On some integral operators on analytic functions

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Abstract

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1 Introduction and Preliminaries

Let $\mathcal{H}(U)$ be the set of functions which are regular in the unit disc $U$, $A = \{ f \in \mathcal{H}(U) : f(0) = f'(0) - 1 = 0 \}$, $\mathcal{H}_u(U) = \{ f \in \mathcal{H}(U) : f \text{ is univalent in } U \}$ and $S = \{ f \in A : f \text{ is univalent in } U \}$. 

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We denote with $T$ the subset of the functions $f \in S$, which have the form
\begin{equation}
    f(z) = z - \sum_{j=2}^{\infty} a_j z^j, \ a_j \geq 0, \ j \geq 2, \ z \in U
\end{equation}
and with $T^* = T \cap S^*$, $T^*(\alpha) = T \cap S^*(\alpha)$, $T_c = T \cap S_c$ and $T_c(\alpha) = T \cap S_c(\alpha)$, where $0 \leq \alpha < 1$.

**Theorem 1.** [5] For a function $f$ having the form (1) the following assertions are equivalents:

(i) $\sum_{j=2}^{\infty} j a_j \leq 1$;

(ii) $f \in T$;

(iii) $f \in T^*$.

Regarding the classes $T^*(\alpha)$ and $T_c(\alpha)$ with $0 \leq \alpha < 1$, we recall here the following result:

**Theorem 2.** [5] A function $f$ having the form (1) is in the class $T^*(\alpha)$ if and only if:
\begin{equation}
    \sum_{j=2}^{\infty} \frac{j - \alpha}{1 - \alpha} a_j \leq 1,
\end{equation}
and is in the class $T_c(\alpha)$ if and only if:
\begin{equation}
    \sum_{j=2}^{\infty} \frac{j(j - \alpha)}{1 - \alpha} a_j \leq 1.
\end{equation}

**Definition 1.** [1] Let $S^*(\alpha, \beta)$ denote the class of functions having the form (1) which are starlike and satisfy
\begin{equation}
    \left| \frac{zf'(z)}{f(z)} - 1 \right| < \beta
\end{equation}
for $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. And let $C^*(\alpha, \beta)$ denote the class of functions such that $zf'(z)$ is in the class $S^*(\alpha, \beta)$.

**Theorem 3.**[1] A function $f$ having the form (1) is in the class $S^*(\alpha, \beta)$ if and only if:

$$\sum_{j=2}^{\infty} \left\{(j - 1) + \beta (j + 1 - 2\alpha)\right\} a_j \leq 2\beta(1 - \alpha),$$

and is in the class $C^*(\alpha, \beta)$ if and only if:

$$\sum_{j=2}^{\infty} j \left\{(j - 1) + \beta (j + 1 - 2\alpha)\right\} a_j \leq 2\beta(1 - \alpha).$$

Let $D^n$ be the Sălăgean differential operator (see [2]) defined as:

$$D^n : A \to A, \quad n \in \mathbb{N} \quad \text{and} \quad D^0 f(z) = f(z)$$

$$D^1 f(z) = Df(z) = zf'(z), \quad D^n f(z) = D(D^{n-1}f(z)).$$

In [3] the author define the class $T_n(\alpha, \beta)$, from which by choosing different values for the parameters we obtain variously subclasses of analytic functions with negative coefficients (for example $T_n(\alpha, 1) = T_n(\alpha)$ which is the class of $n$-starlike of order $\alpha$ functions with negative coefficients and $T_0(\alpha, \beta) = S^*(\alpha, \beta) \cap T$, where $S^*(\alpha, \beta)$ is the class defined by (4)).

**Definition 2.**[3] Let $\alpha \in [0, 1)$, $\beta \in (0, 1]$ and $n \in \mathbb{N}$. We define the class $S_n(\alpha, \beta)$ of the $n$-starlike of order $\alpha$ and type $\beta$ through

$$S_n(\alpha, \beta) = \{f \in A; \ |J(f, n, \alpha; z)| < \beta\}$$
where \( J(f, n, \alpha; z) = \frac{D^{n+1}f(z) - D^nf(z)}{D^{n+1}f(z) + (1 - 2\alpha)D^nf(z)} \), \( z \in U \). Consequently\( T_n(\alpha, \beta) = S_n(\alpha, \beta) \cap T \).

