Some Results On Janowski Starlike Log-harmonic Mappings Of Complex Order $b$

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Abstract

Let $H(\mathbb{D})$ be a linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z| |z| < 1\}$. A sense-preserving log-harmonic function is the solution of the non-linear elliptic partial differential equation

$$\overline{f_z} = w \overline{f} f_z,$$

where $w(z)$ is analytic, satisfies the condition $|w(z)| < 1$ for every $z \in \mathbb{D}$ and is called the second dilatation of $f$. It has been shown that if $f$ is a non-vanishing log-harmonic mapping then $f$ can be represented by

$$f(z) = h(z)g(z),$$

where $h(z)$ and $g(z)$ are analytic in $\mathbb{D}$ with $h(0) \neq 0$, $g(0) = 1([1])$. If $f$ vanishes at $z = 0$ but it is not identically zero, then $f$ admits the representation

$$f(z) = z |z|^{2\beta} \overline{h(z)g(z)},$$
where $\Re \beta > -\frac{1}{2}$, $h(z)$ and $g(z)$ are analytic in $D$ with $g(0) = 1$ and $h(0) \neq 0$. The class of sense-preserving log-harmonic mappings is denoted by $S_{LH}$. We say that $f$ is a Janowski starlike log-harmonic mapping. If

$$1 + \frac{1}{b} \left( z f' - \frac{zf'}{f} - 1 \right) = \frac{1 + A \phi(z)}{1 + B \phi(z)}$$

where $\phi(z)$ is Schwarz function. The class of Janowski starlike log-harmonic mappings is denoted by $S^*_{LH}(A, B, b)$. We also note that, if $(zh(z))$ is a starlike function, then the Janowski starlike log-harmonic mappings will be called a perturbated Janowski starlike log-harmonic mappings. And the family of such mappings will be denoted by $S^*_{PLH}(A, B, b)$. The aim of this paper is to give some distortion theorems of the class $S^*_{LH}(A, B, b)$.

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1 Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are regular in $D$ and satisfying the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$.

Next, denote by $\mathcal{P}(A, B)$ the family of functions

$$p(z) = 1 + p_1 z + p_2 z^2 + ...$$

regular in $\mathbb{D}$, such that $p(z)$ is in $\mathcal{P}(A, B)$ if and only if

$$(1) \quad p(z) = \frac{1 + A \phi(z)}{1 + B \phi(z)}, \quad -1 \leq B < A \leq 1$$
Some Results On Janowski Starlike Log-harmonic Mappings ...

for some function $\phi(z) \in \Omega$ and for every $z \in \mathbb{D}$. Therefore we have $p(0) = 1$, $\text{Rep}(z) > \frac{1-A}{1-B} > 0$ whenever $p(z) \in \mathcal{P}(A, B)$. Moreover, let $\mathcal{S}^*(A, B)$ denote the family of functions

$$s(z) = z + a_2z^2 + ...$$

regular in $\mathbb{D}$, and such that $s(z)$ is in $\mathcal{S}^*$ if and only if

$$Re \left( z \frac{s'(z)}{s(z)} \right) = p(z) = \frac{1 + \phi(z)}{1 - \phi(z)}, p(z) \in \mathcal{P}(1, -1)$$

Let $S_1(z)$ and $S_2(z)$ be analytic functions in $\mathbb{D}$ with $S_1(0) = S_2(0)$. We say that $S_1(z)$ subordinated to $S_2(z)$ and denote by $S_1(z) \prec S_2(z)$, if $S_1(z) = S_2(\phi(z))$ for some function $\phi(z) \in \Omega$ and every $z \in \mathbb{D}$. If $S_1(z) \prec S_2(z)$, then $S_1(\mathbb{D}) \subset S_2(\mathbb{D})$.[5]

The radius of starlikeness of the class of sense-preserving log-harmonic mapping is

$$r_s = \sup \left\{ r | \text{Re} \left( \frac{zf_z - zf_{\overline{z}}}{f} \right) > 0, 0 < r < 1 \right\}.$$

Finally, let $H(D)$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$. A sense-preserving log-harmonic mapping is the solution of the non-linear elliptic partial differential equation

$$F_z = w(z) \frac{f_z}{f},$$

where $w(z) \in H(\mathbb{D})$ is the second dilatation of $f$ such that $|w(z)| < 1$ for every $z \in \mathbb{D}$. It has been shown that if $f$ is a non-vanishing log-harmonic mapping, then $f$ can be expressed as

$$f = h(z)g(z)$$
where \( h(z) \) and \( g(z) \) are analytic functions in \( \mathbb{D} \).

