On a subclass of analytic functions with negative coefficient associated with convolution structure \(^1\)

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Abstract

The main object of this paper is to study the subclass \(SC(\gamma, \lambda, \beta)\) of analytic univalent functions with negative coefficients in unit disc \(U = \{z : |z| < 1\}\). Further coefficient estimates, distortion theorem and integral operators for this class are also obtained. We also discuss radii of convexity and closure properties for functions belonging to the class \(SC(\gamma, \lambda, \beta)\).

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1 Introduction

Let \(\mathcal{A}\) denote the class of the functions

\[
f(z) = z + \sum_{k=2}^{\infty} a_k z^k
\]

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which are analytic in the unit disk \( U = \{ z : |z| < 1 \} \).

A function \( f \in A \) is said to belong to the class \( A \) of \textit{Starlike} functions of order \( \alpha \) \((0 \leq \alpha < 1)\), if it is satisfies, for \( z \in U \), the conditions

\[
\text{Re} \left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha.
\]

We denote this class by \( S^\star(\alpha) \). Further, \( f \in A \) is said to be convex function of order \( \alpha \) in \( U \), if it satisfies

\[
\text{Re} \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > \alpha, \quad z \in U,
\]

for some \( \alpha \) \((0 \leq \alpha < 1)\). We denote this class \( K(\alpha) \).

Let \( T \) denote subclass of \( A \), consisting functions of the form

\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0.
\]

The function

\[
S_\alpha(z) = z(1-z)^{-2(1-\alpha)}, \quad \alpha(0 \leq \alpha \leq 1)
\]

is the familiar extremal function for the class \( S^\star(\alpha) \), setting

\[
C(\alpha,k) = \Pi_{i=2}^{k} (i-2\alpha) \over (k-1)! , k \geq 2,
\]

using (5) and (6) we can write

\[
S_\alpha(z) = z + \sum_{k=2}^{\infty} C(\alpha,k) z^k.
\]

Clearly, \( C(\alpha,k) \) is a decreasing function in \( \alpha \), and that

\[
\lim_{k \to \infty} C(\alpha,k) = \begin{cases} \infty, & \alpha < 1/2 \\ 1, & \alpha = 1/2 \\ 0, & \alpha > 1/2. \end{cases}
\]
If we now define \( g(z) \) as
\[
g(z) = z + \sum_{k=2}^{\infty} b_k z^k
\]
then the Hadamard product (or convolution) of two analytic functions \( f(z) \) and \( g(z) \), where \( f(z) \), \( g(z) \) is given by equations (1) and (9) respectively, is defined as
\[
(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k
\]

For a function \( f(z) \) in \( A \), we can define the differential operator \( D^n \), introduced by Salagean [9] as
\[
D^0 f(z) = f(z)
\]
\[
D^1 f(z) = D f(z) = zf'(z) = z + \sum_{k=2}^{\infty} k a_k z^k
\]
\[
D^2 f(z) = D(D f(z)) = z + \sum_{k=2}^{\infty} k^2 a_k z^k
\]
\[
D^n f(z) = D(D^{n-1} f(z)) = z + \sum_{k=2}^{\infty} k^n a_k z^k
\]

We also define a subclass of \( A \) consisting of functions \( f(z) \), denoted by \( SC(\gamma, \lambda, \beta) \) which satisfy the following condition
\[
\Re \left[ 1 + \frac{1}{\gamma} \left( \frac{z \lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)'(z)}{\lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)'(z)} - 1 \right) \right] > \beta,
\]
\((0 \leq \lambda \leq 1, 0 \leq \beta < 1; \gamma \in C, z \in U)\).

**Special case of class.**

(a) When \( \lambda = 0 \), and \( \alpha = 1/2 \) then our class reduces in class of starlike
functions of order $\beta$.

(b) When $\lambda = 0$, then our class reduces in class of starlike functions of complex order $\gamma$.

(c) When $\alpha = 1/2$ then this class reduces in class defined and studied by Altintas, Irmak, Owa and Srivastava [5].

2 Coefficient estimates.

Theorem 1 Let the function $f(z) \in A$ is in the class $SC(\gamma, \lambda, \beta)$, if and only if

$$\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)a_k \leq \gamma(1-\beta)$$  \hspace{1cm} (13)

Proof. Assume that the inequality (13) holds true, then

$$\left| \frac{1}{\gamma} \left( \frac{z [\lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)'(z)]}{\lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)(z)} - 1 \right) \right|$$

$$= \left| \frac{1}{\gamma} \left( \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] (1 - k)C(\alpha, k)a_k z^{k-1} \right) \left( 1 - \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] C(\alpha, k)a_k z^{k-1} \right) \right| \leq (1 - \beta).$$

Hence, by using the maximum modulus principle, $f(z) \in SC(\gamma, \lambda, \beta)$. Conversely, assume that the function $f(z)$ defined by (1) is in the class $SC(\gamma, \lambda, \beta)$.

