Hankel determinant for $p$-valently starlike and convex functions of order $\alpha$

Toshio Hayami, Shigeyoshi Owa

Abstract
For $p$-valently starlike and convex functions $f(z)$ in the open unit disk $\mathbb{U}$, the upper bounds of the functional $|a_{p+2} - \mu a_{p+1}^2|$, defined by using the second Hankel determinant $H_2(n)$ due to J. W. Noonan and D. K. Thomas (Trans. Amer. Math. Soc. 223(2) (1976), 337-346), are discussed.

2000 Mathematics Subject Classification: Primary 30C45
Key words and phrases: Hankel determinant, $p$-valently starlike function, $p$-valently convex function.

1 Introduction

Let $A_p$ denote the class of functions $f(z)$ of the form

$$f(z) = z^p + \sum_{n=p+1}^{\infty} a_n z^n \quad (p \in \mathbb{N} = \{1, 2, 3, \cdots \})$$
which are analytic in the open unit disk $U = \{z \in \mathbb{C} : |z| < 1\}$.

Furthermore, let $\mathcal{P}$ denote the class of functions $p(z)$ of the form

$$p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k$$

which are analytic in $U$ and satisfy

$$\text{Re } p(z) > 0 \quad (z \in U).$$

Then we say that $p(z) \in \mathcal{P}$ is the Carathéodory function (cf. [1]).

If $f(z) \in \mathcal{A}_p$ satisfies the following condition

$$\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha$ ($0 \leq \alpha < p$), then $f(z)$ is said to be $p$-valently starlike of order $\alpha$ in $U$. We denote by $\mathcal{S}^*_p(\alpha)$ the subclass of $\mathcal{A}_p$ consisting of functions $f(z)$ which are $p$-valently starlike of order $\alpha$ in $U$. Similarly, we say that $f(z)$ belongs to the class $\mathcal{K}_p(\alpha)$ of $p$-valently convex functions of order $\alpha$ in $U$ if $f(z) \in \mathcal{A}_p$ satisfies the following inequality

$$\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (z \in U)$$

for some $\alpha$ ($0 \leq \alpha < p$).

As usual, in the present investigation, we write

$$\mathcal{S}^*_p = \mathcal{S}^*_p(0), \quad \mathcal{K}_p = \mathcal{K}_p(0), \quad \mathcal{S}^*(\alpha) = \mathcal{S}^*_1(\alpha) \quad \text{and} \quad \mathcal{K}(\alpha) = \mathcal{K}_1(\alpha).$$
Remark 1. For a function \( f(z) \in A_p \), it follows that
\[
f(z) \in K_p(\alpha) \text{ if and only if } \frac{zf'(z)}{p} \in S_p^*(\alpha)
\]
and
\[
f(z) \in S_p^*(\alpha) \text{ if and only if } \int_0^z \frac{pf(\zeta)}{\zeta} d\zeta \in K_p(\alpha).
\]

Example 1.
\[
f(z) = \frac{z^p}{(1 - z)^{2(p - \alpha)}} \in S_p^*(\alpha)
\]
and
\[
f(z) = z^p F_1(2(p - \alpha), p; p + 1; z) \in K_p(\alpha)
\]
where \( F_1(a, b; c; z) \) represents the hypergeometric function.

In [7], Noonan and Thomas stated the \( q \)-th Hankel determinant as
\[
H_q(n) = \det \begin{pmatrix}
a_n & a_{n+1} & \cdots & a_{n+q-1} \\
a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{pmatrix} \quad (n, q \in \mathbb{N} = \{1, 2, 3, \ldots \}).
\]
This determinant is discussed by several authors. For example, we can know that the Fekete and Szegö functional \( |a_3 - a_2^2| = |H_2(1)| \) and they consider the further generalized functional \( |a_3 - \mu a_2^2| \), where \( \mu \) is some real number (see, [2]). Moreover, we also know that the functional \( |a_2a_4 - a_3^2| \) is equivalent to \( |H_2(2)| \).

