p.45] Let $f, g \in H(U)$. We say that the function $f$ is subordinate to the function $g$, and we denote this by $f(z) \prec g(z)$, if there is a function $w \in H(U)$, with $w(0) = 0$ and $|w(z)| < 1$, $z \in U$, such that

$$f(z) = g[w(z)], \ z \in U.$$  

**Remark 1.** If $f(z) \prec g(z)$, then $f(0) = g(0)$ and $f(U) \subseteq g(U)$.

**Theorem 1.** [3, p.4], [4, p.46] Let $f, g \in H(U)$ and let $g$ be a univalent function in $U$. Then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(U) \subseteq g(U)$. 
Theorem 2. [3, Theorem 2.4f., 4, p.212] Let \( p \in H[a, n] \) with \( \Re a > 0 \) and let \( P : U \to \mathbb{C} \) be a function with \( \Re P(z) > 0, z \in U \). If
\[
\Re [p(z) + P(z)zp'(z)] > 0, z \in U;
\]
then \( \Re p(z) > 0, z \in U \).

Definition 2. [3, p.46], [4, p.228] Let \( c \in \mathbb{C} \) with \( \Re c > 0 \) and \( n \in \mathbb{N}^* \). We consider
\[
C_n = C_n(c) = \frac{n}{\Re c} \left| c \right| \sqrt{1 + \frac{2\Re c}{n} + \Im c}.
\]

If the univalent function \( R : U \to \mathbb{C} \) is given by \( R(z) = \frac{2C_nz}{1 - z^2} \), then we will denote by \( R_{c,n} \) the "Open Door" function, defined as
\[
R_{c,n}(z) = R \left( \frac{z + b}{1 + bz} \right) = 2C_n \frac{(z + b)(1 + bz)}{(1 + bz)^2 - (z + b)^2},
\]
where \( b = R^{-1}(c) \).

Lemma 1. [3, p.35], [4, pg. 209] Let \( \psi : \mathbb{C}^2 \times U \to \mathbb{C} \) be a function that satisfies the condition
\[
\Re \psi(\rho, \sigma; z) \leq 0,
\]
when \( \rho, \sigma \in \mathbb{R}, \sigma \leq -\frac{n}{2}(1 + \rho^2), z \in U, n \geq 1 \).

If \( p \in H[1, n] \) and
\[
\Re \psi(p(z), zp'(z); z) > 0, \quad z \in U,
\]
then
\[
\Re p(z) > 0, \quad z \in U.
\]

Theorem 3. [3, Theorem 2.5c.] Let \( \Phi, \varphi \in H[1, n] \) with \( \Phi(z) \neq 0, \varphi(z) \neq 0 \), for \( z \in U \). Let \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) with \( \beta \neq 0, \alpha + \delta = \beta + \gamma \) and \( \Re (\alpha + \delta) > 0 \). Let the function \( f(z) = z + a_{n+1}z^{n+1} + \cdots \in A_n \) and suppose that
\[
\alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta < R_{\alpha + \delta, n}(z).
\]

If \( F = I_{\alpha, \beta, \gamma, \delta}^\Phi \varphi (f) \) is defined by
\[
F(z) = I_{\alpha, \beta, \gamma, \delta}^\Phi \varphi (f)(z) = \left[ \frac{\beta + \gamma}{z^2\Phi(z)} \int_0^z f^\alpha(t)\varphi(t)t^{\delta-1}dt \right]^{\frac{1}{\beta}},
\]
then \( F \in A_n \) with \( \frac{F(z)}{z} \neq 0, \ z \in U, \) and

\[
\text{Re} \left[ \beta \frac{zF'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0, \ z \in U.
\]

All powers in (1) are principal ones.

**Theorem 4.** [3, Lemma 1.2c.] Let \( n \geq 0 \) be an integer and let \( \gamma \in \mathbb{C}, \) with \( \text{Re}\,\gamma > -n. \) If \( f(z) = \sum_{m=n}^{\infty} a_m z^m \) is analytic in \( U \) and \( F \) is defined by

\[
F(z) = I[f](z) = \frac{1}{z^{\gamma}} \int_0^z f(\zeta) \zeta^{\gamma-1} d\zeta = \int_0^1 f(tz) t^{\gamma-1} dt,
\]

then \( F(z) = \sum_{m=n}^{\infty} \frac{a_m z^m}{m+\gamma} \) is analytic in \( U. \)

## 2 Main results

**Theorem 5.** Let \( p \in \mathbb{N}^*, \ \Phi, \varphi \in H[1,p] \) with \( \Phi(z)\varphi(z) \neq 0, \ z \in U. \) Let \( \alpha, \beta, \gamma, \delta \in \mathbb{C} \) with \( \beta \neq 0, \delta + p\beta = \gamma + p\alpha \) and \( \text{Re} (\gamma - p\beta) > 0. \) Let \( g \in \Sigma_p \) and suppose that

\[
\alpha \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta < R_{\delta-p\alpha,p}(z), \ z \in U.
\]

If \( G = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g) \) is defined by

\[
G(z) = J^{\Phi,\varphi}_{p,\alpha,\beta,\gamma,\delta}(g)(z) = \left[ \frac{\gamma - p\beta}{z^{\gamma} \Phi(z)} \int_0^z g^\alpha(t) \varphi(t) t^{\delta-1} dt \right]^{1/p},
\]

then \( G \in \Sigma_p \) with \( z^p G(z) \neq 0, \ z \in U, \) and

\[
\text{Re} \left[ \beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0, \ z \in U.
\]

All powers in (2) are principal ones.

