Characterizations of weak Cauchy $sn$-symmetric spaces

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Abstract

This paper proves that a space $X$ is a weak Cauchy $sn$-symmetric space iff it is a sequentially-quotient, $\pi$-image of a metric space, which answers a question posed by Z. Li.

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1 Introduction

$sn$-symmetric spaces is an important generalization of symmetric spaces. Recently, Y. Ge and S. Lin [10] investigate $sn$-symmetric spaces and obtained some interesting results. However, how characterize $sn$-symmetric spaces as images of metric spaces? This question is still open. As is well known, each

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weak Cauchy symmetric space can be characterized as a quotient, \(\pi\)-image of a metric space [11]. By viewing this result, Z. Li posed the following question [12, Question 3.2].

**Question 1** How characterize weak Cauchy sn-symmetric spaces by means of certain \(\pi\)-images of metric spaces?

In this paper, we prove that a space \(X\) is a weak Cauchy sn-symmetric space iff it is a sequentially-quotient, \(\pi\)-image of a metric space, which answers Question 1 affirmatively.

Throughout this paper, all spaces are assumed to be Hausdorff, and all mappings are continuous and onto. \(\mathbb{N}\) denotes the set of all natural numbers.

Let \(P\) be a subset of a space \(X\) and \(\{x_n\}\) be a sequence in \(X\) converging to \(x\). \(\{x_n\}\) is eventually in \(P\) if \(\{x_n : n > k\} \cup \{x\} \subset P\) for some \(k \in \mathbb{N}\); it is frequently in \(P\) if \(\{x_{n_k}\}\) is eventually in \(P\) for some subsequence \(\{x_{n_k}\}\) of \(\{x_n\}\).

Let \(\mathcal{P}\) be a family of subsets of a space \(X\) and \(x \in X\). \(\bigcup \mathcal{P}\) and \(\bigcap \mathcal{P}\) denote the union \(\bigcup \{P : P \in \mathcal{P}\}\) and the intersection \(\bigcap \{P : P \in \mathcal{P}\}\), respectively. \((\mathcal{P})_x = \{P \in \mathcal{P} : x \in P\}\) and \(st(x, \mathcal{P}) = \bigcup (\mathcal{P})_x\). A sequence \(\{P_n : n \in \mathbb{N}\}\) of subsets of a space \(X\) is abbreviated to \(\{P_n\}\). A point \(b = (\beta_n)_{n \in \mathbb{N}}\) of a Tychonoff-product space is abbreviated to \((\beta_n)\).

### 2 Definitions and Remarks

**Definition 1** ([4]) Let \(X\) be a space and \(x \in X\). \(P\) is called a sequential neighborhood of \(x\), if each sequence \(\{x_n\}\) converging to \(x\) is eventually in \(P\).

**Remark 1** ([5]) \(P\) is a sequential neighborhood of \(x\) iff each sequence \(\{x_n\}\) converging to \(x\) is frequently in \(P\).
Definition 2 ([6]) Let $\mathcal{P}$ be a family of subsets of a space $X$ and $x \in X$. $\mathcal{P}$ is called a network at $x$ in $X$, if $x \in \bigcap \mathcal{P}$ and for each neighborhood $U$ of $x$, there exists $P \in \mathcal{P}$ such that $P \subset U$. Moreover, $\mathcal{P}$ is called an sn-network at $x$ in $X$ if in addition each element of $\mathcal{P}$ is also a sequential neighborhood of $x$.

Definition 3 Let $X$ be a set. A non-negative real valued function $d$ defined on $X \times X$ is called a $d$-function on $X$ if $d(x, x) = 0$ and $d(x, y) = d(y, x)$ for any $x, y \in X$.

Let $d$ be a $d$-function on a space $X$. For $x \in X$ and $n \in \mathbb{N}$, put $S_n(x) = \{y \in X : d(x, y) < 1/n\}$.

Definition 4 ([10]) $(X, d)$ is called an sn-symmetric space and $d$ is called an sn-symmetric on $X$, if $\{S_n(x) : n \in \mathbb{N}\}$ is an sn-network at $x$ in $X$ for each $x \in X$.

For subsets $A$ and $B$ of an sn-symmetric space $(X, d)$, we write $d(A) = \sup\{d(x, y) : x, y \in A\}$ and $d(A, B) = \inf\{d(x, y) : x \in A$ and $y \in B\}$.

Definition 5 ([11]) Let $(X, d)$ be an sn-symmetric space.

1. A sequence $\{x_n\}$ in $X$ is called $d$-Cauchy if for each $\varepsilon > 0$, there exists $k \in \mathbb{N}$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > k$.