On the other hand, if \( f \) vanishes at \( z = 0 \) and at no other point, then \( f \) admits the representation,

\[
(5) \quad f = z |z|^{2\beta} h(z) g(z),
\]

where \( \Re\beta > -1/2 \), \( h(z) \) and \( g(z) \) are analytic in \( \mathbb{D} \) with \( g(0) = 1 \) and \( h(0) \neq 0 \). We note that the class of log-harmonic mappings is denoted by \( \mathcal{S}_{LH} \).

Let \( f = zh(z)g(z) \) be an element of \( \mathcal{S}_{LH} \). We say that \( f \) is a Janowski starlike log-harmonic mapping if

\[
(6) \quad 1 + \frac{1}{b} \left( \frac{zf_z - \overline{z}f_z}{f} - 1 \right) = p(z) = \frac{1 + A\phi(z)}{1 + B\phi(z)}, p(z) \in \mathcal{P}(A, B)
\]

where \(-1 \leq B < A \leq 1\), \( b \neq 0 \) and complex and denote by \( \mathcal{S}_{LH}^*(A, B, b) \) the set of all starlike log-harmonic mappings. Also we denote \( \mathcal{S}_{PLH}^*(A, B, b) \) the class of all functions in \( \mathcal{S}_{LH}^*(A, B, b) \) for which \((zh(z)) \in \mathcal{S}^*(A, B)\) for all \( z \in \mathbb{D} \).

We note that if we give special values to \( b \), then we obtain important subclasses of Janowski starlike log-harmonic mappings

i. For \( b = 0 \), we obtain the class of starlike log-harmonic mappings.

ii. For \( b = 1 - \alpha \), \( 0 \leq \alpha < 1 \), we obtain the class of starlike log-harmonic mappings of order \( \alpha \).

iii. For \( b = e^{-i\lambda} \cos \lambda, |\lambda| < \frac{\pi}{2} \), we obtain the class of \( \lambda \)-spirallike log-harmonic mappings.
iv. For $b = (1 - \alpha)e^{-i\lambda}\cos\lambda$, $0 \leq \alpha < 1$, $|\lambda| < \frac{\pi}{2}$, we obtain the class of $\lambda-$ spirallike log-harmonic mappings of order $\alpha$.

2 Main results

Theorem 1 Let $f = zh(z)g(z)$ be an element of $S^*_\text{PLH}(A, B, b)$. Then

\begin{equation}
 f = zh(z)g(z) \in S^*_\text{LH}(A, B, b) \text{ iff } \begin{cases} 
 z\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)} < \frac{b(A-B)z}{1+Bz}; & B \neq 0, \\
 z\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)} < bAz; & B = 0.
\end{cases}
\end{equation}

Proof. Let $f \in S^*_\text{LH}(A, B, b)$. Using the principle of subordination then we have

\[ 1 + \frac{1}{b} \left( \frac{zfz - zf_\infty}{f} - 1 \right) = 1 + \frac{1}{b} \left( \frac{z\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)}}{1 + A\phi(z)} \right) = \begin{cases} 
 \frac{1 + A\phi(z)}{1 + B\phi(z)}; & B \neq 0, \\
 1 + A\phi(z); & B = 0,
\end{cases} \]

iff $z\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)} = \begin{cases} 
 b(A-B)\phi(z); & B \neq 0, \\
 bA\phi(z); & B = 0,
\end{cases}$

iff $z\frac{h'(z)}{h(z)} - \frac{g'(z)}{g(z)} < \begin{cases} 
 \frac{b(A-B)z}{1+Bz}; & B \neq 0, \\
 bAz; & B = 0.
\end{cases}$