**Proof.** Let \( g \in \Sigma_p \) be of the form \( g(z) = \frac{a_{-p}}{z^p} + \sum_{k=0}^{\infty} a_k z^k, \ z \in U, \ a_{-p} \neq 0. \) It’s easy to see that the function \( f(z) = \frac{z^{p+1}g(z)}{a_{-p}} \) belongs to the class \( A_p. \)
After a simple computation we have

\[ \frac{zf'}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} = \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \alpha(p + 1), \]

hence

\[ \alpha \frac{zf'(z)}{f(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \delta - \alpha(p + 1) < R_{\delta - p\alpha, p}(z). \]

By denoting \( \delta - \alpha(p + 1) = \delta_1 \) and \( \gamma - \beta(p + 1) = \gamma_1 \), after using the fact that \( \delta + p\beta = \gamma + p\alpha \) and \( \text{Re} (\gamma - p\beta) > 0 \), we obtain that \( \alpha + \delta_1 = \beta + \gamma_1 \) and \( \text{Re} (\beta + \gamma_1) > 0 \).

Now we remark that the conditions of Theorem 3 are satisfied for the functions \( f, \Phi, \varphi \) and the numbers \( \alpha, \beta, \gamma_1, \delta_1 \), so, we obtain that

\[ F(z) = f^{\Phi, \varphi}_{\alpha, \beta, \gamma_1, \delta_1}(f)(z) = \left[ \frac{\beta + \gamma_1}{z^{\gamma_1} \Phi(z)} \int_0^z f^\alpha(t) \varphi(t) t^{\delta_1 - 1} dt \right]^{\frac{1}{\beta}} \in A_p, \]

with \( \frac{F(z)}{z} \neq 0, z \in U \), and

(3) \[ \text{Re} \left[ \frac{\beta zf'(z)}{F(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma_1 \right] > 0, z \in U. \]

It’s not difficult to see that

(4) \[ F^\beta(z)(a-p)^\alpha = G^\beta(z)z^{\beta(p+1)}, \]

where

\[ G(z) = J_{\alpha, \beta, \gamma_1, \delta_1}(g)(z) = \left[ \frac{\gamma - p\beta}{z^{\gamma_1} \Phi(z)} \int_0^z g^\alpha(t) \varphi(t) t^{\delta_1 - 1} dt \right]^{\frac{1}{\beta}}. \]

Since \( \frac{F(z)}{z} \neq 0, z \in U \), we have from (4), \( z^pG(z) \neq 0, z \in U \).

Using the logarithmic differential and the multiplying with \( z \) for (4), we obtain

\[ \beta \frac{zf'(z)}{F(z)} = \beta \frac{zG'(z)}{G(z)} + \beta(p + 1), z \in U. \]

From this last equality and (3), we get

\[ \text{Re} \left[ \beta \frac{zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0, z \in U. \]

Taking \( \alpha = \beta \) and \( \gamma = \delta \) in the above theorem and using the notation \( J_{p, \delta, \gamma}^{\Phi, \varphi} \) instead of \( J_{p, \beta, \alpha, \gamma_1, \delta_1}^{\Phi, \varphi} \), we obtain the next corollary:
Corollary 1. Let $p \in \mathbb{N}^*$, $\Phi, \varphi \in H[1, p]$ with $\Phi(z)\varphi(z) \neq 0$, $z \in U$. Let $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\Re (\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and

$$
\beta \frac{zg'(z)}{g(z)} + \frac{z\varphi'(z)}{\varphi(z)} + \gamma < R_{\gamma - p\beta, p}(z),
$$

then

$$
G(z) = J_{p, \beta, \gamma}(g)(z) = \left[ \frac{\gamma - p\beta}{z\gamma \Phi(z)} \int_0^z g^\beta(t)\varphi(t)t^{\gamma-1}dt \right]^{\frac{1}{\beta}} \in \Sigma_p,
$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$
\Re \left[ \frac{\beta zG'(z)}{G(z)} + \frac{z\Phi'(z)}{\Phi(z)} + \gamma \right] > 0, \ z \in U.
$$

Considering $\Phi = \varphi \equiv 1$ in Corollary 1, and using the notation $J_{p, \beta, \gamma}$ instead of $J_{p, \beta, \gamma, \gamma}$, we obtain:

