On Class of Hypergeometric Meromorphic Functions with Fixed Second Positive Coefficients

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Abstract

In the present paper, we consider the class of hypergeometric meromorphic functions $\Sigma^*(A, B, k, c)$ with fixed second positive coefficient. The object of the present paper is to obtain the coefficient estimates, convex linear combinations, distortion theorems, and radii of starlikeness and convexity for $f$ in the class $\Sigma^*(A, B, k, c)$.

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1 Introduction

Let $\Sigma$ denote the class of meromorphic functions $f$ normalized by

\[ f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \]  

that are analytic and univalent in the punctured unit disk $U = \{ z : 0 < |z| < 1 \}$. For $0 \leq \beta < 1$, we denote by $S^*(\beta)$ and $k(\beta)$, the subclasses of $\Sigma$ consisting of all meromorphic functions that are, respectively, starlike of order $\beta$ and convex of order $\beta$ in $U$ (cf. e.g., [[1, 3, 5, 16]]).

For functions $f_j(z)(j = 1; 2)$ defined by

\[ f_j(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,j} z^n, \]

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

\[ (f_1 * f_2) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n. \]

Let us define the function $\tilde{\phi}(a, c; z)$ by

\[ \tilde{\phi}(a, c; z) = \frac{1}{z} + \sum_{n=0}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n z^n, \]

for $c \neq 0, -1, -2, ..., \$ and $a \in \mathbb{C}/\{0\}$, where $(\lambda)_n = \lambda(\lambda + 1)_{n-1}$ is the Pochhammer symbol. We note that

\[ \tilde{\phi}(a, c; z) = \frac{1}{z} {2F1}(1, a, c; z) \]

where

\[ {2F1}(b, a, c; z) = \sum_{n=0}^{\infty} \frac{(b)_n (a)_n}{(c)_n n!} z^n \]
is the well-known Gaussian hypergeometric function. Corresponding to the function $\tilde{\phi}(a, c; z)$, using the Hadamard product for $f \in \Sigma$, we define a new linear operator $L^*(a, c)$ on $\Sigma$ by

$$L^*(a, c)f(z) = \tilde{\phi}(a, c; z) * f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}} a_n z^n.$$  \hfill (5)

The meromorphic functions with the generalized hypergeometric functions were considered recently by Dziok and Srivastava [6], [7], Liu [10], Liu and Srivastava [11], [12],[13], Cho and Kim [4].

For a function $f \in L^*(a, c)f(z)$ we define

$$I_0(L^*(a, c)f(z)) = L^*(a, c)f(z),$$

and for $k = 1, 2, 3, ...$,

$$I_k(L^*(a, c)f(z)) = z (I^{k-1}L^*(a, c)f(z))' + \frac{2}{z}$$

$$= \frac{1}{z} + \sum_{n=1}^{\infty} n^k \frac{(a)_{n+1}}{(c)_{n+1}} a_n z^n.$$ \hfill (6)

We note that $I_k(L^*(a, a)f(z))$ studied by Frasin and Darus [8].

It follows from (5) that

$$z (L(a, c)f(z))' = aL(a + 1, c)f(z) - (a + 1) L(a, c)f(z).$$ \hfill (7)

Also, from (6) and (7) we get

$$z (I^kL(a, c)f(z))' = aI^kL(a + 1, c)f(z) - (a + 1) I^kL(a, c)f(z).$$ \hfill (8)

Now, let $-1 \leq B < A \leq 1$ and for all $z \in U$, a function $f \in \Sigma$ is said to be a member of the class $\Sigma^*(A, B, k)$ if it satisfies

$$\left| \frac{z (I^kL^*(a, c)f(z))' + I^kL^*(a, c)f(z)}{Bz (I^kL^*(a, c)f(z))' + A (I^kL^*(a, c)f(z))} \right| < 1.$$
Note that, for \( a = c \), \( \Sigma^* (1 - 2\alpha, -1, k) \) with \( 0 \leq \alpha < 1 \), is the class introduced and studied in [8]. In the following section, we will state a result studied previously by Ghanim, Darus and Swaminathan [9].

