A new univalent integral operator defined by Al-Oboudi differential operator

Serap Bulut

Abstract

In [3], Breaz and Breaz gave an univalence condition of the integral operator $G_{n,\alpha}$ introduced in [2]. The purpose of this paper is to give univalence condition of the generalized integral operator $G_{n,m,\alpha}$ defined in [4]. Our results generalize the results of [3].

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1 Introduction

Let $A$ denote the class of all functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k$$

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which are analytic in the open unit disk \( U = \{ z \in \mathbb{C} : |z| < 1 \} \), and

\[ S = \{ f \in A : f \text{ is univalent in } U \}. \]

For \( f \in A \), Al-Oboudi [1] introduced the following operator:

\begin{align*}
(2) \quad D^0 f(z) &= f(z), \\
(3) \quad D^1 f(z) &= (1-\delta) f(z) + \delta z f'(z) = D_\delta f(z), \quad \delta \geq 0 \\
(4) \quad D^n f(z) &= D_\delta (D^{n-1} f(z)), \quad (n \in \mathbb{N} := \{ 1, 2, 3, \ldots \}).
\end{align*}

If \( f \) is given by (1), then from (3) and (4) we see that

\begin{equation}
D^n f(z) = z + \sum_{k=2}^{\infty} [1 + (k - 1)\delta]^n a_k z^k, \quad (n \in \mathbb{N}_0 := \mathbb{N} \cup \{ 0 \}),
\end{equation}

with \( D^n f(0) = 0 \).

**Remark 1** When \( \delta = 1 \), we get Sălăgean’s differential operator [9].

The following results will be required in our investigation.

**General Schwarz Lemma.** [5] Let the function \( f \) be regular in the disk \( U_R = \{ z \in \mathbb{C} : |z| < R \} \), with \(|f(z)| < M\) for fixed \( M \). If \( f \) has one zero with multiplicity order bigger than \( m \) for \( z = 0 \), then

\[ |f(z)| \leq \frac{M}{R^m} |z|^m \quad (z \in U_R). \]

The equality can hold only if

\[ f(z) = e^{i\theta} \frac{M}{R^m} z^m, \]

where \( \theta \) is constant.
Theorem A. [7] Let $\alpha$ be a complex number with $\text{Re}\alpha > 0$ and $f \in A$. If $f$ satisfies
\[
1 - \frac{|z|^{2\text{Re}\alpha}}{\text{Re}\alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq 1 \quad (z \in \mathbb{U}),
\]
then, for any complex number $\beta$ with $\text{Re}\beta \geq \text{Re}\alpha$, the integral operator
\[
F_{\beta}(z) = \left\{ \beta \int_{0}^{z} t^{\beta-1} f'(t) dt \right\}^{\frac{1}{\beta}}
\]
is in the class $S$.

Theorem B. [6] Let $f \in A$ satisfy the following inequality:
\[
\left| \frac{z^2 f'(z)}{(f(z))^2} - 1 \right| \leq 1 \quad (z \in \mathbb{U}).
\]
Then $f$ is univalent in $\mathbb{U}$.

Theorem C. [8] Assume that $g \in A$ satisfies condition (6), and let $\alpha$ be a complex number with
\[
|\alpha - 1| \leq \frac{\text{Re}\alpha}{3}.
\]
If
\[
|g(z)| \leq 1, \quad \forall z \in \mathbb{U}
\]
then the function
\[
G_{\alpha}(z) = \left\{ \alpha \int_{0}^{z} (g(t))^{\alpha-1} dt \right\}^{\frac{1}{\alpha}}
\]
is of class $S$.

In [2], Breaz and Breaz considered the integral operator
\[
G_{n,\alpha}(z) := \left\{ n (\alpha - 1) + 1 \right\} \int_{0}^{z} (g_1(t))^{\alpha-1} \cdots (g_n(t))^{\alpha-1} dt \right\}^{\frac{1}{n(\alpha-1)+1}},
\]
($g_1, \ldots, g_n \in A$), and proved that the function $G_{n,\alpha}$ is univalent in $\mathbb{U}$.