Corollary 2. Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$ and $\Re (\gamma - p\beta) > 0$. If $g \in \Sigma_p$ and

$$
\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma - p\beta, p}(z),
$$

then

$$
G(z) = J_{p, \beta, \gamma}(g)(z) = \left[ \frac{\gamma - p\beta}{z\gamma \Phi(z)} \int_0^z g^\beta(t)\varphi(t)t^{\gamma-1}dt \right]^{\frac{1}{\beta}} \in \Sigma_p,
$$

with $z^p G(z) \neq 0$, $z \in U$, and

$$
\Re \left[ \frac{\beta zG'(z)}{G(z)} + \gamma \right] > 0, \ z \in U.
$$

Let $p \in \mathbb{N}^*$, $\beta, \gamma \in \mathbb{C}$ with $\beta \neq 0$, $g \in \Sigma_p$, $G = J_{p, \beta, \gamma}(g)$ and let us denote $P(z) = \frac{-zG'(z)}{G(z)}$, $z \in U$. If we suppose that $P \in H(U)$, we obtain from

$$
G(z) = \left[ \frac{\gamma - p\beta}{z\gamma} \int_0^z t^{\gamma-1}g^\beta(t)dt \right]^{\frac{1}{\beta}}, \ z \in \dot{U},
$$

that

\begin{equation}
(5) \quad P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \ z \in U.
\end{equation}

Theorem 6. Let $p \in \mathbb{N}^*$, $\lambda \in \mathbb{C}$ with $\Re \lambda > p$. If $g \in \Sigma_p$, then $J_{p, \lambda}(g) \in \Sigma_p$, where $J_{p, \lambda}(g)(z) = J_{p, 1, \lambda}(g)(z) = \frac{\lambda - 1}{z^\lambda} \int_0^z g(t)t^{\lambda-1}dt$. 

Proof. Let \( g \) be of the form \( g(z) = \frac{a_{-p}}{z^p} + a_0 + a_1z + \cdots, z \in \hat{U}, a_{-p} \neq 0. \) Since \( g \in \Sigma_p \) we have \( z^p g \in H[a_{-p}, p]. \) Let us denote \( f(z) = z^pg(z), z \in U, \) and \( \gamma = \lambda - p. \)

We know that \( \Re \lambda > p, \) so, \( \Re \gamma > 0, \) and using Theorem 4 for \( f \) and \( \gamma \) we get that

\[
F(z) = \frac{1}{z^\gamma} \int_0^z f(t)t^{\gamma-1}dt
\]

is analytic in \( U, \) so \( F \in H \left[ \frac{a_{-p}}{\gamma}, p \right]. \) It’s easy to see that

\[
F(z) = \frac{1}{z^{\lambda-p}} \int_0^z g(t)t^{\lambda-1}dt = z^p \frac{1}{\lambda-1} J_{p,\lambda}(g)(z),
\]

therefore \( J_{p,\lambda}(g) \in \Sigma_p. \)

Remark 2. Let \( p \in \mathbb{N}^*, \lambda \in \mathbb{C} \) with \( \Re \lambda > p. \) From the above theorem, it’s easy to see that we have \( J_{p,\lambda}(g) \in \Sigma_{p,0}, \) when \( g \in \Sigma_{p,0}. \)

For the next results we need the following lemmas:

Lemma 2. Let \( n \in \mathbb{N}^*, \alpha, \beta \in \mathbb{R}, \gamma \in \mathbb{C} \) with \( \Re [\gamma - \alpha \beta] \geq 0. \) If \( P \in H[P(0), n] \) with \( P(0) \in \mathbb{R} \) and \( P(0) > \alpha, \) then we have

\[
\Re \left[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] > \alpha \Rightarrow \Re P(z) > \alpha, z \in U.
\]

Proof. If we take \( R(z) = \frac{P(z) - \alpha}{P(0) - \alpha}, \) we have \( R(z) \in H[1, 1] \) and from

\[
\Re \left[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] > \alpha, z \in U,
\]

since \( P(0) - \alpha > 0, \) we obtain

\[
\Re \left[ R(z) + \frac{zR'(z)}{\gamma - \beta \alpha - \beta (P(0) - \alpha)R(z)} \right] > 0, z \in U.
\]

Now let us put

\[
\psi(R(z), zR'(z); z) = R(z) + \frac{zR'(z)}{\gamma - \beta \alpha - \beta (P(0) - \alpha)R(z)}.
\]

We have \( \Re \psi(R(z), zR'(z); z) > 0, z \in U. \)
To apply Lemma 1 we need to show that \( \text{Re} \, \psi(p; \sigma; z) \leq 0 \), when \( p \in \mathbb{R}, \sigma \leq -1 + \frac{\rho^2}{2}, \) \( z \in U \). We have

\[
\text{Re} \, \psi(p; \sigma; z) = \text{Re} \left( \frac{\sigma}{\gamma - \beta \alpha - \beta(P(0) - \alpha) \rho} \right) = \text{Re} \left( \frac{\sigma}{\gamma_1 + i \gamma_2 - \beta \alpha - \beta(P(0) - \alpha) \rho} \right) =
\]

\[
\frac{\sigma \gamma_1 - \beta \alpha}{(\gamma_1 - \beta \alpha)^2 + (\gamma_2 - \beta(P(0) - \alpha) \rho)^2} \leq 0, \quad z \in U, \quad \rho \in \mathbb{R}, \quad \sigma \leq -1 + \frac{\rho^2}{2}, \quad \gamma_1 = \text{Re} \, \gamma \geq \alpha \beta.
\]

Applying now Lemma 1 we obtain \( \text{Re} \, R(z) > 0, \) \( z \in U \), hence \( \text{Re} \, P(z) > \alpha \).