## 2 Preliminary results

For the class \( \Sigma^* (A, B, k) \), Ghanim, Darus and Swaminathan [9] showed:

**Theorem 1** Let the function \( f \) be defined by (5). If

\[
\sum_{n=1}^{\infty} n^k \frac{|(a)_{n+1}|}{|c|_{n+1}} \left| \left( a n + 1 \right) (1 - B) + (1 - A) \right| a_n \leq A - B,
\]

where \( k \in N_0, -1 \leq B < A \leq 1 \), then \( f \in \Sigma^* (A, B, k) \).

In view of Theorem 1, we can see that the function \( f \) given by (5) is in the class \( \Sigma^* (A, B, k) \) satisfying

\[
a_n \leq \frac{|(c)_{n+1}| (A - B)}{|(a)_{n+1}| n^k (n (1 - B) + (1 - A))}, \quad (n \geq 1, \ k \in N_0).
\]

In view of (9), we can see that the function \( f \) defined by (5) is in the class \( \Sigma^* (A, B, k) \) satisfying the coefficient inequality

\[
\frac{|(a)_{2}|}{|(c)_{2}|} a_1 \leq \frac{(A - B)}{(2 - (A + B))}.
\]

Hence we may take

\[
\frac{|(a)_{2}|}{|(c)_{2}|} a_1 = \frac{(A - B) c}{(2 - (A + B))}, \quad \text{for some } c (0 < c < 1).
\]
Making use of (12), we now introduce the following class of functions:

Let \( \Sigma^*(A, B, k, c) \) denote the class of functions \( f \) in \( \Sigma^*(A, B, k) \) of the form

\[
    f(z) = \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \sum_{n=2}^{\infty} \frac{|(c)_{n+1}|}{|(a)_{n+1}|} |a_n| z^n
\]

with \( 0 < c < 1 \).

In this paper we obtain coefficient estimates, convex linear combination, distortion theorem, and radii of starlikeness and convexity for \( f \) to be in the class \( \Sigma^*(A, B, k, c) \).

There are many studies regarding the fixed second coefficient see for example: Aouf and Darwish [2], Silverman and Silvia [14], and Uralegaddi [15], few to mention. We shall use similar techniques to prove our results.

### 3 Coefficient inequalities

**Theorem 2** A function \( f \) defined by (13) is in the class \( \Sigma^*(A, B, k, c) \), if and only if,

\[
    \sum_{n=2}^{\infty} n^k \frac{|(c)_{n+1}|}{|(a)_{n+1}|} (n (1 - B) + (1 - A)) |a_n| \leq (A - B) (1 - c).
\]

The result is sharp.

**Proof.** By putting

\[
    \frac{|(a)_{2}|}{|(c)_{2}|} a_1 = \frac{(A - B) c}{(2 - (A + B))}, \quad 0 < c < 1
\]

in (9), the result is easily derived. The result is sharp for function
\[ f_n(z) = \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \] 
\[ \frac{|(c)_{n+1}|}{|(a)_{n+1}|} \left( A - B \right) \frac{(1 - c)}{n^k (n (1 - B) + (1 - A))} z^n, \quad n \geq 2. \]

**Corollary 1** Let the function \( f \) given by (13) be in the class \( \Sigma^*(A, B, k, c) \), then
\[ a_n \leq \frac{(c)_{n+1}}{(a)_{n+1}} \frac{(A - B) (1 - c)}{n^k (n (1 - B) + (1 - A))}, \quad n \geq 2. \]

The result is sharp for the function \( f \) given by (16).

## 4 Growth and distortion theorems

A growth and distortion property for function \( f \) to be in the class \( \Sigma^*(A, B, k, c) \) is given as follows:

**Theorem 3** If the function \( f \) defined by (13) is in the class \( \Sigma^*(A, B, k, c) \) for \( 0 < |z| = r < 1 \), then we have
\[ \frac{1}{r} - \frac{(A - B) c}{(2 - (A + B))} r - \frac{(A - B) (1 - c)}{(3 - (2B + A))} r^2 \leq |f(z)| \]
\[ \leq \frac{1}{r} + \frac{(A - B) c}{(2 - (A + B))} r + \frac{(A - B) (1 - c)}{(3 - (2B + A))} r^2, \]
with equality for
\[ f_2(z) = \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \frac{(A - B) (1 - c)}{(3 - (2B + A))} z^2. \]
Proof. Since $\Sigma^*(A, B, k, c)$, Theorem 2 yields to the inequality

\begin{equation}
\left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \leq \frac{(A - B) (1 - c)}{n^k (n (1 - B) + (1 - A))}, \quad n \geq 2.
\end{equation}

