A note on the Bernstein’s cubature formula \(^1\)

Dan Bărbosu, Ovidiu T. Pop

Abstract

The Bernstein’s cubature formula is revisited and the evaluation of it’s remainder term is corrected.

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1 Preliminaries

Let us to denote \(\mathbb{N} = \{1, 2, \ldots \} \) and \(\mathbb{N}_0 = \mathbb{N} \cup \{0\}\). The Bernstein’s bivariate operator \(B_{m,n} : C([0, 1] \times [0, 1]) \to C([0, 1] \times [0, 1])\) is defined for any \(f \in C([0, 1] \times [0, 1])\), any \((x, y) \in [0, 1] \times [0, 1]\) and any \(m, n \in \mathbb{N}\) by:

\[
(B_{m,n} f)(x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x)p_{n,j}(y) f \left( \frac{k}{m}, \frac{j}{n} \right),
\]

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where

\[ p_{m,k}(x) = \binom{m}{k} x^k (1 - x)^{m-k} \]

and

\[ p_{n,j}(y) = \binom{n}{j} y^j (1 - y)^{n-j} \]

are the fundamental Bernstein’s polynomials.

Many approximation properties of the operator (1) are well known [1].

Let \( f \in C([0,1] \times [0,1]) \) be given. The following

\[ f = B_{m,n}f + R_{m,n}f \]

is known as the "Bernstein bivariate approximation formula", \( R_{m,n}f \) denoting the remainder term.

In [14], pp. 325, is mentioned the following:

"If \( f \in C^{(2,2)}([0,1] \times [0,1]) \) the remainder term of (4) can be expressed under the form

\[ (R_{m,n}f)(x, y) = -\frac{x(1-x)}{2m} f^{(2,0)}(x, \eta) - \frac{y(1-y)}{2n} f^{(0,2)}(\xi, y) \]
\[ + \frac{xy(1-x)(1-y)}{4mn} f^{(2,2)}(\xi, \eta)." \]

Next, using (4) with the expression of remainder term from (5), the Bernstein’s cubature formula is constructed.

In our recent paper [4], was obtained the correct form for the remainder term of (4) when the approximated function \( f \) belong to \( C([0,1] \times [0,1]) \) and an upper bound estimation for \( R_{m,n}f \) for the case when \( f \) is "sufficiently" differentiable on \([0,1] \times [0,1]\).
Let $X$ be a linear space, $L_1, L_2 : X \to X$ be projectors, $I : X \to X$ be the identity operator and $R_1, R_2 : X \to X$ be the remainder operators associated to $L_1$ and respectively $L_2$. If $L_1$ and $L_2$ commute on $X$, the following decomposition of the identity operator

\begin{equation}
I = L_1 L_2 + R_1 \oplus R_2
\end{equation}

with

\begin{equation}
R_1 \oplus R_2 = R_1 + R_2 - R_1 R_2
\end{equation}

is well known [6], [7].

Suppose now that $X := C([0, 1] \times [0, 1])$, $L_1 := B^x_m$, $L_2 := B^y_n$, where $B^x_m$, $B^y_n$ denote the parametrical extensions [1] of the Bernstein’s univariate operator, i.e.

\begin{equation}
(B^x_m f) (x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f \left( \frac{k}{m}, y \right),
\end{equation}

\begin{equation}
(B^y_n f) (x, y) = \sum_{k=0}^{m} \sum_{j=0}^{n} p_{m,k}(x) p_{n,j}(y) f \left( x, \frac{j}{n} \right).
\end{equation}

It is well known [1] that (8) and (9) are not projectors. Is also well known [1] that for $f \in C^{2,2}([0, 1] \times [0, 1])$ the remainder operators associated to (8) and (9) are defined respectively by

\begin{equation}
(R^x_{m,n} f) (x, y) = -\frac{x(1-x)}{2m} f^{(2,0)}(x, \eta)
\end{equation}

\begin{equation}
(R^y_{m,n} f) (x, y) = -\frac{y(1-y)}{2n} f^{(0,2)}(\xi, y)
\end{equation}
for any \((x, y) \in [0, 1] \times [0, 1]\) and any \(m, n \in \mathbb{N}\), where \((\xi, \eta) \in ]0,1]\times]0,1].\)