Remark 2 Note that for $n = 1$, we obtain the integral operator $G_{\alpha}$ defined by (7).
Theorem D. [3] Let $g_i \in \mathcal{A}, \forall i = 1, \ldots, n, n \in \mathbb{N}$, satisfy the properties

$$\left| \frac{z^2 g_i'(z)}{(g_i(z))^2} - 1 \right| < 1, \forall z \in \mathbb{U}, \forall i = 1, \ldots, n$$

and $\alpha \in \mathbb{C}$ with

$$|\alpha - 1| \leq \frac{\text{Re}\alpha}{3n}.$$ 

If

$$|g_i(z)| \leq 1, \forall z \in \mathbb{U}, \forall i = 1, \ldots, n,$$

then the function $G_{n,\alpha}$ defined by (8) is univalent.

In [4], the author introduced a new general integral operator by means of the Al-Oboudi differential operator as follows.

Definition 1 [4] Let $n \in \mathbb{N}, m \in \mathbb{N}_0$ and $\alpha \in \mathbb{C}$. We define the integral operator $G_{n,m,\alpha}$ by

$$G_{n,m,\alpha}(z) := \left\{ n (\alpha - 1) + 1 \int_0^z \prod_{j=1}^n (D^m g_j(t))^{\alpha-1} dt \right\}^{\frac{1}{n(\alpha - 1) + 1}} (z \in \mathbb{U}),$$

where $g_1, \ldots, g_n \in \mathcal{A}$ and $D^m$ is the Al-Oboudi differential operator.

Remark 3 In the special case $n = 1$, we obtain the integral operator

$$G_{m,\alpha}(z) := \left\{ \alpha \int_0^z (D^m g(t))^{\alpha-1} dt \right\}^{\frac{1}{\alpha}} (z \in \mathbb{U}).$$

Remark 4 If we set $m = 0$ in (9) and (10), then we obtain the integral operators defined in (8) and (7), respectively.
2 Main Results

Theorem 1 Let $M_j \geq 1$, each of the functions $g_j \in A \ (j \in \{1, \ldots, n\})$ satisfies the inequality

$$(11) \quad \left| \frac{z^2 (D^m g_j(z))'}{(D^m g_j(z))^2} - 1 \right| \leq 1 \quad (z \in U; \ m \in \mathbb{N}_0).$$

and $\alpha \in \mathbb{C}$ with

$$|\alpha - 1| \leq \frac{\text{Re}\alpha}{\sum_{j=1}^n (2M_j + 1)}, \quad \text{Re} (n (\alpha - 1) + 1) \geq \text{Re}\alpha > 0.$$

If

$$|D^m g_j(z)| \leq M_j \quad (z \in U; \ j \in \{1, \ldots, n\}),$$

then the integral operator $G_{n,m,\alpha}$ defined by (9) is in the univalent function class $S$.

Proof. Since $g_j \in A \ (j \in \{1, \ldots, n\})$, by (5), we have

$$\frac{D^m g_j(z)}{z} = 1 + \sum_{k=2}^{\infty} [1 + (k - 1)\delta] a_{k,j} z^{k-1} \quad (m \in \mathbb{N}_0)$$

and

$$\frac{D^m g_j(z)}{z} \neq 0$$

for all $z \in U$.

Also we note that

$$G_{n,m,\alpha}(z) = \left\{ n(\alpha - 1) + 1 \int_0^z t^{n(\alpha - 1)} \prod_{j=1}^n \left( \frac{D^m g_j(t)}{t} \right)^{\alpha - 1} \ dt \right\}^{\frac{1}{n(\alpha - 1) + 1}}.$$

Define a function

$$f(z) = \int_0^z \prod_{j=1}^n \left( \frac{D^m g_j(t)}{t} \right)^{\alpha - 1} \ dt.$$
Then we obtain

\begin{equation}
\frac{d^n f(z)}{dz^n} = \prod_{j=1}^{n} \left( \frac{D^m g_j(z)}{z} \right)^{\alpha - 1}.
\end{equation}

It is clear that \( f(0) = f'(0) - 1 = 0 \).