**Theorem 4.** [3] Let \( f \) be a function having the form (1). Then \( f \in T_n(\alpha, \beta) \) if and only if

\[
\sum_{j=2}^{\infty} j^n [j - 1 + \beta(j + 1 - 2\alpha)] a_j \leq 2\beta(1 - \alpha).
\]

**2 Main results**

From [4] we have the following definitions:

Let \( f(z) \in T, f(z) = z - \sum_{n=2}^{\infty} a_n z^n \), satisfies \( V_\mu(f)(z) = \int_0^1 \mu(t) \frac{f(tz)}{t} dt \), where \( \mu \) is a real-valued, non-negative weight function normalized so that \( \int_0^1 \mu(t) dt = 1 \).

If \( \mu(t) = \frac{(c + 1)^{\delta}}{\mu(\delta)} t^c \left( \log \frac{1}{t} \right)^{\delta-1} \quad (c > -1; \delta > 0) \), which gives the Komatu operator. Then we have

\[
V_\mu(f)(z) = z - \sum_{n=2}^{\infty} \left( \frac{c + 1}{c + n} \right)^\delta a_n z^n.
\]

**Remark 1.** We notice that \( 0 < \left( \frac{c + 1}{c + n} \right)^\delta < 1 \), where \( c > -1, \delta > 0 \) and \( j \geq 2 \).

**Remark 2.** It is easy to prove, by using Theorem 1 and Remark 1, that for \( F(z) \in T \) and \( f(z) = V_\mu(F)(z) \), we have \( f(z) \in T \), where \( V_\mu \) is the integral operator defined by (8).
Theorem 5. Let \( F(z) \) be in the class \( T^*(\alpha), \alpha \in [0,1), F(z) = z - \sum_{j=2}^{\infty} a_j z^j \), \( a_j \geq 0, j \geq 2 \). Then \( f(z) = V_\mu(F)(z) \in T^*(\alpha) \), where \( V_\mu \) is the integral operator defined by (8).

Proof. From Remark 2 we obtain \( f(z) = V_\mu(F)(z) \in T \).

We have \( f(z) = z - \sum_{j=2}^{\infty} b_j z^j \), where \( b_j = \left( \frac{c+1}{c+j} \right)^j a_j z^j \). By using Remark 1 we obtain

\[
\frac{j - \alpha}{1 - \alpha} b_j < \frac{j - \alpha}{1 - \alpha} a_j, \text{ for } j = 2, 3, \ldots, 0 \leq \alpha < 1,
\]

and thus \( \sum_{j=2}^{\infty} \frac{j - \alpha}{1 - \alpha} b_j \leq \sum_{j=2}^{\infty} \frac{j - \alpha}{1 - \alpha} a_j \leq 1 \). This mean (see Theorem 2) that \( f(z) = V_\mu(F)(z) \in T^*(\alpha) \).

Similarly (by using Remark 2 and the Theorems 2, 3 and 4) we obtain:

Theorem 6. Let \( F(z) \) be in the class \( T^c(\alpha), \alpha \in [0,1), F(z) = z - \sum_{j=2}^{\infty} a_j z^j \), \( a_j \geq 0, j \geq 2 \). Then \( f(z) = V_\mu(F)(z) \in T^c(\alpha) \), where \( V_\mu \) is the integral operator defined by (8).

Theorem 7. Let \( F(z) \) be in the class \( C^*(\alpha, \beta), \alpha \in [0,1), \beta \in (0,1], F(z) = z - \sum_{j=2}^{\infty} a_j z^j \), \( a_j \geq 0, j \geq 2 \). Then \( f(z) = V_\mu(F)(z) \in C^*(\alpha, \beta) \), where \( V_\mu \) is the integral operator defined by (8).

Theorem 8. Let \( F(z) \) be in the class \( T_n(\alpha, \beta), \alpha \in [0,1), \beta \in (0,1], n \in \mathbb{N}, F(z) = z - \sum_{j=2}^{\infty} a_j z^j \), \( a_j \geq 0, j \geq 2 \). Then \( f(z) = V_\mu(F)(z) \in T_n(\alpha, \beta) \), where \( V_\mu \) is the integral operator defined by (8).

Remark 3. By choosing \( \beta = 1 \), respectively \( n = 0 \), in the above theorem, we obtain the similarly results for the classes \( T_n(\alpha) \) and \( S^*(\alpha, \beta) \).
References