Thus, for $0 < |z| = r < 1$

\begin{align*}
|f(z)| & \leq \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \sum_{n=2}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n z^n \\
|z| & = r
\end{align*}

\begin{align*}
& \leq \frac{1}{r} + \frac{(A - B) c}{(2 - (A + B))} r + r^2 \sum_{n=2}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \\
& \leq \frac{1}{r} + \frac{(A - B) c}{(2 - (A + B))} r + (A - B) (1 - c) r^2
\end{align*}

and

\begin{align*}
|f(z)| & \geq \frac{1}{z} - \frac{(A - B) c}{(2 - (A + B))} z - \sum_{n=2}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n z^n, \quad (|z| = r)
\end{align*}

\begin{align*}
& \geq \frac{1}{r} - \frac{(A - B) c}{(2 - (A + B))} r - r^2 \sum_{n=2}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \\
& \geq \frac{1}{r} - \frac{(A - B) c}{(2 - (A + B))} r - \frac{(A - B) (1 - c)}{(3 - (2B + A))} r^2
\end{align*}

Thus the proof of the theorem is complete.

**Theorem 4** If the function $f(z)$ defined by (13) is in the class $\Sigma^*(A, B, k, c)$ for $0 < |z| = r < 1$, then we have

\[
\frac{1}{r^2} - \frac{(A - B) c}{(2 - (A + B))} - \frac{(A - B) (1 - c)}{(3 - (2B + A))} r \leq |f'(z)|
\]
\[ \frac{1}{r^2} + \frac{(A-B) c}{(2 - (A + B))} + \frac{(A-B) (1-c)}{(3 - (2B + A))} r. \]

with equality for
\[ f_2(z) = \frac{1}{z} + \frac{(A-B) c}{(2 - (A + B))} z + \frac{(A-B) (1-c)}{(3 - (2B + A))} z^2. \]

**Proof.** From Theorem 2, it follows that
\[ n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \leq \frac{(A-B) (1-c)}{n^{k-1} (n(1-B) + (1-A))}, \quad n \geq 2. \]

Thus, for \(0 < |z| = r < 1\), and making use of (19), we obtain
\[ |f'(z)| \leq \left| -\frac{1}{z^2} \right| + \frac{(A-B) c}{(2 - (A + B))} + \sum_{n=2}^{\infty} n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n |z|^{n-1}, \quad (|z| = r) \]

\[ \leq \frac{1}{r^2} + \frac{(A-B) c}{(2 - (A + B))} + r \sum_{n=2}^{\infty} n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \]

\[ \leq \frac{1}{r^2} + \frac{(A-B) c}{(2 - (A + B))} + \frac{(A-B) (1-c)}{(3 - (2B + A))} r. \]

and
\[ |f'(z)| \geq \left| -\frac{1}{z^2} \right| - \frac{(A-B) c}{(2 - (A + B))} - \sum_{n=2}^{\infty} n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n |z|^{n-1}, \quad (|z| = r) \]

\[ \geq \frac{1}{r^2} - \frac{(A-B) c}{(2 - (A + B))} - r \sum_{n=2}^{\infty} n \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n \]

\[ \geq \frac{1}{r^2} - \frac{(A-B) c}{(2 - (A + B))} - \frac{(A-B) (1-c)}{(3 - (2B + A))} r. \]

The proof is complete.
5 Radii of Starlikeness and Convexity

The radii of starlikeness and convexity for the class $\Sigma^*(A, B, k, c)$ is given by the following theorem:

**Theorem 5** If the function $f$ given by (13) is in the class $\Sigma^*(A, B, k, c)$, then $f$ is starlike of order $\delta (0 \leq \delta \leq 1)$ in the disk $|z| < r_1(A, B, k, c, \delta)$ where $r_1(A, B, k, c, \delta)$ is the largest value for which

$$
(3 - \delta) (A - B) c r^2 + \frac{(n + 2 - \delta) (A - B) (1 - c)}{n k (n (1 - B) + (1 - A))} r^{n+1} \leq (1 - \delta)
$$

for $n \geq 2$. The result is sharp for function $f_n(z)$ given by (16).