Theorem 2 Let $f = zh(z)g(z)$ be an element of $S^*_\text{PLH}(A, B, b)$. Then

\[ \begin{cases} 
 G(A, B, -r) \leq \left| \frac{h(z)}{g(z)} \right| \leq G(A, B, r); & B \neq 0, \\
 G_1(A, -r) \leq \left| \frac{h(z)}{g(z)} \right| \leq G_1(A, r); & B = 0,
\end{cases} \]
where

\[
G(A, B, r) = \frac{(1+B)r}{(1-B)(|z| + Br)}; \quad B \neq 0,
\]

\[
G_1(A, r) = \frac{(1+r)A}{A/r}; \quad B = 0.
\]

**Proof.** The function \((\frac{1+A}{1+B^2z})\) maps \(|z| = r\) on to circle with the centre \(C(r) = (\frac{1-ABr^2}{1-Br^2}, 0)\) and the raadius \(p(r) = \frac{(A-B)r}{1-Br^2}\). Therefore using the definition of subordination and Theorem 1, we get

\[
\left| \left(1 + \frac{1}{b} \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - \frac{1-ABr^2}{1-Br^2} \right) \right| \leq \frac{(A-B)r}{1-Br^2}; \quad B \neq 0,
\]

\[
\left| \left(1 + \frac{1}{b} \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) - 0 \right) \right| \leq Ar; \quad B = 0.
\]

The inequality (9) takes the form,

\[
\left\{ \begin{array}{l}
\left(\frac{B(A-B)Rebz^2-|b|(A-B)r}{1+B^2z^2} \right) \leq Re \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) \leq \frac{b(A-B)r}{1-Br^2} \\
-\frac{A|b|r}{1-r^2} \leq Re \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) \leq \frac{A|b|r}{1-r^2} \end{array} \right; \quad B \neq 0,
\]

on the other hand we have,

\[
Re \left( z \frac{h'(z)}{h(z)} - z \frac{g'(z)}{g(z)} \right) = r \frac{\partial}{\partial r} \left( \log |h(z)| - \log |g(z)| \right)
\]

Thus the inequality (10) can be written in the form,

\[
\left\{ \begin{array}{l}
\left| \frac{B(A-B)Rebz^2-|b|(A-B)r}{(1-Br)(1+Br)} \right| \leq \frac{\partial}{\partial r} \log |h(z)| - g(z) \leq \frac{B(A-B)Rebz^2+|b|(A-B)r}{(1-Br)(1+Br)} ; \quad B \neq 0, \\
-\frac{A|b|}{(1-r)(1+r)} \leq \frac{\partial}{\partial r} \log |h(z)| - g(z) \leq \frac{A|b|}{(1-r)(1+r)} ; \quad B = 0,
\end{array} \right.
\]

integrating both sides of (11) from 0 to r we get (8).
Corollary 1 The radius of starlikeness of the class $S_{PLH}^*$ is

\begin{equation}
 r_s = \begin{cases} 
 \frac{2}{(A-B)|b|+\sqrt{(A-B)^2|b|^2+4|B|^2+(AB-B^2)\operatorname{Re} b}}; & B \neq 0, \\
 \frac{1}{|b|A}; & B = 0.
\end{cases}
\end{equation}

Proof. The inequality (9) can be written in the form

\[
\left| \frac{zf_z - \overline{zf_z}}{f} - \frac{1-(B^2+(AB-B^2)\operatorname{Re} b)r^2-i((AB-B^2)\operatorname{Im} b)r^2)}{1-B^2r^2} \right| \leq \frac{|b|(A-B)r}{1-B^2r^2}, \quad B \neq 0,
\]

\[
\left| \frac{zf_z - \overline{zf_z}}{f} - 1 \right| \leq |b|Ar; \quad B = 0.
\]

Therefore we have

\[
\operatorname{Re} \left( \frac{zf_z - \overline{zf_z}}{f} \right) \geq \begin{cases} 
 \frac{1-(A-B)|b|-(B^2+(AB-B^2)\operatorname{Re} b)r^2}{1-B^2r^2}; & B \neq 0, \\
 1 - |b|Ar; & B = 0,
\end{cases}
\]

which gives (12).