2. $(X, d)$ is called satisfying weak Cauchy condition if each convergent sequence has a $d$-Cauchy subsequence.

3. An sn-symmetric space satisfying weak Cauchy condition is called a weak Cauchy sn-symmetric space.

Remark 2 ([13]) $(X, d)$ satisfies weak Cauchy condition iff for each convergent sequence $L$ in $X$ and for each $\varepsilon > 0$, there exists a subsequence $L'$ of $L$ such that $d(L') < \varepsilon$. 


Definition 6 ([8]) Let $P$ be a cover of a space $X$. $P$ is called a cs$^*$-cover if for each convergent sequence $L$, there exists $P \in P$ such that $L$ is frequently in $P$.

Definition 7 ([14]) Let $\{P_n\}$ be a sequence of covers of a space $X$ such that $P_{n+1}$ refines $P_n$ for each $n \in \mathbb{N}$. $P = \bigcup\{P_n : n \in \mathbb{N}\}$ is called a $\sigma$-strong network of $X$, if $\{st(x, P_n) : n \in \mathbb{N}\}$ is a network at $x$ in $X$ for each $x \in X$. Moreover, if in addition $P_n$ is also a cs$^*$-cover of $X$ for each $n \in \mathbb{N}$, then $P$ is called a $\sigma$-strong network consisting of cs$^*$-covers.

Definition 8 ([7]). Let $f : X \rightarrow Y$ be a mapping. $f$ is called a sequentially-quotient mapping if for each convergent sequence $S$ in $Y$, there exists a convergent sequence $L$ in $X$ such that $f(L)$ is a subsequence of $S$.

Remark 3 Sequentially-quotient mappings are namely presequential mappings in the sense of J. R. Boone (see [2, 3, 9]).

Definition 9 ([10]) Let $(X, d)$ be an sn-symmetric and let $f : X \rightarrow Y$ be a mapping. $f$ is called a $\pi$-mapping, if for each $y \in Y$ and each neighborhood $U$ of $y$ in $Y$, $d(f^{-1}(y), X - f^{-1}(U)) > 0$.

3 The Main Results

Lemma 1 Let $(X, d)$ be an sn-symmetric space, $n \in \mathbb{N}$ and $x \in X$. Put $P_n = \{P \subset X : d(P) < 1/n\}$, then $st(x, P_n) = S_n(x)$.

Proof. If $y \in st(x, P_n)$, then there exists $P \in P_n$ such that $x, y \in P$. So $d(x, y) \leq d(P) < 1/n$, and hence $y \in S_n(x)$. On the other hand, if $y \in S_n(x)$, then $d(x, y) < 1/n$. So $\{x, y\} \in P_n$, thus $y \in st(x, P_n)$. Consequently, $st(x, P_n) = S_n(x)$.
Lemma 2 Let $\mathcal{P} = \bigcup \{ \mathcal{P}_n : n \in \mathbb{N} \}$ be a $\sigma$-strong network of $X$ and $x \in X$. If $P_n \in (\mathcal{P}_n)_x$ for each $n \in \mathbb{N}$, then $\{P_n\}$ is a network at $x$ in $X$.

Proof. Let $x \in U$ with $U$ open in $X$. Since $\mathcal{P}$ is a $\sigma$-strong network of $X$, there exists $m \in \mathbb{N}$ such that $st(x, P_m) \subset U$. Note that $P_m \subset st(x, P_m)$, so $x \in P_m \subset U$. This proves that $\{P_n\}$ is a network at $x$ in $X$.

Lemma 3 Let $\{\mathcal{P}_n\}$ be a sequence of $cs^*$-covers of a space $X$, and $S$ be a sequence in $X$ converging to $x$. Then there is a subsequence $S'$ of $S$ such that for each $n \in \mathbb{N}$, $S'$ is eventually in $P_n$ for some $P_n \in \mathcal{P}_n$.