**Proof.** It is enough to highlight that

$$
\left| \frac{(z) f'(z)}{f(z)} + 1 \right| \leq 1 - \delta
$$

for $|z| < r_1$. We have

$$
(20) \quad \left| \frac{(z) f'(z)}{f(z)} + 1 \right| = \left| \frac{2(A-B)c}{(2-(A+B))} z + \sum_{n=2}^{\infty} (n+1) \frac{(a)_{n+1}}{(c)_{n+1}} |a_n| \frac{z^n}{r} \right| \leq 1 - \delta.
$$

Hence (20) holds true if

$$
(21) \quad \frac{2 (A - B) c}{(2 - (A + B))} r^2 + \sum_{n=2}^{\infty} (n + 1) \left| \frac{(a)_{n+1}}{(c)_{n+1}} |a_n| r^{n+1} \right| \leq (1 - \delta) \left( 1 - \frac{(A - B) c}{(2 - (A + B))} r^2 - \sum_{n=2}^{\infty} \left| \frac{(a)_{n+1}}{(c)_{n+1}} a_n r^{n+1} \right| \right).
$$

or

$$
(22) \quad \frac{(3 - \delta) (A - B) c}{(2 - (A + B))} r^2 + \sum_{n=2}^{\infty} (n + 2 - \delta) \left| \frac{(a)_{n+1}}{(c)_{n+1}} a_n r^{n+1} \right| \leq (1 - \delta)
$$
and it follows that from (14), we may take

\[ a_n \leq \frac{\left| (c)_{n+1} \right| (A - B) (1 - c)}{\left| (a)_{n+1} \right| n^k (n (1 - B) + (1 - A))} \lambda_n, \quad (n \geq 2). \]  

(23)

where \( \lambda_n \geq 0 \) and \( \sum_{n=2}^{\infty} \lambda_n \leq 1. \)

For each fixed \( r \), we choose the positive integer \( n_o = n_o(r) \) for which

\[ \frac{n + 2 - \delta}{n^k (n (1 - B) + (1 - A))} \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| r^{n+1} \]

is maximal. Then it follows that

\[ \sum_{n=2}^{\infty} (n + 2 - \delta) \left| \frac{(a)_{n+1}}{(c)_{n+1}} \right| a_n r^n \leq \frac{(n_o + 2 - \delta) (A - B) (1 - c)}{n_o^k (n_o (1 - B) + (1 - A))} r^{n_o+1}. \]  

(24)

Then \( f \) is starlike of order \( \delta \) in \( 0 < |z| < r_1(A, B, k, c, \delta) \) provided that

\[ \frac{(3 - \delta) (A - B) c}{(2 - (A + B))} r^2 + \frac{(n_o + 2 - \delta) (A - B) (1 - c)}{n_o^k (n_o (1 - B) + (1 - A))} r^{n_o+1} \leq (1 - \delta) \]  

(25)

we find the value \( r_o = r_o(k, \beta, c, \delta, n) \) and the corresponding integer \( n_o(r_o) \) so that

\[ \frac{(3 - \delta) (A - B) c}{(2 - (A + B))} r^2 + \frac{(n_o + 2 - \delta) (A - B) (1 - c)}{n_o^k (n_o (1 - B) + (1 - A))} r^{n_o+1} = (1 - \delta) \]  

(26)

Then this value is the radius of starlikeness of order \( \delta \) for function \( f \) belonging to the class \( \Sigma^*(A, B, k, c) \).
Theorem 6 If the function \( f \) given by (13) is in the class \( \Sigma^*(A,B,k,c) \), then \( f \) is convex of order \( \delta (0 \leq \delta \leq 1) \) in the disk \( |z| < r_2(A,B,k,c,\delta) \) where \( r_2(A,B,k,c,\delta) \) is the largest value for which
\[
\frac{(3-\delta)(A-B)c}{(2-(A+B))} r_2^2 + \frac{(n+2-\delta)(A-B)(1-c)}{n^{k-1}(n(1-B)+(1-A))} r_2^{n+1} \leq (1-\delta).
\]
The result is sharp for function \( f_n \) given by (16).