It is immediately that the operator (1) is the ”tensorial product” [6], [7] of operators (10) and (11), i.e

\[
B_{m,n} = B^{x}_{m}B^{y}_{n}.
\]

Computing the boolean sum of operators (10) and (11) one arrives to the expression (5) which is false, because \(B^{x}_{m}, B^{y}_{n}\) are not projectors and the decomposition formula (6) doesn’t holds.

By the above motives, we corrected (5) as follows.

\textbf{Theorem 1} \cite{2} For any \(f \in C([0, 1] \times [0, 1])\) and any \((x, y) \in [0, 1] \times [0, 1]\) the remainder term of (4) can be expressed under the form:

\[
(R_{m,n}f)(x, y) = -\frac{x(1-x)}{m} \sum_{k=0}^{m-1} \sum_{j=0}^{n} p_{m-1,k}(x)p_{n,j}(y) \left[ x, \frac{k}{m}, \frac{k+1}{m} : f \right] \\
- \frac{y(1-y)}{n} \sum_{k=0}^{m} \sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y) \left[ y, \frac{j}{n}, \frac{j+1}{n} : f \right] \\
+ \frac{xy(1-x)(1-y)}{mn} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y) \left[ x, \frac{k}{m}, \frac{k+1}{m}, y, \frac{j}{n}, \frac{j+1}{n} : f \right].
\]

Note that in (13) the brackets denote bivariate divided differences [2], [4].

In the Section 2, we use the following mean-value theorem for divided differences (see [8]).

\textbf{Theorem 2} Let \(m \in \mathbb{N}, a \leq x_{0} < x_{1} < \cdots < x_{m} \leq b\) distinct knots and \(f : [a, b] \rightarrow \mathbb{R}\) be a given function. If \(f\) is continuous on \([a, b]\) and has a \(m^{th}\)
derivatives on \((a, b)\), then there exists \(\xi \in (a, b)\) such that

\[
(x_0, x_1, \ldots, x_m; f) = \frac{1}{m!} f^{(m)}(\xi).
\]

## 2 Main results

**Theorem 3** Let \(p, q \in \mathbb{N}_0\), \(p + q \geq 1\), \(x_0, x_1, \ldots, x_p \in [a, b]\) and \(y_0, y_1, \ldots, y_q \in [c, d]\) be a distinct knots and \(f : [a, b] \times [c, d] \to \mathbb{R}\) be a function. If \(f(\cdot, y) \in C([a, b])\) for any \(y \in [c, d]\), \(\frac{\partial^p f}{\partial x^p}(\cdot, y)\) exists on \(]a, b[\) for any \(y \in [c, d]\), \(\frac{\partial^p f}{\partial x^p}(x, \ast) \in C([c, d])\) for any \(x \in ]a, b[\) and \(\frac{\partial^{p+q}}{\partial x^p \partial y^q}(x, \ast)\) exists on \(]c, d[\) for any \(x \in ]a, b[\), then there exists \((\xi, \eta) \in ]a, b[ \times ]c, d[\) such that

\[
(x_0, x_1, \ldots, x_p; y_0, y_1, \ldots, y_q; f) = \frac{1}{p!} \partial^p f \left(\xi, \ast\right)\bigg|_y
\]

so the equality (15) holds.