Janteng, Halim and Darus [4] have shown the following theorems.
Theorem 1. Let $f(z) \in S^\ast$. Then
\[ |a_2a_4 - a_3^2| \leq 1. \]

Equality is attained for functions
\[ f(z) = \frac{z}{(1 - z)^2} = z + 2z^2 + 3z^3 + 4z^4 + \cdots \]
and
\[ f(z) = \frac{z}{1 - z^2} = z + z^3 + z^5 + z^7 + \cdots. \]

Theorem 2. Let $f(z) \in K$. Then
\[ |a_2a_4 - a_3^2| \leq \frac{1}{8}. \]

The present paper is motivated by these results and the purpose of this investigation is to find the upper bounds of the generalized functional $|a_{p+2} - \mu a_{p+1}^2|$, defined by the second Hankel determinant, for functions $f(z)$ in the class $S_p^\ast(\alpha)$ and $K_p(\alpha)$, respectively.

2 Preliminary results

In order to discuss our problems, we need some lemmas. The following lemma can be found in [1] or [8].

Lemma 1. If a function $p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \in P$, then
\[ |c_k| \leq 2 \quad (k = 1, 2, 3, \cdots). \]
The result is sharp for
\[ p(z) = \frac{1 + z}{1 - z} = 1 + \sum_{k=1}^{\infty} 2z^k. \]

Using the above, we derive

**Lemma 2.** If a function \( p(z) = p + \sum_{k=1}^{\infty} c_k z^k \) satisfies the following inequality
\[ \text{Re} \ p(z) > \alpha \quad (z \in \mathbb{U}) \]
for some \( \alpha \) \((0 \leq \alpha < p)\), then
\[ |c_k| \leq 2(p - \alpha) \quad (k = 1, 2, 3, \ldots). \]

The result is sharp for
\[ p(z) = \frac{p + (p - 2\alpha)z}{1 - z} = p + \sum_{k=1}^{\infty} 2(p - \alpha)z^k. \]

**Proof.** Let \( q(z) = \frac{p(z) - \alpha}{p - \alpha} = 1 + \sum_{k=1}^{\infty} \frac{c_k}{p - \alpha} z^k \). Noting that \( q(z) \in \mathcal{P} \) and using Lemma 1, we see that
\[ \left| \frac{c_k}{p - \alpha} \right| \leq 2 \quad (k = 1, 2, 3, \ldots) \]
which implies
\[ |c_k| \leq 2(p - \alpha) \quad (k = 1, 2, 3, \ldots). \]
Lemma 3. The power series for \( p(z) = 1 + \sum_{k=1}^{\infty} c_k z^k \) converges in \( \mathbb{U} \) to a function in \( \mathcal{P} \) if and only if the Toeplitz determinants
\[
D_n = \begin{vmatrix}
  2 & c_1 & c_2 & \cdots & c_n \\
  c_1 & 2 & c_1 & \cdots & c_{n-1} \\
  c_2 & c_1 & 2 & \cdots & c_{n-2} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  c_{n-1} & c_{n} & c_{n+1} & \cdots & 2
\end{vmatrix} \quad (n = 1, 2, 3, \cdots),
\]
where \( c_{-k} = \overline{c_k} \), are all non-negative. They are strictly positive except for
\[
p(z) = \sum_{k=1}^{m} \rho_k p_0(e^{it_k}z), \quad \rho_k > 0, \ t_k \ \text{real and} \ t_k \neq t_j \ \text{for} \ k \neq j, \ \text{where} \ p_0(z) = \frac{1+z}{1-z}; \ \text{in this case} \ D_n > 0 \ \text{for} \ n < m - 1 \ \text{and} \ D_n = 0 \ \text{for} \ n \geq m.
\]

This necessary and sufficient condition is due to Carathéodory and Toeplitz, and it can be found in [3]. And then, Libera and Zlotkiewicz [5] (see, also [6]) have given the following result by using this lemma with \( n = 2, 3 \).

Lemma 4. If a function \( p(z) \in \mathcal{P} \), then the representations
\[
\begin{align*}
2c_2 &= c_1^2 + (4 - c_1^2)\zeta \\
4c_3 &= c_1^3 + 2(4 - c_1^2)c_1\zeta - (4 - c_1^2)c_1\zeta^2 + 2(4 - c_1^2)(1 - |\zeta|^2)\eta
\end{align*}
\]
for some complex numbers \( \zeta \) and \( \eta \ (|\zeta| \leq 1, |\eta| \leq 1) \), are obtained.

