A generalization of some classical quadrature formulas

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Abstract

In this paper is obtained a quadrature formula with higher degree of exactness. This formula is a generalization of some classical quadrature rules.

2000 Mathematics Subject Classification: 41A55, 65D30, 65D32

1 Introduction

Let \([\alpha, \beta]\) be an interval on the real axis.

For a positive integer \(n\), a function \(F \in C^n[\alpha, \beta]\) and a point \(\delta \in [\alpha, \beta]\) let denote by \(\Theta_n(F; \delta)(x)\) the corresponding Taylor polynomial, i.e.

\[
\Theta_n(F; \delta)(x) = \sum_{i=0}^{n} \frac{F^{(i)}(\delta)}{i!}(x - \delta)^i.
\]

Lemma 1. (see [1]) If \(F \in C^{n+1}[\alpha, \beta]\) and \(\delta \in [\alpha, \beta]\) then

\[
F(x) - \Theta_n(F; \delta)(x) = \frac{F^{(n+1)}(\xi)}{(n+1)!}(x - \delta)^{n+1}, \ x \in [\alpha, \beta],
\]

where \(\xi\) is a point between \(x\) and \(\delta\).

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Received 16 November, 2009
Accepted for publication (in revised form) 17 December, 2009
In the sequel we consider a parameter $\gamma \in (0, 1]$ and we define the points 
\[ \delta_{1,\gamma} = \frac{\alpha + \beta}{2} - \gamma \frac{\beta - \alpha}{2}, \delta_2 = \frac{\alpha + \beta}{2}, \delta_{3,\gamma} = \frac{\alpha + \beta}{2} + \gamma \frac{\beta - \alpha}{2}. \]

Further we assume that $p$ is a given even integer. Using a remark from the paper [2] and an idea suggested by Al. Lupaş in the paper [3] we consider the following quadrature formula which depend on parameters $\gamma \in (0, 1]$, $\rho \in \mathbb{R}$:

\begin{equation}
\int_{\alpha}^{\beta} F(x)dx = \Lambda_p(F; \gamma, \rho) + \Omega_p(F; \gamma, \rho), \ F \in C^{p+1}[\alpha, \beta],
\end{equation}

where

\begin{equation}
\Lambda_p(F; \gamma, \rho) = \\
= \rho \int_{\alpha}^{\beta} \Theta_{p+1}(F; \delta_2)(x)dx + \frac{1}{2}(1 - \rho) \int_{\alpha}^{\beta} (\Theta_p(F; \delta_{1,\gamma})(x) + \Theta_p(F; \delta_{3,\gamma})(x))dx
\end{equation}

and $\Omega_p(F; \gamma, \rho)$ being the remainder term.

## 2 Main result

**Theorem 1.** The remainder term in the quadrature formula (1) has the following representation

\begin{equation}
\Omega_p(F; \gamma, \rho) = \\
= \rho \int_{\alpha}^{\beta} (F(x) - \Theta_{p+1}(F; \delta_2)(x))dx + \frac{1}{2}(1 - \rho) \int_{\alpha}^{\beta} (F(x) - \Theta_p(F; \delta_{1,\gamma})(x))dx + \\
+ \frac{1}{2}(1 - \rho) \int_{\alpha}^{\beta} (F(x) - \Theta_p(F; \delta_{3,\gamma})(x))dx, \ F \in C^{p+1}[\alpha, \beta].
\end{equation}

The proof is obtained directly from the relations (1), (2).

Further we consider the functions $\varepsilon_k(x) = x^k, k \in \mathbb{N}$. 

Lemma 2. \( \Omega_p(\varepsilon_k; \gamma, \rho) = 0 \) for any \( k \in \{0, ..., p + 1\} \), \( \gamma \in (0, 1] \) and \( \rho \in \mathbb{R} \).

**Proof.** For \( k \in \{0, ..., p\} \) using Lemma 1 we have
\[
\varepsilon_k(x) - \Theta_p(\varepsilon_k; \delta_2)(x) = 0, \quad \varepsilon_k(x) - \Theta_p(\varepsilon_k; \delta_{1, \gamma})(x) = 0, \quad \varepsilon_k(x) - \Theta_p(\varepsilon_k; \delta_{3, \gamma})(x) = 0,
\]
where \( x \in [\alpha, \beta] \).
Therefore from (3) it follows that \( \Omega_p(\varepsilon_k; \gamma, \rho) = 0 \) for any \( k \in \{0, ..., p\} \), \( \gamma \in (0, 1] \) and \( \rho \in \mathbb{R} \).

If \( k = p + 1 \) using Lemma 1 we deduce
\[
\varepsilon_{p+1}(x) - \Theta_p(\varepsilon_{p+1}; \delta_2)(x) = 0, \quad \varepsilon_{p+1}(x) - \Theta_p(\varepsilon_{p+1}; \delta_{1, \gamma})(x) = (x - \delta_{1, \gamma})^{p+1}, \quad \varepsilon_{p+1}(x) - \Theta_p(\varepsilon_{p+1}; \delta_{3, \gamma})(x) = (x - \delta_{3, \gamma})^{p+1},
\]
where \( x \in [\alpha, \beta] \), and substituting in (3) we obtain \( \Omega_p(\varepsilon_{p+1}; \gamma, \rho) = 0 \) for any \( \gamma \in (0, 1] \) and \( \rho \in \mathbb{R} \).