**Lemma 3.** Let \( n \in \mathbb{N}^* \), \( \sigma, \beta \in \mathbb{R} \), \( \sigma, \beta \in \mathbb{C} \) with \( \text{Re} \, \sigma \geq 0 \). If \( P \in H[P(0), n] \) with \( P(0) \in \mathbb{R} \) and \( P(0) < \delta \), then we have

\[
\text{Re} \left[ P(z) + \frac{z P'(z)}{\gamma - \beta P(z)} \right] < \delta \Rightarrow \text{Re} \, P(z) < \delta, \quad z \in U.
\]

**Proof.** Let us denote \( R(z) = -P(z), \sigma = -\delta, \beta_1 = -\beta \). It is easy to see that the conditions from Lemma 2 holds for the function \( R \) and the numbers \( \alpha, \beta_1, \gamma \), so we obtain \( \text{Re} \, R(z) > \alpha, \) \( z \in U \), which is equivalent to \( \text{Re} \, P(z) < \delta, \) \( z \in U \).

Next we will study the properties of the image of a function \( g \in \Sigma^*_p(\alpha, \delta) \) through the integral operator \( J_{p, \beta, \gamma} \) defined by

\[
J_{p, \beta, \gamma}(g)(z) = \left[ \frac{\gamma - p \beta}{z^\gamma} \int_0^z g^\beta(t) t^{\gamma - 1} dt \right]^\frac{1}{\beta}.
\]

**Theorem 7.** Let \( p \in \mathbb{N}^* \), \( \beta > 0, \gamma \in \mathbb{C} \) and \( \alpha < p < \delta \leq \frac{\text{Re} \, \gamma}{\beta} \). If \( g \in \Sigma^*_p(\alpha, \delta) \), then \( G = J_{p, \beta, \gamma}(g) \in \Sigma^*_p(\alpha, \delta) \).

**Proof.** We know that \( g \in \Sigma^*_p(\alpha, \delta) \) is equivalent to

\[
\alpha < \text{Re} \left[ - \frac{z g'(z)}{g(z)} \right] < \delta, \quad z \in U,
\]

so,

\[
\text{Re} \, \gamma - \beta \delta < \text{Re} \left[ \gamma + \beta \frac{z g'(z)}{g(z)} \right] < \text{Re} \, \gamma - \beta \alpha, \quad z \in U, \quad \text{when} \quad \beta > 0.
\]
Because \( \delta \leq \frac{\Re \gamma}{\beta} \), we have \( \Re \left[ \gamma + \beta \frac{zg'(z)}{g(z)} \right] > 0 \), \( z \in U \), and using Corollary 2, we obtain that \( G = J_{\beta, \gamma}(g) \in \Sigma_p \), \( z^p G(z) \neq 0 \), \( z \in U \), and \( \Re \left[ \gamma + \beta \frac{zG'(z)}{G(z)} \right] > 0 \), \( z \in U \).

From (5) we know that

\[
P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)} \quad \text{is analytic in} \quad U.
\]

Using (7) we get

\[
(8) \quad \alpha < \Re \left[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] < \delta, \quad z \in U.
\]

Since \( \alpha < P(0) = p < \delta \) and \( 0 \leq \Re \gamma - \beta \delta < \Re \gamma - \beta \alpha \), we obtain from (8), after applying Lemma 2 and Lemma 3, that

\[
\alpha < \Re P(z) < \delta, \quad z \in U,
\]

which is equivalent to

\[
(9) \quad \alpha < \Re \left[ -\frac{zG'(z)}{G(z)} \right] < \delta, \quad z \in U.
\]

Since \( G \in \Sigma_p \) we get from (9) that \( G \in \Sigma_p^*(\alpha, \delta) \).

We remark that for \( p = 1 \) all members of the class \( \Sigma^*_1(\alpha, \delta) \) are univalent functions, when \( 0 \leq \alpha < 1 < \delta \), so \( G = J_{1, \beta, \gamma}(g) \) is an univalent function when \( g \in \Sigma^*_1(\alpha, \delta) \) and \( 0 \leq \alpha < 1 < \delta \leq \frac{\Re \gamma}{\beta} \), \( \beta > 0 \).