Then we will have

$$Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{z [\lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)'(z)]}{\lambda z(D^n f * S_\alpha)'(z) + (1 - \lambda)(D^n f * S_\alpha)(z)} - 1 \right) \right\} > \beta,$$

$$Re \left\{ 1 + \frac{1}{\gamma} \left( \frac{\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] (1 - k)C(\alpha, k)a_k z^k}{z - \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] C(\alpha, k)a_k z^k} \right) \right\} > \beta,$$
\[
\text{Re} \left[ 1 + \frac{1}{\gamma} \left\{ \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] (1 - k) C(\alpha, k)a_k z^{k-1} \right\} \right] > \beta,
\]
and now when \( z \to 1^- \), we obtain
\[
\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] (1 - k) C(\alpha, k) a_k 
\leq \gamma (\beta - 1)
\]
and finally,
\[
\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k) a_k 
\leq \gamma (1 - \beta).
\]

**Corollary 1** Let the function \( f(z) \) defined by (1) be in the class \( SC(\gamma, \lambda, \beta) \).

Then
\[
a_k \leq \frac{\gamma (1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)}, \quad (k \geq 2)
\]
and the equality is attained for the function \( f(z) \) given by
\[
f(z) = z - \frac{\gamma (1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)} z^k.
\]

3 Distortion Theorem.

**Theorem 2** Let the function \( f(z) \) be in class \( SC(\gamma, \lambda, \beta) \) then for \( 0 \leq |z| = r \)
\[
r - \frac{\gamma (1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)} r^k \leq |f(z)|
\]
\[
\leq r + \frac{\gamma (1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)} r^k.
\]
**Proof.** From equation (15), easy to find that
\[
|z| - \frac{\gamma(1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^k \leq |f(z)|
\]
\[
\leq |z| + \frac{\gamma(1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^k
\]
Now using the fact that $|z| = r < 1$, we obtain the desired result.

**Corollary 2** If the function $f(z)$ is in the class $SC(\gamma, \lambda, \beta)$ then $f(z)$ is included in a disc with centre at the origin and radius $r$, where
\[
r = 1 + \frac{\gamma(1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)}
\]

**Theorem 3** Let the function $f(z)$ be in the class $SC(\gamma, \lambda, \beta)$, then
\[
1 - \frac{\gamma(1 - \beta)}{k^{n-1} [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^{k-1} \leq |f(z)|
\]
\[
\leq 1 + \frac{\gamma(1 - \beta)}{k^{n-1} [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^{k-1}
\]
Where equality holds for the function $f(z)$ given by (15).

\[
1 - \frac{k\gamma(1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^{k-1} \leq |f(z)|
\]
\[
\leq 1 + \frac{k\gamma(1 - \beta)}{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)} |z|^{k-1}
\]
Again using the fact that $|z| = r$, we obtain the desired result.

### 4 Integral Operators

**Theorem 4** Let the function $f(z)$ defined by (1) be in the class $SC(\gamma, \lambda, \beta)$ and let $c$ be a real number such that $c > -1$. Then $F(z)$, defined by
\[
F(z) = \frac{c+1}{z^c} \int_0^z t^{c-1} f(t) dt
\]
Proof. From the representation of $F(z)$, it is obtained that

$$F(z) = z - \sum_{k=2}^{\infty} b_k z^k,$$

where $b_k = \left( \frac{c+1}{k+c} \right) a_k$

Therefore

$$\sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k) b_k$$

$$= \sum_{k=2}^{\infty} k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k) \left( \frac{c+1}{k+c} \right) a_k$$

$$\leq \gamma(\beta - 1),$$

since $f(z)$ belongs to $SC(\gamma, \lambda, \beta)$ so by virtue of Theorem 1, $F(z)$ is also element of $SC(\gamma, \lambda, \beta)$.

Theorem 5 Let the function

$$F(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0$$

be in the class $SC(\gamma, \lambda, \beta)$ and is defined by equation (19). Now if $c > -1$, then $F(z)$ is univalent in $|z| < R^*$, where

$$R^* = \inf \left\{ \frac{k^{n-1} [\lambda k + 1 - \lambda] [k - 1 - \gamma(\beta - 1)] C(\alpha, k)(c+1)}{(c+k)\gamma(1-\beta)} \right\}, k \geq 2$$

The result is sharp.

Proof. From (19) we have

$$f(z) = \frac{z^{1-c} (z^c F(z))'}{c+1} = z - \sum_{k=2}^{\infty} \left( \frac{c+k}{c+1} \right) a_k z^k.$$
In order to obtain the required result, it is sufficient to prove that

\[ |f'(z) - 1| < 1 \text{ for } |z| < R^*. \]

Now since

\[
|f'(z) - 1| = \left| - \sum_{k=2}^{\infty} k \left( \frac{c+k}{c+1} \right) a_k z^{k-1} \right|
\]

\[
\leq \sum_{k=2}^{\infty} k \left( \frac{c+k}{c+1} \right) a_k |z|^{k-1}
\]

\[
\leq \sum_{k=2}^{\infty} \frac{k^n \left[ \lambda k + 1 - \lambda \right] [k-1 - \gamma(\beta - 1)] C(\alpha, k) a_k}{\gamma(1 - \beta)} |z|^{k-1}
\]

