On sufficient condition for starlikeness

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Abstract

In this paper, we give a condition for starlikeness of the integral operator of the form

\[ F(z) = \int_0^z \prod_{k=1}^k \left( \frac{f_1(s)}{s} \right) \frac{z^2}{s} ds. \]

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1 Introduction

Let \( A \) be the class of all analytic functions \( f(z) \) defined in the open unit disk \( U = \{ z \in C : |z| < 1 \} \) and \( S \) the subclass of \( A \) consisting of univalent functions

\[ f(z) = z + \sum_{k=2}^{\infty} a_k z^k \]

\[ S^* = \{ f \in S : \text{Re}(\frac{zf'(z)}{f(z)}) > 0, z \in U \}, \]

\[ M_\alpha = \{ f \in S : \frac{f(z)f'(z)}{z} \neq 0, \text{Re}(\alpha, f; z) > 0, z \in U \} \]

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where \( J(\alpha, f; z) = (1 - \alpha) \frac{zf'(z)}{f(z)} + \alpha \frac{zf''(z)}{f(z)} \) be the class of starlike and \( \alpha - \text{convex} \) functions respectively. Let \( p(z) \) be the class of functions that are regular in \( U \) and of the form:

\[
p(z) = 1 + \sum_{k=1}^{\infty} b_k z^k
\]

Furthermore, let \( h(z) = \frac{1+z}{1-z} \)

Let \( T \) be the univalent [5] subclass of \( A \) consisting of functions \( f(z) \) satisfying

\[
\left| \frac{zf'(z)}{f(z)} - 1 \right| < 1, \quad (z \in U)
\]

Let \( T_n \) be the subclass of \( T \) for which \( f_k(0) = 0 \) (\( k = 2, 3, \ldots, n \)).

Let \( T_{n,\mu} \) be the subclass of \( T_n \) consisting of functions of the form

\[
\int_0^z \prod_{i=1}^k \left( \frac{f_i(s)}{s} \right)^{\frac{1}{\mu}} ds \text{ satisfying: } |\frac{zf'(z)}{f(z)} - 1| < \mu, \quad (z \in U)
\]

for some \( \mu(0 < \mu \leq 1) \).

2 Preliminaries

**Theorem 1** [1] Let \( M \) and \( N \) be analytic in \( U \) with \( M(0) = N(0) = 0 \). If \( N(z) \) maps onto a many-sheeted region which is starlike with respect to the origin and \( \text{Re}\{\frac{M(z)}{N(z)}\} > 0 \) in \( U \), then \( \text{Re}\{\frac{M(z)}{N(z)}\} > 0 \) in \( U \).

**Theorem 2** [6] Let \( f_i \in T_{n,\mu} (i = 1, 2, \ldots, k; k \in N^*) \) be defined by

\[
f_i(z) = z + \sum_{n=2}^{\infty} a_n^i z^n
\]

for all \( i = 1, 2, \ldots, k; \alpha, \beta \in C; R\{\beta\} \geq \gamma \) and \( \gamma = \sum_{i=1}^{k} \frac{1 + (1 + \mu_i)M}{|\alpha|} (M \geq 1, 0 < \mu_i < 1, k \in N^*) \). If \( |f_i(z)| \leq M(z \in U), i = 1, 2, \ldots, k \) then, the integral operator

\[
F_{\alpha,\beta}(z) = \{\beta \int_0^z t^{\beta-1} \prod_{i=1}^k \left( \frac{f_i(t)}{t} \right)^{\frac{1}{\mu_i}} dt \}^\gamma
\]

is univalent.

**Theorem 3** [2] Let \( h \) be convex in \( U \) and \( \text{Re}\{\beta h(z) + \gamma\} > 0, z \in U \). If \( p \in H(U) \) where \( H(U) \) is the class of functions which are analytic in the unit disk, with \( p(0) = h(0) \) and \( p \) satisfies the Briot-Bouquet differential subordinations: \( p(z) + \frac{zp'(z)}{bp(z)+\gamma} < h(z), \quad z \in U \). Then, \( p(z) < h(z), \quad z \in U \).
3 Main Results

We now give the proof of the following results:

**Theorem 4** Let $F_\alpha(z)$ be the function in $U$ defined by

\begin{equation}
F_\alpha(z) = \int_0^z \prod_{i=1}^k \left( \frac{f_i(s)}{s} \right)^{\frac{1}{\beta}} ds, \alpha \in C.
\end{equation}

If $f_i \in S^*$ then, $F(z) \in S^*$ where $f_i$ is as in equation (3) above.