Taking \( \beta = 1 \) in the above theorem and using the notation \( J_{p, \gamma} \) instead of \( J_{1, \beta, \gamma} \), we obtain:

**Corollary 3.** Let \( p \in \mathbb{N}^*, \gamma \in \mathbb{C} \) and \( \alpha < p < \delta \leq \Re \gamma \). If \( g \in \Sigma_p^*(\alpha, \delta) \), then

\[
G = J_{p, \gamma}(g) = \frac{\gamma - p}{z^\gamma} \int_0^z t^{\gamma - 1} g(t) dt \in \Sigma_p^*(\alpha, \delta).
\]

The properties of the integral operator \( J_{1, \gamma} \), were studied by many authors in different papers, from which we remember [1], [2], [5], [6], [7].
Theorem 8. Let $p \in \mathbb{N}^*$, $\beta > 0$, $\gamma \in \mathbb{C}$ and $\alpha < p < \frac{\text{Re} \gamma}{\beta} \leq \delta$.
If $g \in \Sigma^*_{p}(\alpha, \delta)$, with

$$\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma - p\beta, p}(z), \ z \in U,$$

then $G = J_{p, \beta, \gamma}(g) \in \Sigma^*_{p}(\alpha, \delta)$.

Proof. Because $\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma - p\beta, p}(z), \ z \in U$, we obtain from Corollary 2 that $G \in \Sigma_p$, with $z^p G(z) \neq 0, \ z \in U$, and

$$(10) \quad \text{Re} \left[ \beta \frac{zG'(z)}{G(z)} + \gamma \right] > 0, \ z \in U, \text{ where } G = J_{p, \beta, \gamma}(g).$$

Since $\frac{\text{Re} \gamma}{\beta} \leq \delta$, we get from (10),

$$(11) \quad \text{Re} \frac{zG'(z)}{G(z)} + \delta > 0, \ z \in U.$$

From (5) we know that

$$(12) \quad P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \text{ where } P(z) = -\frac{zG'(z)}{G(z)}.$$

Since $g \in \Sigma^*_{p}(\alpha, \delta)$, we obtain from (12) that

$$(13) \quad \alpha < \text{Re} \left[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] < \delta, \ z \in U.$$

Because we know from (11) that $\text{Re} \ P(z) < \delta, \ z \in U$, we have only to verify that $\text{Re} \ P(z) > \alpha$. To show this we will use Lemma 2.

We know that $P$ is analytic in $U$ with $P(0) = p > \alpha$. We also have $\text{Re} \gamma - \alpha \beta > 0$. Since the conditions from Lemma 2 are met, we obtain $\text{Re} \ P(z) > \alpha$, which is equivalent to

$$(14) \quad -\text{Re} \frac{zG'(z)}{G(z)} > \alpha.$$

Since $G \in \Sigma_p$, from (11) and (14) we have $G \in \Sigma^*_{p}(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:
Corollary 4. Let \( p \in \mathbb{N}^* \), \( \beta > 0 \), \( \gamma \in \mathbb{C} \) and \( \alpha < p < \frac{\text{Re} \gamma}{\beta} \). If \( g \in \Sigma_p^*(\alpha) \), with
\[
\beta \frac{z g'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U,
\]
then \( G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha) \).

We make the remark that we can obtain a similar result, without the condition \( \beta \frac{z g'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U \), as it follows:

Theorem 9. Let \( p \in \mathbb{N}^* \), \( \beta > 0 \), \( \gamma \in \mathbb{C} \), \( \alpha < p < \frac{\text{Re} \gamma}{\beta} \) and \( g \in \Sigma_p^*(\alpha) \). Let \( G = J_{p,\beta,\gamma}(g) \). If \( G \in \Sigma_p \) and \( z^p G(z) \neq 0, \quad z \in U \), then \( G \in \Sigma_p^*(\alpha) \).

Proof. Let us denote \( P(z) = -\frac{z G'(z)}{G(z)}, \quad z \in U \). Because \( G \in \Sigma_p \) and \( z^p G(z) \neq 0, \quad z \in U \), we have that \( P \) analytic in \( U \), hence from \( G = J_{p,\beta,\gamma}(g) \) and (5) we have that
\[
P(z) + \frac{z P'(z)}{\gamma - \beta P(z)} = \frac{-z g'(z)}{g(z)}, \quad z \in U.
\]
Since \( g \in \Sigma_p^*(\alpha) \), we obtain from (15) that
\[
\text{Re} \left[ P(z) + \frac{z P'(z)}{\gamma - \beta P(z)} \right] > \alpha, \quad z \in U.
\]
We have to verify that \( \text{Re} P(z) > \alpha \). To show this we will use Lemma 2.

We have \( P \) analytic in \( U \) with \( P(0) = p > \alpha \) and \( \text{Re} \gamma - \alpha \beta > 0 \). Since the conditions from Lemma 2 are met, we obtain \( \text{Re} P(z) > \alpha \), which is equivalent to
\[
-\text{Re} \frac{z G'(z)}{G(z)} > \alpha, \quad z \in U.
\]
Because \( G \in \Sigma_p \), from (17), we get \( G \in \Sigma_p^*(\alpha) \).