Lemma 1 Let $f = z|z|^{2\beta} h(z)g(z) \in S_{LH}$ and let $w(z)$ be the second dilatation of $f$. Then

\begin{equation}
 \frac{||\beta| - |\beta + 1|\, r}{||\beta + 1| - |\beta|\, r} \leq |w(z)| \leq \frac{||\beta| + |\beta + 1|\, r}{||\beta + 1| + |\beta|\, r}.
\end{equation}

This inequality is sharp because the extremal function is

\[
w(z) = e^{i\beta} \frac{e^{i\ell z} - \frac{\beta}{\beta+1}}{1 - \beta+1} e^{i\ell z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}.
\]

Proof. $f = z|z|^{2\beta} h(z)g(z) \in S_{LH}$ and let $1^{2\beta} = 1$. Then $f$ is the solution of the nonlinear elliptic partial differential equation

\[
w(z) = \frac{f_z}{f} \cdot \frac{f}{f_z}
\]
\[ f_z = \left( \frac{1}{z} + \frac{\beta}{z} + \frac{h'(z)}{h(z)} \right) f, \]

\[ f_{\overline{z}} = \left( \frac{\overline{\beta}}{z} + \frac{g'(z)}{g(z)} \right) \overline{f} \]

\[ w(z) = \frac{f_{\overline{z}}}{f} f_z = \frac{\overline{\beta} + z \frac{g'(z)}{g(z)}}{(\beta + 1) + z \frac{h'(z)}{h(z)}}, \quad w(0) = \frac{\overline{\beta}}{\beta + 1}, \quad |w(0)| < 1. \]

On the other hand for \( \text{Re} \beta > -\frac{1}{2} \), we have \( \left| \frac{\overline{\beta}}{\beta + 1} \right| < 1 \). Therefore we can take \( w(0) = c_0 = \left| \frac{\overline{\beta}}{\beta + 1} \right| e^{i\theta}, \quad \theta \in \mathbb{R}. \)

Now consider the function

\[ \phi(z) = \frac{e^{-i\theta} w(z) - \left| \frac{\overline{\beta}}{\beta + 1} \right|}{1 - \left| \frac{\overline{\beta}}{\beta + 1} \right| e^{i\theta} w(z)}, \quad z \in \mathbb{D}, \]

which satisfies the conditions Schwarz lemma and use the estimate \( |\phi(z)| \leq |z| < r \), to get

\[ \left| e^{-i\theta} w(z) - \left| \frac{\overline{\beta}}{\beta + 1} \right| \right| \leq r \left| \frac{\overline{\beta}}{\beta + 1} \right| e^{-i\theta} w(z) - 1 \right|. \]

This is equivalent to

\[ w(z) - \left| \frac{\overline{\beta}}{\beta + 1} \right| \frac{(1 - r^2)}{1 - \left| \frac{\overline{\beta}}{\beta + 1} \right|^2 r^2} \leq r \left( 1 - \left| \frac{\overline{\beta}}{\beta + 1} \right|^2 \right) \]

\[ \left( 1 - \left| \frac{\overline{\beta}}{\beta + 1} \right|^2 r^2 \right) \]
and the equality holds only for a function of the form

\[ w(z) = e^{i\theta} z - \frac{\beta^2}{\beta + 1} e^{i\ell z}, \quad z \in \mathbb{D}, \ell \in \mathbb{R}. \]