But from Theorem 1, we know that

\[
\sum_{k=2}^{\infty} k^n \left[ \lambda k + 1 - \lambda \right] [k-1 - \gamma(\beta - 1)] C(\alpha, k) a_k < 1
\]

From equation (22) and (23) we have

\[
|z|^{k-1} \leq \frac{k^n \left[ \lambda k + 1 - \lambda \right] [k-1 - \gamma(\beta - 1)] C(\alpha, k)}{(c+k)\gamma(1 - \beta)}
\]

or

\[
|z| \leq \left\{ \frac{k^{n-1} \left[ \lambda k + 1 - \lambda \right] [k-1 - \gamma(\beta - 1)] C(\alpha, k)(c+1)}{(c+k)\gamma(1 - \beta)} \right\}^{\frac{1}{k-1}}, (k \geq 2),
\]

we obtain the desired result. The result is sharp for the function

\[
f(z) = z - \frac{(c+k)\gamma(1 - \beta)}{k^n \left[ \lambda k + 1 - \lambda \right] [k-1 - \gamma(\beta - 1)] (c+k)C(\alpha, k)} z^k, (k \geq 2).
\]
5 Radius of Convexity

Theorem 6 If $f(z)$ given by (1) is in class $SC(\gamma, \lambda, \beta)$ then $f(z)$ is convex in $|z| < R_p$, where

$$R_p = \inf \left\{ \frac{k^{n-2} [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k) a_k}{\gamma (1 - \beta)} \right\}^{\frac{1}{k-1}}$$

The result is sharp.

Proof. In order to establish the required result it is sufficient to show that

$$\left | \frac{zf'(z)}{f'(z)} \right | < 1, \quad |z| < R_p$$

in view of (1), we have

$$\left | \frac{zf''(z)}{f'(z)} \right | \leq \frac{\sum_{k=2}^{\infty} k(k-1)a_k |z|^{k-1}}{1 - \sum_{k=2}^{\infty} ka_k |z|^{k-1}}$$

Hence, we obtain

$$\sum_{k=2}^{\infty} k^2 a_k |z|^{k-1} \leq 1$$

but from Theorem 1, we know that

$$\sum_{k=2}^{\infty} \frac{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k) a_k}{\gamma (1 - \beta)} < 1$$

and thus from (26) and (27) we have

$$k^2 |z|^{k-1} \leq \frac{k^n [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)}{\gamma (1 - \beta)}$$

or

$$|z| \leq \left\{ \frac{k^{n-2} [\lambda k + 1 - \lambda] [k - 1 - \gamma (\beta - 1)] C(\alpha, k)}{\gamma (1 - \beta)} \right\}^{\frac{1}{k-1}}$$

Hence, $f(z)$ is convex in $|z| < R_p$. The result is sharp and is given by (25).
6 Closure Theorem

Theorem 7 Let the function $f_j(z), (j = 1, 2, m)$ be defined by

$$f_j(z) = z - \sum_{k=2}^{\infty} a_{kj} z^k \quad (a_{kj} > 0)$$

for $z \in U$, be in the class $SC(\gamma, \lambda, \beta)$ then the function $h(z)$ defined by

$$h(z) = z - \sum_{k=2}^{\infty} b_k z^k$$

also belongs to the class $SC(\gamma, \lambda, \beta)$, where

$$b_k = \frac{1}{m} \sum_{j=1}^{m} a_{kj}$$

Proof. Since $f_j(z) \in SC(\gamma, \lambda, \beta)$, it follows from Theorem 1, that

$$\sum_{k=2}^{\infty} k^n \left[ \lambda k + 1 - \lambda \right] \left[ k - 1 - \gamma (\beta - 1) \right] C(\alpha, k)a_{kj} < \gamma (1 - \beta), \quad (j = 1, 2, m).$$

Therefore,

$$\sum_{k=2}^{\infty} k^n \left[ \lambda k + 1 - \lambda \right] \left[ k - 1 - \gamma (\beta - 1) \right] C(\alpha, k)b_k$$

$$= \sum_{k=2}^{\infty} k^n \left[ \lambda k + 1 - \lambda \right] \left[ k - 1 - \gamma (\beta - 1) \right] C(\alpha, k) \left( \frac{1}{m} \sum_{j=1}^{m} a_{kj} \right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left( \sum_{k=2}^{\infty} k^n \left[ \lambda k + 1 - \lambda \right] \left[ k - 1 - \gamma (\beta - 1) \right] C(\alpha, k)a_{kj} \right)$$

$$< \gamma (1 - \beta).$$

Hence by Theorem 1, $h(z) \in SC(\gamma, \lambda, \beta)$ also.

Theorem 8 The class $SC(\gamma, \lambda, \beta)$ is closed under linear combination.

Proof. Employing same techniques used by Silverman [14] with the aid of Theorem 8, it can be easily proved.
References


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