Since we know from Theorem 6 that for \( p \in \mathbb{N}^* \), \( \gamma \in \mathbb{C} \) with \( \text{Re} \gamma > p \), we have \( J_{p,\gamma}(g) \in \Sigma_p \) when \( g \in \Sigma_p \), we obtain for the above theorem, taking \( \beta = 1 \), the next corollary:

Corollary 5. Let \( p \in \mathbb{N}^* \), \( \gamma \in \mathbb{C} \) and \( \alpha < p < \text{Re} \gamma \). If \( g \in \Sigma_p^*(\alpha) \) with \( z^p J_{p,\gamma}(g)(z) \neq 0, \quad z \in U \), then \( G = J_{p,\gamma}(g) \in \Sigma_p^*(\alpha) \).
Taking $\beta = 1$ in Theorem 8, we get:

**Corollary 6.** Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ and $\alpha < p < \text{Re}\gamma \leq \delta$. If $g \in \Sigma_p^\star(\alpha, \delta)$, with

$$\frac{zg'(z)}{g(z)} + \gamma < R_{\gamma - p,p}(z), \quad z \in U,$$

then $G = J_{p,\gamma}(g) \in \Sigma_p^\star(\alpha, \delta)$.

**Theorem 10.** Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\frac{\text{Re}\gamma}{\beta} \leq \alpha < p < \delta$. If $g \in \Sigma_p^\star(\alpha, \delta)$, then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^\star(\alpha, \delta)$.

**Proof.** We know that $g \in \Sigma_p^\star(\alpha, \delta)$ is equivalent to

$$\alpha < \text{Re}\left[\frac{-zg'(z)}{g(z)}\right] < \delta, \quad z \in U, \quad (18)$$

so,

$$\text{Re}\gamma - \beta \alpha < \text{Re}\left[\gamma + \beta \frac{zg'(z)}{g(z)}\right] < \text{Re}\gamma - \beta \delta, \quad z \in U, \quad \text{when} \quad \beta < 0.$$

Because $\alpha \geq \frac{\text{Re}\gamma}{\beta}$, we have $\text{Re}\left[\gamma + \beta \frac{zg'(z)}{g(z)}\right] > 0$, $z \in U$, and using Corollary 2, we obtain that $G = J_{p,\beta,\gamma}(g) \in \Sigma_p$, $z^p G(z) \neq 0$, $z \in U$ and $\text{Re}\left[\gamma + \beta \frac{zG'(z)}{G(z)}\right] > 0$, $z \in U$.

From (5) we know that

$$P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = \frac{zG'(z)}{G(z)} \quad \text{is analytic in } U.$$

We will use the same idea as at the proof of Theorem 7.

Using (18) we get

$$\alpha < \text{Re}\left[P(z) + \frac{zP'(z)}{\gamma - \beta P(z)}\right] < \delta, \quad z \in U, \quad (19)$$

Since $\alpha < P(0) = p < \delta$ and $\text{Re}\gamma - \beta \delta > \text{Re}\gamma - \beta \alpha \geq 0$, we obtain from (19), after applying Lemma 2 and Lemma 3, that

$$\alpha < \text{Re}P(z) < \delta, \quad z \in U,$$
which is equivalent to
\[
\alpha < \text{Re} \left[ -\frac{zG'(z)}{G(z)} \right] < \delta, \quad z \in U.
\]

Since \( G \in \Sigma_p \) we have from (20) that \( G \in \Sigma_p^*(\alpha, \delta) \).

If we consider \( \delta \to \infty \), in the above theorem, we obtain the next corollary:

**Corollary 7.** Let \( p \in \mathbb{N}^* \), \( \beta < 0 \), \( \gamma \in \mathbb{C} \) and \( \frac{\text{Re} \gamma}{\beta} \leq \alpha < p \). Then we have
\[
g \in \Sigma_p^*(\alpha) \Rightarrow G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha).
\]

**Theorem 11.** Let \( p \in \mathbb{N}^* \), \( \beta < 0 \), \( \gamma \in \mathbb{C} \) and \( \alpha \leq \frac{\text{Re} \gamma}{\beta} < p < \delta \).

If \( g \in \Sigma_p^*(\alpha, \delta) \), with
\[
\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U,
\]
then \( G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha, \delta) \).

**Proof.** Because \( \beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U \), we obtain from Corollary 2 that \( G \in \Sigma_p \) with \( z^p G(z) \neq 0, \quad z \in U \), and
\[
\text{Re} \left[ \beta \frac{zG'(z)}{G(z)} + \gamma \right] > 0, \quad z \in U, \quad \text{where} \quad G = J_{p,\beta,\gamma}(g).
\]

Since \( \alpha \leq \frac{\text{Re} \gamma}{\beta}, \quad \text{and} \quad \beta < 0 \), we get from (21) that
\[
\text{Re} \frac{zG'(z)}{G(z)} + \alpha < 0, \quad z \in U.
\]

From (5) we know that
\[
P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} = -\frac{zg'(z)}{g(z)}, \quad \text{where} \quad P(z) = -\frac{zG'(z)}{G(z)}.
\]

Since \( g \in \Sigma_p^*(\alpha, \delta) \), we obtain from (23) that
\[
\alpha < \text{Re} \left[ P(z) + \frac{zP'(z)}{\gamma - \beta P(z)} \right] < \delta.
\]
Because we know from (22) that $\text{Re} P(z) > \alpha$, $z \in U$, we have only to verify that $\text{Re} P(z) < \delta$.