From the inequality (14)) we have then

\[
|w(z)| = |e^{-i\theta} w(z)| \begin{aligned}
&\geq \left| \frac{\beta^2}{\beta + 1} \left( 1 - r^2 \right) - r \left( 1 - \frac{\beta^2}{\beta + 1} \right)^2 \right| = \left| \frac{\beta^2}{\beta + 1} - r \right|,
&\geq \left| \frac{\beta^2}{\beta + 1} - r \right|.
\end{aligned}
\]

\[
|w(z)| = |e^{-i\theta} w(z)| \begin{aligned}
&\leq \left| \frac{\beta^2}{\beta + 1} \left( 1 - r^2 \right) - r \left( 1 - \frac{\beta^2}{\beta + 1} \right)^2 \right| = \left| \frac{\beta^2}{\beta + 1} + r \right|.
&\leq \left| \frac{\beta^2}{\beta + 1} + r \right|.
\end{aligned}
\]

Lemma 2 \( \phi(z) \in S^*(A, B) \) ise

\[
\begin{cases}
\frac{1-Ar}{r(1-Br)} \leq \left| \frac{\phi'(z)}{\phi(z)} \right| \leq \frac{1+Ar}{r(1+Br)}, & B \neq 0; \\
\frac{1-Ar}{r} \leq \left| \frac{\phi'(z)}{\phi(z)} \right| \leq \frac{1+Ar}{r}, & B = 0;
\end{cases}
\]

we get the result.

Proof. \( \phi(z) = z.h(z) \in S^*(A, B) \);

\[
\begin{cases}
z \frac{\phi'(z)}{\phi(z)} - \frac{1-ABr^2}{1-Br^2} \leq \frac{(A-B)r}{1-Br^2}, & B \neq 0; \\
z \frac{\phi'(z)}{\phi(z)} - 1 \leq Ar, & B = 0;
\end{cases}
\]

these inequalities can be written. Then we have,

\[
\begin{cases}
\frac{1-Ar}{1-Br} \leq \left| z \frac{\phi'(z)}{\phi(z)} \right| \leq \frac{1+Ar}{1+Br}, & B \neq 0; \\
1 - Ar \leq \left| z \frac{\phi'(z)}{\phi(z)} \right| \leq 1 + Ar, & B = 0;
\end{cases}
\]

And if we divide by \(|z| = r\) each of these ;we obtain the result easily.
Lemma 3 \( f(z) = zh(z)g(z) = \phi(z)g(z) \in S^*_\text{LH}(A, B) \) ve \( \phi(z) = zh(z) \in S^*(A, B) \)

\[
\begin{cases}
-\frac{1-Ar}{1-Br} < \left| \frac{g'(z)}{g(z)} \right| < \frac{1+Ar}{1+Br}, & B \neq 0; \\
-(1-Ar) < \left| \frac{g'(z)}{g(z)} \right| < 1+Ar, & B = 0;
\end{cases}
\]

we get the result.

Proof. \( f(z) = \phi(z)g(z) \) and we use the second dilatation function’s elliptic differential solution,

\[
w(z) = \frac{g'(z)}{g(z)} \frac{\phi'(z)}{\phi(z)}
\]

we hold this result. Therefore, \( w(z) \) function is analytic at \( \mathbb{D} \) disc; \( |w(z)| < 1 \) (sense-preserving) and \( w(0) = 0 \); because of Schwarz Lemma,

\[-r < |w(z)| < r\]

We can write these inequalities. Then we have,

\[-r < \left| \frac{g'(z)}{g(z)} \frac{\phi'(z)}{\phi(z)} \right| < r\]

And if we use Lemma (11), we can take the result easily.

Theorem 3 Let \( f = zh(z)g(z) \) be an element of \( S^*_\text{PLH}(A, B, b) \) then

\[
F(A, B, -r) \leq |g(z)| \leq F(A, B, r),
\]

\[
F_1(A, -r) \leq |g(z)| \leq F_1(A, r),
\]
where
\[ F(A, B, r) = \frac{1}{(1 + Br)^{\frac{B-A}{A}}} \cdot \frac{(1 + Br)^{\frac{B-A}{B}}}{(1 - Br)^{\frac{B-A}{B}}} \]

\[ F_1(A, r) = e^{Ar} \cdot \frac{(1 - r)^{\frac{|-b|}{2}}}{(1 + r)^{\frac{|-b|}{2}}} \]