By virtue of Lemma 4, we have
Lemma 5. If a function \( p(z) = p + \sum_{k=1}^{\infty} c_k z^k \) satisfies \( \text{Re} p(z) > \alpha \) (\( z \in \mathbb{U} \)) for some \( \alpha \) (0 \( \leq \alpha < p \)), then
\[
2(p - \alpha)c_2 = c_1^2 + \{4(p - \alpha)^2 - c_1^2\}\zeta
\]
\[
4(p - \alpha)^2c_3 = c_1^3 + 2\{4(p - \alpha)^2 - c_1^2\}c_1\zeta - \{4(p - \alpha)^2 - c_1^2\}c_1\zeta^2
\]
\[
+ 2(p - \alpha)\{4(p - \alpha)^2 - c_1^2\}(1 - |\zeta|^2)\eta
\]
for some complex numbers \( \zeta \) and \( \eta \) (\( |\zeta| \leq 1, |\eta| \leq 1 \)).

Proof. Since \( q(z) = \frac{p(z) - \alpha}{p - \alpha} = 1 + \sum_{k=1}^{\infty} \frac{c_k}{p - \alpha} z^k \in \mathcal{P} \), replacing \( c_2 \) and \( c_3 \) by \( \frac{c_2}{p - \alpha} \) and \( \frac{c_3}{p - \alpha} \) in Lemma 4, respectively, we immediately have the relations of the lemma.

We also need the next remark.

Remark 2. If \( f(z) \in S_p^*(\alpha) \), then there exists a function \( p(z) = p + \sum_{k=1}^{\infty} c_k z^k \) such that \( \text{Re} p(z) > \alpha \) (\( z \in \mathbb{U} \)) and
\[
 zf'(z) = f(z)p(z)
\]
which implies that
\[
p + \sum_{n=p+1}^{\infty} n a_n z^{n-p} = p + \sum_{n=p+1}^{\infty} \left( \sum_{l=p}^{n} a_l c_{n-l} \right) z^{n-p}
\]
where \( a_p = 1 \) and \( c_0 = p \). Therefore, we have the following relation
\[
(n - p)a_n = \sum_{l=p}^{n-1} a_l c_{n-l} \quad (n \geq p + 1).
\]
3 Main results

In this section, we begin with the upper bound of $|a_{p+2} - \mu a_{p+1}^2|$ for $p$-valently starlike functions of order $\alpha$ below.

**Theorem 3.** If a function $f(z) \in S_p^*(\alpha)$ $(0 \leq \alpha < p)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} 
(p - \alpha) \{(2(p - \alpha) + 1) - 4(p - \alpha)\mu\} & \left( \mu \leq \frac{1}{2} \right) \\
p - \alpha & \left( \frac{1}{2} \leq \mu \leq \frac{p + 1 - \alpha}{2(p - \alpha)} \right) \\
(p - \alpha) \{4(p - \alpha)\mu - 2(p - \alpha) + 1\} & \left( \mu \geq \frac{p + 1 - \alpha}{2(p - \alpha)} \right) 
\end{cases}$$

with equality for

$$f(z) = \begin{cases} 
\frac{z^p}{(1 - z)^2(p - \alpha)} & \left( \mu \leq \frac{1}{2} \text{ or } \mu \geq \frac{p + 1 - \alpha}{2(p - \alpha)} \right) \\
\frac{z^p}{(1 - z^2)^{p-\alpha}} & \left( \frac{1}{2} \leq \mu \leq \frac{p + 1 - \alpha}{2(p - \alpha)} \right). 
\end{cases}$$

**Proof.** If $f(z) \in S_p^*(\alpha)$, then we have the equation (3) which means that $a_{p+1} = c_1$ and $a_{p+2} = \frac{c_2 + c_1^2}{2}$. Thus, by the inequality (1) and the representation (2), we can suppose that $c_1 = c$ $(0 \leq c \leq 2(p - \alpha))$ without
loss of generality and we derive

\[ |a_{p+2} - \mu a_{p+1}^2| = \left| \frac{c_2 + c^2}{2} - \mu c^2 \right| \]

\[ = \frac{1}{2} \left| (1 - 2\mu)c^2 + \frac{c^2 + \{4(p - \alpha)^2 - c^2\} \zeta}{2(p - \alpha)} \right| \]

\[ = \frac{1}{4(p - \alpha)} \left| \{2(p - \alpha) - 4(p - \alpha)\mu + 1\}c^2 + \{4(p - \alpha)^2 - c^2\} \zeta \right| \]

\[ \equiv A(\zeta). \]