To show this we will use Lemma 3.

We know that $P$ is analytic in $U$ with $P(0) = p < \delta$. Also we have $\text{Re} \gamma - \delta \beta > 0$. Since the conditions from Lemma 3 are met, we obtain $\text{Re} P(z) < \delta$, which is equivalent to

$$-\text{Re} \frac{zG'(z)}{G(z)} < \delta. \quad (25)$$

From (22) and (25), since $G \in \Sigma_p$, we have $G \in \Sigma_p^*(\alpha, \delta)$.

If we consider $\delta \to \infty$, in the above theorem, we obtain the next corollary:

**Corollary 8.** Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$ and $\alpha \leq \frac{\text{Re} \gamma}{\beta} < p$.

If $g \in \Sigma_p^*(\alpha)$, with

$$\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U,$$

then $G = J_{p,\beta,\gamma}(g) \in \Sigma_p^*(\alpha)$.

We make the remark that we can obtain a similar result, without the condition $\beta \frac{zg'(z)}{g(z)} + \gamma < R_{\gamma-p\beta,p}(z), \quad z \in U$, as it follows:

**Theorem 12.** Let $p \in \mathbb{N}^*$, $\beta < 0$, $\gamma \in \mathbb{C}$, $\alpha \leq \frac{\text{Re} \gamma}{\beta} < p$ and $g \in \Sigma_p^*(\alpha)$. Let $G = J_{p,\beta,\gamma}(g)$. If $G \in \Sigma_p$ and $z^p G(z) \neq 0, \quad z \in U$, then $G \in \Sigma_p^*(\alpha)$.

We omit the proof because it is similar to that of Theorem 9.

The next results concern the sets $\Sigma K_p(\alpha, \delta), \Sigma C_{p,0}(\alpha, \delta; \varphi)$ and the operator $J_{p,\gamma} = J_{p,1,\gamma}$.

**Theorem 13.** Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\text{Re} \gamma > p$ and let $\alpha < p < \delta \leq \text{Re} \gamma$. If $g \in \Sigma K_p(\alpha, \delta)$ and $z^p J_{p,\gamma}(g)(z) \neq 0, \quad z \in U$, then

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha, \delta).$$

**Proof.** Let us denote $G = J_{p,\gamma}(g)$. We know from Theorem 6 that $G \in \Sigma_p$.

Let $P(z) = -1 - \frac{zG''(z)}{G'(z)}$, $z \in U$. Since $G \in \Sigma_p$ and $z^p G'(z) \neq 0, \quad z \in U$, we have $P \in H(U)$. 

Using the definition of the operator $J_{p,\gamma}$ and the logarithmic differential, two times, we obtain

$$P(z) + \frac{zP'(z)}{\gamma - P(z)} = -1 - \frac{zg''(z)}{g'(z)}, \quad z \in U. \quad (26)$$

From $g \in \Sigma K_p(\alpha, \delta)$, we have

$$\alpha < \text{Re} \left[ -1 - \frac{zg''(z)}{g'(z)} \right] < \delta, \quad z \in U,$$

so, using (26), we obtain

$$\alpha < \text{Re} \left[ P(z) + \frac{zP'(z)}{\gamma - P(z)} \right] < \delta, \quad z \in U. \quad (27)$$

Since $\alpha < P(0) = p < \delta$ and $0 \leq \text{Re} \gamma - \delta < \text{Re} \gamma - \alpha$, we obtain from (27), after applying Lemma 2 and Lemma 3 (in the case $\beta = 1$), that

$$\alpha < \text{Re} P(z) < \delta, \quad z \in U,$$

which is equivalent to

$$\alpha < \text{Re} \left[ -1 - \frac{zG''(z)}{G'(z)} \right] < \delta, \quad z \in U. \quad (28)$$

Since $G = J_{p,\gamma}(g) \in \Sigma_p$, we have from (28), that $J_{p,\gamma}(g) \in \Sigma K_p(\alpha, \delta)$.

From the proof of the above theorem we remark that we also have the next result.