Proof. Since $\mathcal{P}_1$ is a $cs^*$-cover of $X$ and $S$ is a convergent sequence in $X$, there is a subsequence $S_1$ of $S$ such that $S_1 \cup \{x\} \subset P_1$ for some $P_1 \in \mathcal{P}_1$. Put $x_1$ is the first term of $S_1$. Similarly, $\mathcal{P}_2$ is a $cs^*$-cover of $X$ and $S_1$ is a convergent sequence in $X$, there is a subsequence $S_2$ of $S_1$ such that $S_2 \cup \{x\} \subset P_2$ for some $P_2 \in \mathcal{P}_2$. Put $x_2$ is the second term of $S_2$. Assume that $x_1, x_2, \cdots, x_{n-1}$, $S_1, S_2, \cdots, S_{n-1}$, and $P_1, P_2, \cdots, P_{n-1}$ have been constructed as the above method. we construct $x_n$, $S_n$ and $P_n$ as follows. Since $\mathcal{P}_n$ is a $cs^*$-cover of $X$ and $S_{n-1}$ is a convergent sequence in $X$, there is a subsequence $S_n$ of $S_{n-1}$ such that $S_n \cup \{x\} \subset P_n$ for some $P_n \in \mathcal{P}_n$. Put $x_n$ is the $n$-th term of $S_n$. By the inductive method, we construct $x_n$, $S_n$ and $P_n$ for each $n \in \mathbb{N}$. Put $S' = \{x_n\}$, then $S'$ is a subsequence of $S$. For each $n \in \mathbb{N}$, $\{x_k, x\} \in S_k \subset S_n \subset P_n$ for all $k > n$, so $S'$ is eventually in $P_n$.

Now we give the main theorem in this paper.

Theorem 1 The following are equivalent for a space $X$.

(1) $X$ is a weak Cauchy $sn$-symmetric space.

(2) $X$ has a $\sigma$-strong network consisting of $cs^*$-covers.

(3) $X$ is a sequentially-quotient, $\pi$-image of a metric space.
Proof. (1) $\implies$ (2): Let $(X, d)$ be a weak Cauchy $sn$-symmetric space. For each $n \in \mathbb{N}$, put $P_n = \{ P \subset X : d(P) < 1/n \}$. By Lemma 1, $st(x, P_n) = S_n(x)$ for each $x \in X$ and each $n \in \mathbb{N}$. $\{ st(x, P_n) : n \in \mathbb{N} \}$ is a network at $x$ in $X$ for each $x \in X$ because $\{ S_n(x) : n \in \mathbb{N} \}$ is a network at $x$ in $X$. It is clear that $P_{n+1} \subset P_n$, so $P_{n+1}$ refines $P_n$. Thus $\{ P_n \}$ is a $\sigma$-strong network of $X$.

Let $n \in \mathbb{N}$ and $L = \{ x_k \}$ be a sequence in $X$ converging to $x$. It suffices to prove that $L$ is frequently in $P$ for some $P \in P_n$. Without loss of generality, we may assume that $d(x, x_k) < 1/n$ for each $k \in \mathbb{N}$. Since $(X, d)$ satisfying weak Cauchy condition, by Remake 2.7, there exists a subsequence $L'$ of $L$ such that $d(L') < 1/n$. Put $P = L' \cup \{ x \}$, then $d(P) < 1/n$, and hence $L$ is frequently in $P \in P_n$.

(2) $\implies$ (3): Let $X$ have a $\sigma$-strong network $P = \bigcup \{ P_n : n \in \mathbb{N} \}$ consisting of $cs^*$-covers. For each $n \in \mathbb{N}$, put $P_n = \{ P_\beta : \beta \in \Lambda_n \}$, and $\Lambda_n$ is endowed with discrete topology. Put

$$M = \{ b = (\beta_n) \in \prod_{n \in \mathbb{N}} \Lambda_n : \{ P_{\beta_n} \} \text{ is a network at some } x_b \text{ in } X \}.$$ 

Claim 1. $M$ is a metric space:

In fact, $\Lambda_n$, as a discrete space, is a metric space for each $n \in \mathbb{N}$. So $M$, which is a subspace of the Tychonoff-product space $\prod_{n \in \mathbb{N}} \Lambda_n$, is a metric space.

The metric $d$ on $M$ can be described as follows. Let $b = (\beta_n), c = (\gamma_n) \in M$. If $b = c$, then $d(b, c) = 0$. If $b \neq c$, then $d(b, c) = 1/\min \{ n \in \mathbb{N} : \beta_n \neq \gamma_n \}$.

Claim 2. Let $b = (\beta_n) \in M$. Then there exists unique $x_b \in X$ such that $\{ P_{\beta_n} \}$ is a network at $x_b$ in $X$:

The existence comes from the construction of $M$, we only need to prove the uniqueness. Let $\{ P_{\beta_n} \}$ be a network at both $x_b$ and $x'_b$ in $X$, then $\{ x_b, x'_b \} \subset P_{\beta_n}$ for each $n \in \mathbb{N}$. If $x_b \neq x'_b$, then there exists an open neighborhood $U$
of $x_b$ such that $x'_b \notin U$. Because $\{P_{\beta_n}\}$ is a network at $x_b$ in $X$, there exists $n \in \mathbb{N}$ such that $x_b \in P_{\beta_n} \subset U$, thus $x'_b \notin P_{\beta_n}$, a contradiction. This proves the uniqueness.