**Proof.** Applying the method of parametric extension (see [3]) and the mean-value theorem for one dimensional divided differences, there exist \(\xi \in ]a, b[\) and respectively \(\eta \in ]c, d[\), such that

\[
\left[ x_0, x_1, \ldots, x_p; y_0, y_1, \ldots, y_q; f \right] = \frac{1}{p!} \left[ y_0, y_1, \ldots, y_q; \frac{\partial^p f}{\partial x^p}(\xi, \ast) \right]_y
\]

so the equality (15) holds.
Remark 1 In the conditions of Theorem 3, if $p = 0$ then $q \in \mathbb{N}$, and we consider that $f$ has the properties that $f(x_0, \ast) \in C([c, d])$ and $\frac{\partial^q f}{\partial y^q}(x_0, \ast)$ exists on $]c, d[. If $q = 0$, then we consider similarly above conditions about function $f$.

Theorem 4 Let $p, q \in \mathbb{N}_0$, $p + q \geq 1$, $x_0, x_1, \ldots, x_p \in [a, b]$ and $y_0, y_1, \ldots, y_q \in [c, d]$ be a distinct knots. If $f : [a, b] \times [c, d] \rightarrow \mathbb{R}$ is a function with the property that $f \in C^{(p, q)}([a, b] \times [c, d])$, then exists $(\xi, \eta) \in ]a, b[ \times ]c, d[\) such that

$$\left[ x_0, x_1, \ldots, x_p \ y_0, y_1, \ldots, y_q ; f \right] = \frac{1}{p!q!} \frac{\partial^{p+q} f}{\partial x^p \partial y^q} (\xi, \eta).$$

Proof. It results from Theorem 3.

Theorem 5 Let $f : [0, 1] \times [0, 1] \rightarrow \mathbb{R}$ be a function.

If $f(\cdot, y) \in C^1([0, 1])$ for any $y \in [0, 1]$, exists $\frac{\partial^2 f}{\partial x^2}(\cdot, y)$ on $]0, 1[$ for any $y \in [0, 1]$, $\frac{\partial^2 f}{\partial x^2}(x, \ast) \in C^1([0, 1])$ for any $x \in ]0, 1[\), exists $\frac{\partial^4 f}{\partial x^2 \partial y^2}(x, \ast)$ on $]0, 1[\)$ for any $x \in ]0, 1[\$, then for any $(x, y) \in [0, 1] \times [0, 1]$, any $m, n \in \mathbb{N}$, there exist $(\xi_i(k, j), \eta_i(k, j)) \in [0, 1] \times [0, 1]$, $i \in \{1, 2, 3\}$, such that

$$\left( R_{m,n}f \right)(x, y) = \frac{x(1 - x)}{2m} \sum_{k=0}^{m-1} \sum_{j=0}^{n} \frac{\partial^2 f}{\partial x^2} (\xi_1(k, j), \eta_1(k, j)) +$$

$$\frac{y(1 - y)}{2n} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} \frac{\partial^2 f}{\partial y^2} (\xi_2(k, j), \eta_2(k, j)) +$$

$$\frac{xy(1 - x)(1 - y)}{4mn} \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} \frac{\partial^4 f}{\partial x^2 \partial y^2} (\xi_3(k, j), \eta_3(k, j)).$$
If $\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^4 f}{\partial x^2 \partial y^2}$ are bounded on $]0,1[\times]0,1[$, the following inequalities

$$|(R_{m,n}f)(x, y)| \leq \frac{x(1-x)}{2m} M_1(f) + \frac{y(1-y)}{2n} M_2(f) + \frac{xy(1-x)(1-y)}{4mn} M_3(f)$$

and

$$|(R_{m,n}f)(x, y)| \leq \left(\frac{1}{8m} + \frac{1}{8n} + \frac{1}{64mn}\right) M(f)$$

hold, for any $(x, y) \in [0,1] \times [0,1]$ and any $m, n \in \mathbb{N}$, where

$$M_1(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^2 f}{\partial x^2}(x, y) \right|,$$

$$M_2(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^2 f}{\partial y^2}(x, y) \right|,$$

$$M_3(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x, y) \right|,$$

and

$$M(f) = \max\{M_1(f), M_2(f), M_3(f)\}.$$

**Proof.** In the relation (13) we apply Theorem 3 and the relation (17) results. Because $x(1-x) \leq \frac{1}{4}, y(1-y) \leq \frac{1}{4},$