Lemma 3. \( \Omega_p(\varepsilon_k; \gamma, \overline{p}_p(\gamma)) = 0 \) for any \( k \in \{p + 2, p + 3\} \) and \( \gamma \in (0, 1] \), where
\[
\overline{p}_p(\gamma) = \frac{\gamma(p + 3)((1 + \gamma)^{p+2} - (1 - \gamma)^{p+2}) - (1 + \gamma)^{p+3} - (1 - \gamma)^{p+3}}{\gamma(p + 3)((1 + \gamma)^{p+2} - (1 - \gamma)^{p+2}) - (1 + \gamma)^{p+3} - (1 - \gamma)^{p+3} + 2}.
\]

The proof is similar with Lemma 2, taking into account Lemma 1 and Theorem 1.

**Theorem 2.** The degree of exactness in the quadrature formula (1) for \( \rho := \overline{p}_p(\gamma) \) is at least \( p + 3 \) for any \( \gamma \in (0, 1] \), i.e. \( \Omega_p(Q; \gamma, \overline{p}_p(\gamma)) = 0 \), for all \( Q : [\alpha, \beta] \to \mathbb{R} \) polynomials of degree at least \( p + 3 \).

The proof follows directly from Lemma 2 and Lemma 3.

**Theorem 3.** The quadrature formula (1) for \( \rho := \overline{p}_p(\gamma) \) has the following representation
\[
\int_{\alpha}^{\beta} F(x)dx = \sum_{i=0}^{p} \sigma_{p,i}(\gamma, \overline{p}_p(\gamma)) F^{(i)}(\delta_{1, \gamma}) + \sum_{i=0}^{p/2} \tau_{p,2i}(\gamma, \overline{p}_p(\gamma)) F^{(2i)}(\delta_2) + \sum_{i=0}^{p} \varphi_{p,i}(\gamma, \overline{p}_p(\gamma)) F^{(i)}(\delta_{3, \gamma}) + \Omega_p(F; \gamma, \overline{p}_p(\gamma)), \quad F \in C^{p+1}[\alpha, \beta],
\]
where the coefficients \((\sigma_{p,i}(\gamma, \overline{p}_p(\gamma)))_{i \in \{0, \ldots, p\}}, (\tau_{p,2i}(\gamma, \overline{p}_p(\gamma)))_{i \in \{0, \ldots, p/2\}}, (\varphi_{p,i}(\gamma, \overline{p}_p(\gamma)))_{i \in \{0, \ldots, p\}} \in \mathbb{R}\) are given by

\[
\sigma_{p,i}(\gamma, \overline{p}_p(\gamma)) = \frac{1}{2(i + 1)!}(1 - \overline{p}_p(\gamma))(1 + \gamma)^{i+1} + (-1)^i(1 - \gamma)^{i+1}\left(\frac{\beta - \alpha}{2}\right)^{i+1}, \ i \in \{0, \ldots, p\},
\]

\[
\tau_{p,2i}(\gamma, \overline{p}_p(\gamma)) = \frac{2}{(2i + 1)!}\overline{p}_p(\gamma)\left(\frac{\beta - \alpha}{2}\right)^{2i+1}, \ i \in \{0, \ldots, p/2\},
\]

\[
\varphi_{p,i}(\gamma, \overline{p}_p(\gamma)) = \frac{(-1)^i}{2(i + 1)!}(1 - \overline{p}_p(\gamma))(1 + \gamma)^{i+1} + (-1)^i(1 - \gamma)^{i+1}\left(\frac{\beta - \alpha}{2}\right)^{i+1} = (-1)^i\sigma_{p,i}(\gamma, \overline{p}_p(\gamma)), \ i \in \{0, \ldots, p\}.
\]

The proof is obtained from the relations (1), (2) considering \(\rho := \overline{p}_p(\gamma)\).

In the sequel we consider the particular case \(p = 0\) and for some values of the parameter \(\gamma \in (0, 1]\) we obtain some classical quadrature rules.

For \(\gamma = 1\) one obtains the Simpson quadrature formula

\[
\int_{\alpha}^{\beta} F(x)dx = \frac{\beta - \alpha}{6} \left(F(\alpha) + 4F\left(\frac{\alpha + \beta}{2}\right) + F(\beta)\right) - \frac{(\beta - \alpha)^5}{2880}F^{(4)}(\xi), \ \xi \in [\alpha, \beta].
\]

For \(\gamma = \frac{2}{3}\) one obtains the Maclaurin quadrature formula

\[
\int_{\alpha}^{\beta} F(x)dx = \frac{3(\beta - \alpha)}{8} \left(F\left(\frac{5\alpha + \beta}{6}\right) + \frac{2}{3}F\left(\frac{\alpha + \beta}{2}\right) + F\left(\frac{\alpha + 5\beta}{6}\right)\right) + \ldots
\]
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\[ + \frac{7(\beta - \alpha)^5}{51840} F^{(4)}(\xi), \; \xi \in [\alpha, \beta]. \]

For \( \gamma = \frac{1}{2} \) one obtains the "open" Newton-Cotes quadrature formula

\[
\int_{a}^{b} F(x)dx = \frac{2(\beta - \alpha)}{3} \left( F\left(\frac{3\alpha + \beta}{4}\right) - \frac{1}{2} F\left(\frac{\alpha + \beta}{2}\right) + F\left(\frac{\alpha + 3\beta}{4}\right) \right) +
\]

\[ + \frac{7(\beta - \alpha)^5}{23040} F^{(4)}(\xi), \; \xi \in [\alpha, \beta]. \]

References


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