Applying the triangle inequality, we deduce

\[ A(\zeta) \leq \frac{1}{4(p - \alpha)} \left[ |(2(p - \alpha) + 1) - 4(p - \alpha)\mu|c^2 + \{4(p - \alpha)^2 - c^2\} \right] \]

\[ = \left\{ \begin{array}{l}
\frac{1}{4(p - \alpha)} \left[ 2(p - \alpha)(1 - 2\mu)c^2 + 4(p - \alpha)^2 \right] \quad \left( \mu \leq \frac{2(p - \alpha) + 1}{4(p - \alpha)} \right) \\
\frac{1}{4(p - \alpha)} \left[ 2\{2(p - \alpha)\mu - (p + 1 - \alpha)\}c^2 + 4(p - \alpha)^2 \right] \quad \left( \mu \geq \frac{2(p - \alpha) + 1}{4(p - \alpha)} \right) \\
(p - \alpha) \left\{ (2(p - \alpha) + 1) - 4(p - \alpha)\mu \right\} \quad \left( \mu \leq \frac{1}{2}, c = 2(p - \alpha) \right) \\
\end{array} \right\} \\
\]

\[ \equiv \left\{ \begin{array}{l}
p - \alpha \quad \left( \frac{1}{2} \leq \mu \leq \frac{2(p - \alpha) + 1}{4(p - \alpha)}, c = 0 \right) \\
p - \alpha \quad \left( \frac{2(p - \alpha) + 1}{4(p - \alpha)} \leq \mu \leq \frac{p + 1 - \alpha}{2(p - \alpha)}, c = 0 \right) \\
(p - \alpha) \left\{ (4(p - \alpha)\mu - (2(p - \alpha) + 1) \right\} \quad \left( \mu \geq \frac{p + 1 - \alpha}{2(p - \alpha)}, c = 2(p - \alpha) \right). \\
\end{array} \right\} \]
Equality is attained for functions $f(z) \in S^*_p(\alpha)$ defined by

$$\frac{zf'(z)}{f(z)} = p(z) = \frac{p + (p - 2\alpha)z}{1 - z}$$

for the case $c_1 = c = 2(p - \alpha)$, $\zeta = 1$ and $c_2 = 2(p - \alpha)$, or

$$\frac{zf'(z)}{f(z)} = p(z) = \frac{p + (p - 2\alpha)z^2}{1 - z^2}$$

for the case $c_1 = c = 0$, $\zeta = 1$ and $c_2 = 2(p - \alpha)$.

Taking $\alpha = 0$ or $p = 1$ in Theorem 3, we obtain the following corollaries, respectively.

**Corollary 1.** If a function $f(z) \in S^*_p$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} 
  p \{2p + 1\} & (\mu \leq \frac{1}{2}) \\
  \mu \{2p + 1\} & (\mu = \frac{1}{2}) \\
  p \{2p + 1\} & (\mu \geq \frac{p + 1}{2p})
\end{cases}$$

with equality for

$$f(z) = \begin{cases} 
  \frac{z^p}{(1 - z)^{2p}} & (\mu \leq \frac{1}{2} \text{ or } \mu \geq \frac{p + 1}{2p}) \\
  \frac{z^p}{(1 - z^2)^{p}} & (\frac{1}{2} \leq \mu \leq \frac{p + 1}{2p})
\end{cases}.$$
Corollary 2. If a function \( f(z) \in S^*(\alpha) \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
(1 - \alpha) \{ (3 - 2\alpha) - 4(1 - \alpha)\mu \} & \left( \mu \leq \frac{1}{2} \right) \\
1 - \alpha & \left( \frac{1}{2} \leq \mu \leq \frac{2 - \alpha}{2(1 - \alpha)} \right) \\
(1 - \alpha) \{ 4(1 - \alpha)\mu - (3 - 2\alpha) \} & \left( \mu \geq \frac{2 - \alpha}{2(1 - \alpha)} \right)
\end{cases}
\]

with equality for

\[
f(z) = \begin{cases} 
z (1 - z)^2(1 - \alpha) & \left( \mu \leq \frac{1}{2} \text{ or } \mu \geq \frac{2 - \alpha}{2(1 - \alpha)} \right) \\
z & \left( \frac{1}{2} \leq \mu \leq \frac{2 - \alpha}{2(1 - \alpha)} \right)
\end{cases}
\]