The equality (12) implies that

\[ \ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \ln \left( \frac{D^m g_j(z)}{z} \right) \]

or equivalently

\[ \ln f'(z) = (\alpha - 1) \sum_{j=1}^{n} \left( \ln D^m g_j(z) - \ln z \right). \]

By differentiating above equality, we get

\[ \frac{f''(z)}{f'(z)} = (\alpha - 1) \sum_{j=1}^{n} \left( \frac{(D^m g_j(z))'}{D^m g_j(z)} - \frac{1}{z} \right). \]

Hence we obtain

\[ \frac{zf''(z)}{f'(z)} = (\alpha - 1) \sum_{j=1}^{n} \left( \frac{z (D^m g_j(z))'}{D^m g_j(z)} - 1 \right), \]

which readily shows that

\[ \frac{1 - |z|^{2\Re \alpha}}{\Re \alpha} \left| \frac{zf''(z)}{f'(z)} \right| \leq \frac{1 - |z|^{2\Re \alpha}}{\Re \alpha} |\alpha - 1| \sum_{j=1}^{n} \left( \left| \frac{z (D^m g_j(z))'}{D^m g_j(z)} \right| + 1 \right) \]

\[ \leq \frac{|\alpha - 1|}{\Re \alpha} \sum_{j=1}^{n} \left( \left| \frac{z^2 (D^m g_j(z))'}{(D^m g_j(z))^2} \right| \left| \frac{D^m g_j(z)}{z} \right| + 1 \right). \]

From the hypothesis, we have \( |g_j(z)| \leq M_j (j \in \{1, \ldots, n\} ; z \in \mathbb{U}) \), then by the General Schwarz Lemma, we obtain that

\[ |g_j(z)| \leq M_j |z| \quad (j \in \{1, \ldots, n\} ; z \in \mathbb{U}). \]
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Then we find

\[
1 - \left| z \right|^{2 \Re \alpha} \frac{\left| f''(z) \right|}{f'(z)} \leq \frac{\left| \alpha - 1 \right|}{\Re \alpha} \sum_{j=1}^{n} \left( \left| \frac{z^2 (D^m g_j(z))'}{(D^m g_j(z))^2} - 1 \right| M_j + M_j + 1 \right)
\]

\[
\leq \frac{\left| \alpha - 1 \right|}{\Re \alpha} \sum_{j=1}^{n} (2M_j + 1) \leq 1
\]

since \( \left| \alpha - 1 \right| \leq \frac{\Re \alpha}{\sum_{j=1}^{n} (2M_j + 1)} \). Applying Theorem A, we obtain that \( G_{n,m,\alpha} \) is in the univalent function class \( S \).

**Corollary 1** Let \( M \geq 1 \), each of the functions \( g_j \in A \) \((j \in \{1, \ldots, n\})\) satisfies the inequality (11) and \( \alpha \in \mathbb{C} \) with

\[
|\alpha - 1| \leq \frac{\Re \alpha}{(2M + 1)n}, \quad \Re \left( n (\alpha - 1) + 1 \right) \geq \Re \alpha > 0.
\]

If

\[
|D^m g_j(z)| \leq M \quad (z \in \mathbb{U}; \ j \in \{1, \ldots, n\}),
\]

then the integral operator \( G_{n,m,\alpha} \) defined by (9) is in the univalent function class \( S \).

**Proof.** In Theorem 1, we consider \( M_1 = \cdots = M_n = M \).

**Corollary 2** Let each of the functions \( g_j \in A \) \((j \in \{1, \ldots, n\})\) satisfies the inequality (11) and \( \alpha \in \mathbb{C} \) with

\[
|\alpha - 1| \leq \frac{\Re \alpha}{3n}, \quad \Re \left( n (\alpha - 1) + 1 \right) \geq \Re \alpha > 0.
\]

If

\[
|D^m g_j(z)| \leq 1 \quad (z \in \mathbb{U}; \ j \in \{1, \ldots, n\}),
\]

then the integral operator \( G_{n,m,\alpha} \) defined by (9) is in the univalent function class \( S \).
Proof. In Corollary 1, we consider $M = 1$.

Remark 5 If we set $m = 0$ in Corollary 2, then we have Theorem D.

Corollary 3 Let the function $g \in A$ satisfies the inequality (11) and $\alpha \in \mathbb{C}$ with

$$|\alpha - 1| \leq \frac{\text{Re} \alpha}{3}, \text{Re} \alpha > 0.$$ 

If

$$|D^m g(z)| \leq 1 \quad (z \in \mathbb{U}),$$

then the integral operator $G_{m, \alpha}$ defined by (10) is in the univalent function class $S$.

Proof. In Corollary 2, we consider $n = 1$.

Remark 6 If we set $m = 0$ in Corollary 3, then we have Theorem C.

References


Serap Bulut
Kocaeli University
Civi Aviation College
Arslanbey Campus
41285 İzmit-Kocaeli, Turkey
E-mail: serap.bulut@kocaeli.edu.tr