**Proof.** By differentiating (5), we obtain:

\[
\frac{zF'(z)}{F(z)} = \prod_{i=1}^k \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\beta}} ds.
\]

Let

\begin{equation}
M = zF'(z), N(z) = F(z)
\end{equation}

From (5) and (6) we have:

\[
\frac{M'(z)}{N'(z)} = 1 + \frac{zF''(z)}{F'(z)}, \quad \frac{M'(z)}{N'(z)} = 1 + \frac{\sum_{i=1}^k \frac{1}{\alpha} \left( \frac{zf_i(z)}{f(z)} - 1 \right)}{\prod_{i=1}^k \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\beta}}}
\]

\[
|\frac{M'(z)}{N'(z)} - 1| = \frac{\sum_{i=1}^k \frac{1}{\alpha} \left| \frac{zf_i(z)}{f(z)} - 1 \right|}{\prod_{i=1}^k \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\beta}}} \leq \frac{\sum_{i=1}^k \left| \frac{1}{\alpha} \left( \frac{zf_i(z)}{f(z)} - 1 \right) \right|}{\prod_{i=1}^k \left( \frac{f_i(z)}{z} \right)^{\frac{1}{\beta}}}.
\]

By hypothesis $f_i \in S^*$. This means that $|\frac{zf_i(z)}{f(z)} - 1| < 1$, which implies that $|\frac{M'(z)}{N'(z)} - 1| < 1$. Thus $Re\{\frac{M'(z)}{N'(z)}\} > 0$ and by Theorem 1, $Re\{\frac{M(z)}{N(z)}\} > 0$. This implies that $Re\{\frac{zF'(z)}{F(z)}\} > 0$. Hence $F \in S^*$.

**Remark 1** The integral in (5) is equivalent to that in (4) of section 2 with $\beta = 1$.

Let $S = \{ f : U \to C \} \cap S$. Let $F(z) \in U$ be defined by

\begin{equation}
F(z) = \int_0^z \prod_{i=1}^k \left( \frac{f_i(s)}{s} \right)^{\frac{1}{\beta}} ds.
\end{equation}
Theorem 5 Let \( z \in U, \alpha \in C, \text{Re} \alpha > 0 \) and \( m_\alpha = M_\alpha \cap s \). If \( F \in m_\alpha \), then \( F \in S^* \) that is \( m_\alpha \subset S^* \).

Proof. From (6) above, we have \( \frac{F(z)F'(z)}{z} \neq 0 \) and for \( F \in m_\alpha \), we have

\[
\text{Re} J(\alpha, f; z) = \text{Re} \left\{ (1 - \alpha) \frac{zF'(z)}{F(z)} + \alpha \left( 1 + \frac{zF''(z)}{F'(z)} \right) \right\}
\]

for \( p(z) = \frac{zF'(z)}{F(z)} \), \( \frac{zp'(z)}{p(z)} = 1 + \frac{zF''(z)}{F'(z)} - p(z) \). This implies that

\[
1 + \frac{zF''(z)}{F'(z)} = \frac{zp'(z)}{p(z)} + p(z)
\]

using (7) and (9) in (8), we obtain

\[
\text{Re} J(\alpha, f; z) = \text{Re} \left\{ (1 - \alpha)p(z) + \alpha \left( \frac{zp'(z)}{p(z)} + p(z) \right) \right\}.
\]

Simplifying (10), we obtain \( \text{Re} J(\alpha, f; z) = \text{Re} \left\{ p(z) + \alpha \left( \frac{zp'(z)}{p(z)} \right) \right\} \)

\( p(0) + \frac{\alpha p'(0)}{p(0)} = 1 \) and \( p(0) = h(0) = 1 \). Thus, using Theorem 3 with \( \beta = 1 \) and \( \gamma = 0 \), we have \( p(z) + \frac{\alpha zp'(z)}{p(z)} < h(z) = \frac{1 + z}{1 - z} \). This implies that \( p(z) < h(z) \).

That is \( \text{Re} \{ p(z) \} > 0 \). Thus, \( \text{Re} \{ \frac{zF'(z)}{F(z)} \} > 0 \). Hence, \( F \in S^* \).

References


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