Also, by Corollary 1 and Corollary 2, we readily know

Corollary 3. If a function \( f(z) \in S^* \), then

\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
3 - 4\mu & \left( \mu \leq \frac{1}{2} \right) \\
1 & \left( \frac{1}{2} \leq \mu \leq 1 \right) \\
4\mu - 3 & \left( \mu \geq 1 \right)
\end{cases}
\]

with equality for

\[
f(z) = \begin{cases} 
z & \left( \mu \leq \frac{1}{2} \text{ or } \mu \geq 1 \right) \\
z & \left( \frac{1}{2} \leq \mu \leq 1 \right)
\end{cases}
\]
Next, in consideration of Remark 1, we derive the upper bounds of $|a_{p+2} - \mu a_{p+1}^2|$ for $p$-valently convex functions.

**Theorem 4.** If a function $f(z) \in \mathcal{K}_p(\alpha)$ $(0 \leq \alpha < p)$, then

$$|a_{p+2} - \mu a_{p+1}^2| \leq$$

$$\left\{ \begin{array}{c}
\frac{p(p - \alpha)}{p + 2} \left( \frac{(p + 1)^2}{2p(p + 2)} \right) \leq \mu \leq \frac{(p + 1)^2}{2p(p + 2)(p - \alpha)} \\
\frac{p(p - \alpha)}{p + 2} \left( \frac{(p + 1)^2}{2p(p + 2)} \right) \leq \mu \leq \frac{(p + 1)^2(p + 1 - \alpha)}{2p(p + 2)(p - \alpha)} \\
\frac{p(p - \alpha)}{(p + 1)^2(p + 2)} \left( \mu \geq \frac{(p + 1)^2(p + 1 - \alpha)}{2p(p + 2)(p - \alpha)} \right)
\end{array} \right.$$  

with equality for

$$f(z) = \left\{ \begin{array}{c}
z^p F_1(2(p - \alpha), p; p + 1; z) \left( \mu \leq \frac{(p + 1)^2}{2p(p + 2)} \text{ or } \mu \geq \frac{(p + 1)^2(p + 1 - \alpha)}{2p(p + 2)(p - \alpha)} \right) \\
z^p F_1 \left( \frac{p}{2}, p - \alpha; 1 + \frac{p}{2}; z^2 \right) \left( \frac{(p + 1)^2}{2p(p + 2)} \leq \mu \leq \frac{(p + 1)^2(p + 1 - \alpha)}{2p(p + 2)(p - \alpha)} \right)
\end{array} \right.$$  

**Proof.** Noting that $f(z) \in \mathcal{K}_p(\alpha)$ if and only if
Hankel determinant for $p$-valently starlike and convex functions . . . 41

\[
\frac{zf'(z)}{p} = z^p + \sum_{n=p+1}^{\infty} \frac{n}{p} a_n z^n \in S_p^*(\alpha) \text{ and using Theorem 3, we see that }
\]

\[
\left| \frac{p+2}{p} a_{p+2} - \nu \frac{(p+1)^2}{p^2} a_{p+1}^2 \right| \leq \begin{cases} 
(p - \alpha) \{(2(p - \alpha) + 1) - 4(p - \alpha)\nu\} \\
p - \alpha \\
(p - \alpha) \{4(p - \alpha)\nu - (2(p - \alpha) + 1)\}, 
\end{cases}
\]

that is, that \[
\left| a_{p+2} - \frac{(p+1)^2}{p(p+2)} \nu a_{p+1}^2 \right| \leq \begin{cases} 
\frac{p(p - \alpha) \{(2(p - \alpha) + 1) - 4(p - \alpha)\nu\}}{p + 2} & \left( \nu \leq \frac{1}{2} \right) \\
\frac{p(p - \alpha)}{p + 2} & \left( \frac{1}{2} \leq \nu \leq \frac{p + 1 - \alpha}{2(p - \alpha)} \right) \\
\frac{p(p - \alpha) \{4(p - \alpha)\nu - (2(p - \alpha) + 1)\}}{p + 2} & \left( \nu \geq \frac{p + 1 - \alpha}{2(p - \alpha)} \right).
\end{cases}
\]

Now, putting \[
\frac{(p + 1)^2}{p(p + 2)} \nu = \mu,
\]
the proof of the theorem is completed.