We define $f : M \rightarrow X$ as follows: for each $b = (\beta_n) \in M$, put $f(b) = x_b$, where $\{P_{\beta_n}\}$ is a network at $x_b$ in $X$. By Claim 2, $f$ is definable.

Claim 3. $f$ is onto:

Let $x \in X$. For each $n \in \mathbb{N}$, there exists $\beta_n \in \Lambda_n$ such that $P_{\beta_n} \in (\mathcal{P}_n)_x$ because $\mathcal{P}$ is a cover of $X$. Since $\mathcal{P}$ is a $\sigma$-strong network of $X$, $\{P_{\beta_n}\}$ is a network at $x$ in $X$ by Lemma 2. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$. This proves that $f$ is onto.

Claim 3. $f$ is continuous:

Let $b = (\beta_n) \in M$ and let $f(b) = x$. If $U$ is an open neighborhood of $x$, then there exists $k \in \mathbb{N}$ such that $x \in P_{\beta_k} \subset U$ because $\{P_{\beta_n}\}$ is a network at $x$ in $X$. Put $V = ((\prod \{\Lambda_n : n < k\}) \times \{\beta_k\} \times (\prod \{\Lambda_n : n > k\})) \cap M$, then $V$ is an open neighborhood of $b$. Let $c = (\gamma_n) \in V$, then $\{P_{\gamma_n}\}$ is a network at $f(c)$ in $X$, so $f(c) \in P_{\gamma_n}$ for each $n \in \mathbb{N}$. Note that $\gamma_k = \beta_k$, $f(c) \in P_{\gamma_k} = P_{\beta_k}$. This proves that $f(V) \subset P_{\beta_k}$, and hence $f(V) \subset U$. So $f$ is continuous.

Claim 4. $f$ is a $\pi$-mapping.

Let $x \in U$ with $U$ open in $X$. Since $\mathcal{P}_n$ is a $\sigma$-strong network of $X$, there exists $n \in \mathbb{N}$ such that $st(x, \mathcal{P}_n) \subset U$. It suffices to prove that $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$. Let $b = (\beta_n) \in M$. If $d(f^{-1}(x), b) < 1/2n$, then there is $c = (\gamma_n) \in f^{-1}(x)$ such that $d(b, c) < 1/n$, so $\beta_k = \gamma_k$ if $k \leq n$. Notice that $x = f(c) \in P_{\gamma_n} \subset \mathcal{P}_n$ and $f(b) \in P_{\beta_n} = P_{\gamma_n}$, so $f(b) \in st(x, \mathcal{P}_n) \subset U$, thus $b \in f^{-1}(U)$. This proves that $d(f^{-1}(x), b) \geq 1/2n$ if $b \in M - f^{-1}(U)$, so $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/2n > 0$.

Claim 5. $f$ is a sequentially-quotient mapping.
Let $S$ be a sequence in $X$ converging to $x \in X$. By Lemma 3, there exists a subsequence $S' = \{x_k\}$ of $S$ such that for each $n \in \mathbb{N}$, $S'$ is eventually in $P_{\beta_n}$ for some $\beta_n \in \Lambda_n$. Note that $x \in P_{\beta_n}$ for each $n \in \mathbb{N}$. Put $b = (\beta_n)$, then $b \in M$ and $f(b) = x$ by Lemma 2. For each $k \in \mathbb{N}$, we pick $b_k \in f^{-1}(x_k)$ as follows. For each $n \in \mathbb{N}$, if $x_k \in P_{\beta_n}$, put $\beta_{k_n} = \beta_n$; if $x_k \notin P_{\beta_n}$, pick $\beta_{k_n} \in \Lambda_n$ such that $x_k \in P_{\beta_{k_n}}$. Put $b_k = (\beta_{k_n}) \in \prod_{n \in \mathbb{N}} \Lambda_n$, then $b_k \in M$ and $f(b_k) = x_k$ by Lemma 2. Put $L = \{b_k\}$, then $L$ is a sequence in $M$ and $f(L) = S'$. It suffices to prove that $L$ converges to $b$. Let $b \in U$, where $U$ is an element of base of $M$. By the definition of Tychonoff-product spaces, we may assume $U = ((\prod \{\beta_n : n \leq m\}) \times (\prod \{\Lambda_n : n > m\})) \cap M$, where $m \in \mathbb{N}$. For each $n \leq m$, $S'$ is eventually in $P_{\beta_n}$, so there is $k(n) \in N$ such that $x_k \in P_{\beta_n}$ for all $k > k(n)$, thus $\beta_{k_n} = \beta_n$. Put $k_0 = \max\{k(1), k(2), \ldots, k(m), m\}$, then $b_k \in U$ for all $k > k_0$, so $L$ converges to $b$.