**Theorem 14.** Let $p \in \mathbb{N}^*$, $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{C}$ with $\alpha < p < \text{Re} \gamma$. If $g \in \Sigma K_p(\alpha)$ and $z^{p+1}J_{p,\gamma}(g)(z) \neq 0$, $z \in U$, then

$$J_{p,\gamma}(g) \in \Sigma K_p(\alpha).$$

**Theorem 15.** Let $p \in \mathbb{N}^*$, $\gamma \in \mathbb{C}$ with $\text{Re} \gamma > p$, and $\alpha < 1 \leq p < \delta \leq \text{Re} \gamma$. Let $\varphi$ be a function in $\Sigma K_{p,0}(\alpha, \delta)$ and $g \in \Sigma C_{p,0}(\alpha, \delta; \varphi)$ such that $z^{p+1}J_{p,\gamma}'(\varphi) \neq 0$, $z \in U$, then

$$J_{p,\gamma}(g) \in \Sigma C_{p,0}(\alpha, \delta; \Phi),$$

where $\Phi = J_{p,\gamma}(\varphi)$. 
Proof. From $g \in \Sigma_{p,0}(\alpha, \delta; \varphi)$, we have

$$\alpha < \Re \left( \frac{g'(z)}{\varphi'(z)} \right) < \delta, \; z \in U.$$  

(29)

Let $G = J_{p, \gamma}(g)$. We know from Remark 2 that $G, \Phi \in \Sigma_{p,0}$.

From $G = J_{p, \gamma}(g)$ and $\Phi = J_{p, \gamma}(\varphi)$, we get

$$\gamma G(z) + zG''(z) = (\gamma - p)g(z)$$

and

$$\gamma \Phi(z) + z\Phi'(z) = (\gamma - p)\varphi(z), \; z \in \hat{U},$$

hence

$$(\gamma + 1)G'(z) + zG'''(z) = (\gamma - p)g'(z)$$

and

$$(\gamma + 1)\Phi'(z) + z\Phi''(z) = (\gamma - p)\varphi'(z).$$

Let us denote

$$p(z) = \frac{G'(z)}{\Phi'(z)}, \; z \in U.$$

Since $G, \Phi \in \Sigma_{p,0}$ and $z^{p+1}\Phi'(z) \neq 0, \; z \in U$, we have $p \in H(U)$. Of course, $p(0) = 1$.

From $p(z)\Phi'(z) = G'(z)$, we get $G''(z) = p'(z)\Phi'(z) + p(z)\Phi''(z)$, so, the equality

$$(\gamma + 1)G'(z) + zG'''(z) = (\gamma - p)g'(z), \; z \in U,$$

can be rewritten as

$$(\gamma + 1)p(z)\Phi'(z) + z[p'(z)\Phi'(z) + p(z)\Phi''(z)] = (\gamma - p)g'(z).$$

(30)

Using the equality $(\gamma + 1)\Phi'(z) + z\Phi''(z) = (\gamma - p)\varphi'(z)$, we obtain from (30) that

$$p(z) + \frac{zp'(z)}{\gamma + 1 + z\Phi''(z)} = \frac{g'(z)}{\varphi'(z)}, \; z \in U,$$

which is equivalent to

$$p(z) + \frac{zp'(z)}{P(z)} = \frac{g'(z)}{\varphi'(z)},$$

where $P(z) = \gamma + 1 + z\Phi''(z)$.

Since $\alpha < \Re \left( \frac{g'(z)}{\varphi'(z)} \right) < \delta, \; z \in U$, we obtain

$$\alpha < \Re \left[ p(z) + \frac{zp'(z)}{P(z)} \right] < \delta, \; z \in U.$$  

(31)
Let us denote \( p_1(z) = p(z) - \alpha \) and \( p_2(z) = \delta - p(z) \). Using now (31), we have

\[
\text{Re} \left[ p_k(z) + \frac{zp_k'(z)}{P(z)} \right] > 0, \quad z \in U, \quad k = 1, 2.
\]

(32)

It is easy to see that \( p_k(0) > 0 \), so, to apply Theorem 2 we need only to verify that \( \text{Re} \, P(z) > 0, \quad z \in U \), where \( P(z) = \gamma + 1 + \frac{z\Phi''(z)}{\Phi'(z)} \).

As we know that \( \varphi \in \Sigma K_p,0(\alpha, \delta) \) with \( z^{p+1} \, J_{p,\gamma}(\varphi)(z) \neq 0, \quad z \in U \), we obtain from Theorem 13 that

\[
\Phi = J_{p,\gamma}(\varphi) \in \Sigma K_p,0(\alpha, \delta),
\]

which is equivalent to

\[
\alpha < \text{Re} \left[ -1 - \frac{z\Phi''(z)}{\Phi'(z)} \right] < \delta, \quad z \in U,
\]

hence

\[
\text{Re} \, \gamma - \delta < \text{Re} \, P(z) < \text{Re} \, \gamma - \alpha, \quad z \in U.
\]

Since \( \text{Re} \, \gamma \geq \delta \), we get \( \text{Re} \, P(z) > 0, \quad z \in U \), and we can now apply Theorem 2 to obtain \( \text{Re} \, p_k(z) > 0, \quad z \in U, \quad k = 1, 2 \). Therefore, we have

\[
\alpha < \text{Re} \left( \frac{G'(z)}{\Phi'(z)} \right) < \delta, \quad z \in U.
\]

(33)

Since we know that \( G \in \Sigma_p,0 \) and \( \Phi \in \Sigma K_p,0(\alpha, \delta) \), we have from (33) that \( G = J_{p,\gamma}(g) \in \Sigma K_p,0(\alpha, \delta; \Phi) \).

References


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