Proof. \( f \in S_{lh}^{*}(A, B) \) then we have \( h(z) \) function satisfies, \( h(0) = 1 \) and has a Taylor formula \( h(z) = 1 + \sum_{n=1}^{\infty} a_n z^n \) We know that, \( \phi(z) = zh(z) \in S^{*}(A, B) \) from starlikeness radius formula,

\[ (19) \]
\[ \Re \left( \frac{\phi'(z)}{\phi(z)} \right) = \Re \left( \frac{(zh(z))'}{zh(z)} \right) = \Re \left( \frac{(zh(z))'}{h(z)} \right) = \Re(1 + \frac{h'(z)}{h(z)}) > 0 \]

satisfied. And also for Janowski Starlike logharmonic mappings,

\[ (20) \]
\[ \left| \frac{(zh(z))'}{h(z)} - \frac{1}{1-B^2r^2} \right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0; \]
\[ \left| \frac{(zh(z))'}{h(z)} - 1 \right| \leq Ar, \quad B = 0; \]

Then we have,

\[ (21) \]
\[ \left| 1 + \frac{zh'(z)}{h(z)} - \frac{1-ABr^2}{1-B^2r^2} \right| \leq \frac{(A-B)r}{1-B^2r^2}, \quad B \neq 0; \]
\[ \left| 1 + \frac{zh'(z)}{h(z)} - 1 \right| \leq Ar, \quad B = 0; \]

we can write these inequalities.

\[ -|z| \leq Rez \leq |z| \]
if we use this inequality:

$$\begin{align*}
\frac{1}{(1-Br)^{\frac{B}{B-A}}} & \leq |h(z)| \leq \frac{1}{(1+Br)^{\frac{B}{B-A}}}, \quad B \neq 0; \\
\exp(-Ar) & \leq |h(z)| \leq \exp(Ar), \quad B = 0;
\end{align*}$$

(22)

Then we have the result.

**Corollary 2** Let $f = zh(z)\overline{g(z)} \in S_{LH}^*(A, B, b)$ and let $(zh(z)) \in S^*(A, B)$. Then

$$\begin{align*}
-(\frac{1-Ar}{1-Br})F(A, B, -r) & < |g'(z)| < (\frac{1+Ar}{1+Br})F(A, B, r), \quad B \neq 0; \\
-(1 - Ar)F_1(A, -r) & < |g'(z)| < (1 + Ar)F_1(A, r), \quad B = 0;
\end{align*}$$

(23)

**Proof.** Follows immediately from Lemma (12) and Theorem 3.

**Corollary 3** Let $f = zh(z)\overline{g(z)} \in S_{LH}^*(A, B, b)$. Then

$$\begin{align*}
\frac{1}{(1-Br)^{\frac{B}{B-A}}} \left[ \frac{1-(A-B)r-ABr^2}{1-B^2r^2} \right] & \leq |h(z) + zh'(z)| \\
& \leq \frac{1}{(1+Br)^{\frac{B}{B-A}}} \left[ \frac{1+(A-B)r-ABr^2}{1-B^2r^2} \right], \quad B \neq 0; \\
e^{-Ar}(1 - Ar) & \leq |h(z) + zh'(z)| \leq e^{Ar}(1 + Ar), \quad B = 0;
\end{align*}$$

(24)

**Proof.** This result is a simple consequence of Lemma (12).

**Corollary 4** Let $f = zh(z)\overline{g(z)} \in S_{LH}^*(A, B, b)$. Then

$$\begin{align*}
\frac{r}{(1-Br)^{\frac{B}{B-A}}}F(A, B, -r) & \leq |f| \leq \frac{r}{(1+Br)^{\frac{B}{B-A}}}F(A, B, r), \quad B \neq 0; \\
r.e^{-Ar}.F_1(A, -r) & \leq |f| \leq r.e^{Ar}.F_1(A, r), \quad B = 0;
\end{align*}$$

(25)

**Proof.** This result is a simple consequence of Theorem 3.
References


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