By the above Claims, $X$ is a sequentially-quotient, $\pi$-image of a metric space.

(3) $\implies$ (1): Let $f$ be a sequentially-quotient, $\pi$-mapping from a metric space $(M, d)$ onto $X$. Put $d'(x, y) = d(f^{-1}(x), f^{-1}(y))$ for each $x, y \in X$. It is clear that $d'$ is a $d$-function on $X$. For $b \in M$, $x \in X$ and $n \in \mathbb{N}$, put $S_n(b) = \{c \in M : d(b, c) < 1/n\}$ and $S_n'(x) = \{y \in X : d'(x, y) < 1/n\}$.

Claim 1. $\{S_n'(x) : n \in \mathbb{N}\}$ is a network at $x$ in $X$ for each $x \in X$:

Let $U$ be an open neighborhood of $x$ in $X$. Since $f$ is a $\pi$-mapping, there exists $n \in \mathbb{N}$ such that $d(f^{-1}(x), M - f^{-1}(U)) \geq 1/n$. If $y \notin U$, then $f^{-1}(y) \subset M - f^{-1}(U)$, hence $d'(x, y) = d(f^{-1}(x), f^{-1}(y)) \geq d(f^{-1}(x), M - f^{-1}(U)) \geq 1/n$, so $y \notin S_n'(x)$. This proves that $S_n'(x) \subset U$.

Claim 2. Let $x \in X$ and $n \in \mathbb{N}$. Then $S_n'(x)$ is a sequential neighborhood of $x$:
Let \( \{x_m\} \) be a sequence converging to \( x \). By Remark 1, it suffices to prove that \( \{x_m\} \) is frequently in \( S'_n(x) \). Since \( f \) is sequentially-quotient, there exists a sequence \( \{b_k\} \) converging to \( b \in f^{-1}(x) \) such that each \( f(b_k) = x_{m_k} \). Pick \( k_0 \in \mathbb{N} \) such that \( d(b, b_k) < 1/n \) for all \( k \geq k_0 \). So \( d'(x, x_{m_k}) = d(f^{-1}(x), f^{-1}(x_{m_k})) \leq d(b, b_k) < 1/n \) for all \( k \geq k_0 \), and hence \( x_{m_k} \in S'_n(x) \) for all \( k \geq k_0 \). Thus \( \{x_{m_k}\} \) is eventually in \( S'_n(x) \), that is, \( \{x_m\} \) is frequently in \( S'_n(x) \).

Claim 3. \((X, d')\) satisfies weak Cauchy condition:

Let \( \{x_n\} \) be a convergent sequence in \( X \). Since \( f \) is sequentially-quotient, there exists a convergent sequence \( L = \{b_k\} \) in \( M \) such that \( f(b_k) = x_{n_k} \) for each \( k \in \mathbb{N} \). It suffices to prove that \( x_{n_k} \) is a \( d \)-Cauchy subsequence. Let \( \varepsilon > 0 \). Note that each convergent sequence in metric space \((M, d)\) is a \( d \)-Cauchy sequence. So there exists \( k_0 \in \mathbb{N} \) such that \( d(b_i, b_j) < \varepsilon \) for all \( i, j > k_0 \). Thus \( d'(x_{n_i}, x_{n_j}) = d(f^{-1}(x_{n_i}), f^{-1}(x_{n_j})) \leq d(b_i, b_j) < \varepsilon \) for all \( i, j > k_0 \). This proves that \( x_{n_k} \) is a \( d \)-Cauchy subsequence.

By the above Claims, \( d' \) is an \( sn \)-symmetric on \( X \) and \((X, d')\) satisfies weak Cauchy condition. So \( X \) is a weak Cauchy \( sn \)-symmetric space.

Remark 4 “\( \sigma \)-strong network” in Theorem 1 can be replaced by “point-star network”, where the concept of “point-star networks” is obtained by omitting “\( \mathcal{P}_{n+1} \) refines \( \mathcal{P}_n \) for each \( n \in \mathbb{N} \)” in the Definition 7 [13].

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References


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