Proof. By using the same technique in the proof of theorem (5) we can show that
\[
\left| \frac{(z) f''(z)}{f'(z)} + 2 \right| \leq (1-\delta).
\]
for \( |z| < r_2 \) with the aid of Theorem 2. Thus, we have the assertion of Theorem 6.

6 Convex Linear Combination

Our next result involves a linear combination of function of the type (13).

Theorem 7 If
\[
(27) \quad f_1(z) = \frac{1}{z} + \frac{(A-B)c}{(2-(A+B))} z
\]
and
\[
(28) \quad f_n = \frac{1}{z} + \frac{(A-B)c}{(2-(A+B))} z + \sum_{n=2}^{\infty} \left| \frac{(c)_{n+1}}{(a)_{n+1}} \right| \frac{(A-B)(1-c)}{(n(1-B)+(1-A))} z^n, \quad n \geq 2.
\]
Then $f \in \Sigma^*(A, B, k, c)$ if and only if it can expressed in the form

\begin{equation}
\label{eq:29}
f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z)
\end{equation}

where $\lambda_n \geq 0$ and $\sum_{n=2}^{\infty} \lambda_n \leq 1$.

**Proof.** From (27), (28) and (29), we have

$$f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) = \frac{1}{z} + \frac{(A - B) c}{2 - (A + B)} z + \sum_{n=2}^{\infty} \frac{(c)_{n+1}}{(a)_{n+1}} \frac{(A - B) (1 - c) \lambda_n}{(n (1 - B) + (1 - A))} z^n.$$

Since

$$\sum_{n=2}^{\infty} \frac{|(c)_{n+1}|}{|(a)_{n+1}|} \frac{(A - B) (1 - c) \lambda_n}{(n (1 - B) + (1 - A))} \frac{|(a)_{n+1}|}{|(c)_{n+1}|} \frac{(n (1 - B) + (1 - A))}{(A - B) (1 - c)} = \sum_{n=2}^{\infty} \lambda_n = 1 - \lambda_1 \leq 1,$$

it follows from Theorem 2 that the function $f \in \Sigma^*(A, B, k, c)$.

Conversely, let us suppose that $f \in \Sigma^*(A, B, k, c)$. Since

$$a_n \leq \frac{|(c)_{n+1}|}{|(a)_{n+1}|} \frac{(A - B) (1 - c)}{n^k (n (1 - B) + (1 - A))}, \quad (n \geq 2).$$

Setting

$$\lambda_n = \frac{n^k |(a)_{n+1}| (n (1 - B) + (1 - A))}{|(c)_{n+1}| (A - B) (1 - c)} a_n.$$
and

\[ \lambda_1 = 1 - \sum_{n=2}^{\infty} \lambda_n \]

It follows that

\[ f(z) = \sum_{n=2}^{\infty} \lambda_n f_n(z) \]

Thus complete the proof of the theorem.

**Theorem 8** The class \( \Sigma^*(A, B, k, c) \) is closed under linear combination.

**Proof.** Suppose that the function \( f \) be given by (13), and let the function \( g \) be given by

\[ g(z) = \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \sum_{n=2}^{\infty} |b_n| z^n, \quad (b_n \geq 2). \]

Assuming that \( f \) and \( g \) are in the class \( \Sigma^*(A, B, k, c) \), it is enough to prove that the function \( H \) defined by

\[ H(z) = \lambda f(z) + (1 - \lambda) g(z) \quad (0 \leq \lambda \leq 1) \]

is also in the class \( \Sigma^*(A, B, k, c) \).

Since

\[ H(z) = \frac{1}{z} + \frac{(A - B) c}{(2 - (A + B))} z + \sum_{n=2}^{\infty} |a_n\lambda + (1 - \lambda) b_n| z^n, \]

we observe that

\[ \sum_{n=2}^{\infty} \frac{|(a)_{n+1}|}{(c)_{n+1}} \left[ n^k \left( n \left( 1 - B \right) + \left( 1 - A \right) \right) \right] |a_n\lambda + (1 - \lambda) b_n| \leq (A - B) (1 - c). \]

with the aid of Theorem 2. Thus \( H \in \Sigma^*(A, B, k, c) \). Hence the theorem.

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