$$\sum_{k=0}^{m-1} \sum_{j=0}^{n} p_{m-1,k}(x)p_{n,j}(y) = \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} p_{m,k}(x)p_{n-1,j}(y)$$

$$= \sum_{k=0}^{m-1} \sum_{j=0}^{n-1} p_{m-1,k}(x)p_{n-1,j}(y) = 1$$
and transforming into modulus in the relation above and taking into account
that the partial derivatives of \( f \) are bounded on \([0,1]\times[0,1]\), the inequalities
from (18) are obtained.

Integrating the Bernstein’s bivariate approximation formula (4) one ar-
vives to the following Bernstein’s cubature formula

\[
\int_{0}^{1} \int_{0}^{1} f(x,y) \, dx \, dy = \sum_{i=0}^{m} \sum_{j=0}^{n} A_{i,j} \left( \frac{i}{m}, \frac{j}{n} \right) + R_{m,n}[f].
\]

**Theorem 6** [14] The coefficients of the cubature formula (24) are given by the equalities:

\[
A_{ij} = \frac{1}{(m+1)(n+1)}, \quad i = 0, m, \quad j = 0, n.
\]

Regarding the remainder term of (23), we have the following:

**Theorem 7** In the conditions of Theorem 5, the following upper-bound es-
timation for the remainder term of Bernstein’s cubature formula (24) is

\[
|R_{m,n}[f]| \leq \frac{1}{12m} M_1(f) + \frac{1}{12n} M_2(f) + \frac{1}{144mn} M_3(f),
\]

where \( M_1(f) \), \( M_2(f) \) and \( M_3(f) \) were defined at (20), (21) and (22).

**Proof.** The inequality (26) follows by integrating the Bernstein’s bivariate
approximation formula (4) and taking the first inequality (18) into account.

**Theorem 8** Let \( f : [0,1] \times [0,1] \rightarrow \mathbb{R} \) be a function. If \( f \in C^{(2,2)}([0,1] \times [0,1]) \), the relations (17) and (26) hold, where

\[
M_1(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^2 f}{\partial x^2} (x,y) \right|,
\]
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\[ M_2(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^2 f}{\partial y^2}(x,y) \right|, \quad \text{and} \]

\[ M_3(f) = \sup_{(x,y) \in [0,1] \times [0,1]} \left| \frac{\partial^4 f}{\partial x^2 \partial y^2}(x,y) \right|. \]

**Proof.** It results from Theorem 7

**Remark 2** In Theorem 7 we give a new proof for the known inequality (26) (see [14], pp.325). The inequality from (26) is demonstrate in [14] in the conditions of Theorem 8.

**Theorem 9** In the conditions of Theorem 7 or Theorem 8, it follows that

\[ \lim_{m,n \to \infty} \sum_{i=0}^{m} \sum_{j=0}^{n} \frac{1}{(m+1)(n+1)} f \left( \frac{i}{m}, \frac{j}{n} \right) = \int_0^1 \int_0^1 f(x,y) dx dy \]

and the convergence from (27) is uniform.

**Proof.** It results from inequality (26).

**Remark 3** Because the Bernstein’s bivariate operator \( B_{m,n} \) conserve only the lineares functions in \( x \) and respectively \( y \), it follows that the degree of exactness for the cubature formula (24) is \((1,1)\). In the case when the approximated function \( f \) satisfies the hypotheses of Theorem 6, the above affirmation follows directly from the mentioned theorem.

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Dan Bărbosu
North University of Baia Mare
Department of Mathematics and Computer Science
Victoriei 76, 430122 Baia Mare Romania,
e-mail: barbosudan@yahoo.com
Ovidiu T. Pop
National College ”Mihai Eminescu”
5 Mihai Eminescu Street
440014 Satu Mare Romania,
e-mail: ovidiutiberiu@yahoo.com