When $\alpha = 0$ or $p = 1$ in Theorem 4, the following three corollaries are obtained.
Corollary 4. If a function $f(z) \in \mathcal{K}_p$, then
\[
|a_{p+2} - \mu a_{p+1}^2| \leq \begin{cases} 
\frac{p^2}{p+2} \left( \frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)} \right) \\
p^2 \left\{ (2p+1)(p+1)^2 - 4p^2(p+2)\mu \right\} \left( \mu \leq \frac{(p+1)^2}{2p(p+2)} \right) \\
p^2 \left\{ 4p^2(p+2)\mu - (2p+1)(p+1)^2 \right\} \left( \mu \geq \frac{(p+1)^3}{2p^2(p+2)} \right)
\end{cases}
\]
with equality for
\[
f(z) = \begin{cases} 
z^p F_1(2p, p; p+1; z) \quad \left( \mu \leq \frac{(p+1)^2}{2p(p+2)} \text{ or } \mu \geq \frac{(p+1)^3}{2p^2(p+2)} \right) \\
z^p F_1 \left( \frac{p}{2}, p; 1 + \frac{p}{2}; z^2 \right) \left( \frac{(p+1)^2}{2p(p+2)} \leq \mu \leq \frac{(p+1)^3}{2p^2(p+2)} \right)
\end{cases}
\]

Corollary 5. If a function $f(z) \in \mathcal{K}(\alpha)$, then
\[
|a_3 - \mu a_2^2| \leq \begin{cases} 
\frac{1-\alpha}{3} \{(3-2\alpha) - 3(1-\alpha)\mu \} \quad \left( \mu \leq \frac{2}{3} \right) \\
\frac{1-\alpha}{3} \left\{ 3(1-\alpha)\mu - (3-2\alpha) \right\} \left( \mu \geq \frac{2(2-\alpha)}{3(1-\alpha)} \right) \\
\end{cases}
\]
with equality for
\[
f(z) = \begin{cases} 
\frac{1 - (1-z)^{2\alpha-1}}{2\alpha-1} \quad \text{and} \quad \log \left( \frac{1}{1-z} \right) \quad \left( \mu \leq \frac{2}{3} \text{ or } \mu \geq \frac{2(2-\alpha)}{3(1-\alpha)} \right) \\
z^2 F_1 \left( \frac{1}{2}; 1-\alpha; \frac{3}{2}; z^2 \right) \left( \frac{2}{3} \leq \mu \leq \frac{2(2-\alpha)}{3(1-\alpha)} \right)
\end{cases}
\]
Corollary 6. If a function $f(z) \in \mathcal{K}$, then

$$|a_3 - \mu a_2^2| \leq \begin{cases} 
1 - \mu & (\mu \leq \frac{2}{3}) \\
\frac{1}{3} & \left(\frac{2}{3} \leq \mu \leq \frac{4}{3}\right) \\
\mu - 1 & (\mu \geq \frac{4}{3})
\end{cases}$$

with equality for

$$f(z) = \begin{cases} 
z & \left(\mu \leq \frac{2}{3} \text{ or } \mu \geq \frac{4}{3}\right) \\
\frac{1}{2} \log \left(\frac{1+z}{1-z}\right) & \left(\frac{2}{3} \leq \mu \leq \frac{4}{3}\right)
\end{cases}.$$ 

References


Toshio Hayami
Kinki University
Department of Mathematics
Higashi-Osaka, Osaka 577-8502, Japan
e-mail: ha_ya_to112@hotmail.com

Shigeyoshi Owa
Kinki University
Department of Mathematics
Higashi-Osaka, Osaka 577-8502, Japan
e-mail: owa@math.kindai.ac.jp